Orthogonal Space-Time Block Codes with Sphere Packing[*]

Weifeng Su

Department of Electrical Engineering

State University of New York (SUNY) at Buffalo, NY 14260 USA

Email: weifeng@eng.buffalo.edu

Zoltan Safar

US R&D Center, Samsung Electro-Mechanics America, Inc.

Atlanta, GA, USA

Email: zoltan.safar@gmail.com

K. J. Ray Liu

Department of Electrical and Computer Engineering

University of Maryland, College Park, MD 20742, USA

Email: kjrliu@eng.umd.edu

Abstract

Orthogonal designs have received considerable attention in the development of efficient modulation and coding methods for future multi-antenna wireless communication systems due to their special properties. In this correspondence, we propose a class of space-time block codes constructed by combining orthogonal designs with sphere packing for an arbitrary number of transmit antennas. The structure of the orthogonal designs is exploited to guarantee full diversity, and sphere packing is used to improve the coding advantage. Space-time block code construction from block-orthogonal designs is also considered: the full-diversity property is ensured by rotating the sphere packing underlying the code, and the optimal rotation angle is determined for a class of sphere packing. Code design examples are provided for 2 and 4 transmit antennas and various transmission rates. The simulation results show that by jointly designing the symbols in the orthogonal designs, the performance of the block codes can be significantly increased.

Keywords: multiple antennas, diversity product, multiple-input-multiple-output (MIMO) systems, space-time block codes (STBCs), orthogonal designs, sphere packing.

1 Introduction

By employing multiple transmit and receive antennas and developing appropriate space-time (ST) coding and modulation, multiple-input-multiple-output (MIMO) systems can significantly increase data rates in wireless communications. The performance criteria for MIMO ST coding were first derived in [1] and [2], characterizing two quantities: the diversity advantage, which describes the

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asymptotic error rate decrease as a function of the signal to noise ratio (SNR), and the coding advantage, which determines the vertical shift of the error performance curve. Since then, a large number of ST codes have been proposed, for example, [3]-[23].

Orthogonal designs have received considerable attention in designing space-time block codes (STBCs) for MIMO communication systems. The theory of orthogonal designs, which focuses on the construction of square matrices from real or complex variables in such a way that their columns are orthogonal to each other, has a long history in mathematics [24]. The first transmit diversity scheme using orthogonal designs was proposed in [3] to construct STBCs for 2 transmit antennas from a 2 by 2 complex orthogonal design. The idea was extended in [4] to non-square code matrices and more transmit antennas. In [5], it was shown that for linear receivers, the orthogonal signaling structure is optimal in the sense that it maximizes the receiver signal to noise ratio. Results from the theory of amicable orthogonal designs were used to construct STBCs. The design of full-diversity, square, complex STBCs was considered in [6] with the aim to reduce the decoding delay and to maintain the maximum achievable symbol rate. The design procedure was based on the properties of the underlying Clifford algebra. There are different realizations of orthogonal designs [24, 5, 6]. A simple recursive expression for orthogonal designs was given in [7]. If non-square codewords are allowed, the transmission rates of orthogonal STBCs can be improved and a systematic code design method was presented in [8].

To increase the transmission rates, the authors of [9] and [10] also constructed STBCs from block-orthogonal designs which are also well-known as quasi-orthogonal designs. In case of the block-orthogonal designs, the columns of the code matrices are grouped, and the columns within a group are not orthogonal, but the columns belonging to different groups are orthogonal to each other. This design approach increases the symbol rate, but the resulting STBCs, in general, cannot achieve full diversity. Full-diversity block-orthogonal STBCs were proposed in [11, 12, 13]. The full-diversity property was ensured by taking some of the channel symbols from a rotated version of the used constellation with a carefully chosen rotation angle.

This work considers the problem of further improving the performance of STBCs. We propose a class of space-time codes constructed by combining orthogonal designs with sphere packing in a systematic way for an arbitrary number of transmit antennas. In case of the conventional STBCs from orthogonal designs, the symbols are chosen independently from a given constellation. The basic idea of our method is to determine the values of the symbols in the orthogonal designs

2

jointly. The structure of the orthogonal designs is exploited to achieve full diversity, and sphere packing is used to improve the coding advantage. We also construct full-diversity STBCs from block-orthogonal designs with sphere packing. The full diversity is guaranteed by choosing some of the symbols from a rotated version of the used sphere packing. The optimal rotation angle that maximizes the normalized coding advantage will be determined for a class of sphere packing.

The correspondence is organized as follows. Section 2 will introduce the channel model and briefly summarize the relevant results from previous work. Section 3 will describe the STBC design method for orthogonal designs with sphere packing. The code construction method for block-orthogonal designs will be given in Section 4. The simulation results will be provided in Section 5, and some conclusions will be drawn in the last section.

2   Channel Model and STBC Design Criteria

We consider a wireless communication system with M transmit antennas and N receive antennas. The encoder divides the input bit stream into b bit long blocks, and for each block, it selects one space-time codeword from the codeword set of size $L = 2^b$. The selected codeword is then transmitted through the channel over the M transmit antennas and T time slots. Each codeword can be represented as a $T \times M$ matrix

$$C = \left[\begin{array}{cccc} c_1^1 & c_1^2 & \cdots & c_1^M \\ c_2^1 & c_2^2 & \cdots & c_2^M \\ \vdots & \vdots & \ddots & \vdots \\ c_T^1 & c_T^2 & \cdots & c_T^M \end{array} \right]_{T \times M}, \tag{1}$$

where c_t^i denotes the channel symbol transmitted by transmit antenna i, $i = 1, 2, \cdots, M$, at discrete time t, $t = 1, 2, \cdots, T$. The codewords are assumed to satisfy the energy constraint $E||C||_F^2 = MT$, where $||C||_F$ is the Frobenius norm[1] of C, and E stands for the expectation.

The transmission medium is assumed to be flat (frequency non-selective), quasi-static, Rayleigh fading channel, so the channel stays constant during the transmission of one codeword. The MIMO transceiver can be modeled as

$$Y = \sqrt{\frac{\rho}{M}} CH + Z, \tag{2}$$

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[1]The Frobenius norm of C is defined as

$$||C||_F^2 = \text{tr}(C^{\mathcal{H}}C) = \text{tr}(CC^{\mathcal{H}}) = \sum_{t=1}^{T} \sum_{i=1}^{M} |c_t^i|^2.$$

where $Y = \{y_t^j\}_{1 \leq t \leq T, 1 \leq j \leq N}$ is the received signal matrix of size $T \times N$ in which y_t^j is the received signal at receive antenna j at time t, $H = \{h_{i,j}\}_{1 \leq i \leq M, 1 \leq j \leq N}$ is the channel coefficient matrix of size $M \times N$ in which $h_{i,j}$ is the channel coefficient between transmit antenna i and receive antenna j, $Z = \{z_t^j\}_{1 \leq t \leq T, 1 \leq j \leq N}$ is the noise matrix of size $T \times N$, and C is the space-time codeword, as defined in (1). The channel coefficients and noise are modeled as zero-mean, complex Gaussian random variables with unit variance. The factor $\sqrt{\rho/M}$ in (2) ensures that ρ is the average signal to noise ratio (SNR) at each receive antenna, and it is independent of the number of transmit antennas.

Assuming that the channel matrix H is available at the receiver, the maximum likelihood decoding algorithm chooses the decoded codeword $\tilde{C}$ according to $\tilde{C} = \arg \min_C ||Y - \sqrt{\frac{\rho}{M}} CH||_F^2$. The pairwise error probability between two distinct codewords C and $\tilde{C}$ can be upper bounded as [1, 2]

$$P(C \rightarrow \tilde{C}) \leq \frac{1}{2} \left(\prod_{i=1}^{r} \lambda_i \right)^{-N} \cdot \left(\frac{\rho}{4M} \right)^{-rN}, \tag{3}$$

where $r = \text{rank}(C - \tilde{C})$, and $\lambda_1, \lambda_2, \cdots, \lambda_r$ are the nonzero eigenvalues of $(C - \tilde{C})(C - \tilde{C})^{\mathcal{H}}$. The superscript $\mathcal{H}$ stands for the complex conjugate and transpose of a matrix. Based on the pairwise error probability, two code design criteria have been proposed [1, 2]: (i) The minimum rank of the code difference matrix $C - \tilde{C}$ over all distinct codewords C and $\tilde{C}$ should be as large as possible; and (ii) The minimum value of the product $\prod_{i=1}^{r} \lambda_i$ over all distinct codewords C and $\tilde{C}$ should be as large as possible. This quantity is referred to as the coding advantage achieved by the STBC. If the difference matrix $C - \tilde{C}$ is always of full rank, the objective is to maximize the determinant of $(C - \tilde{C})^{\mathcal{H}}(C - \tilde{C})$.

If the matrix $C - \tilde{C}$ is always of full rank for a specific STBC, we say that this STBC achieves full diversity. We consider the design of STBCs of square size, i.e., $T = M$. If a STBC of square size achieves full diversity, the **diversity product**, which is the normalized coding advantage, is given by [15, 16]

$$\zeta = \frac{1}{2\sqrt{M}} \min_{C \neq \tilde{C}} \left| \det(C - \tilde{C}) \right|^{1/M}. \tag{4}$$

Substituting (4) into (3), we obtain $P(C \rightarrow \tilde{C}) \leq \frac{1}{2} (\zeta^2 \rho)^{-MN}$, so it is desirable to maximize the diversity product ζ if the full diversity has been achieved. In [17], it was shown that for a STBC consisting of L codewords, the diversity product is upper bounded by $\zeta \leq \sqrt{\frac{L}{2(L-1)}}$.

3 Orthogonal Designs with Sphere Packing

In this section, we consider the construction of STBCs from orthogonal designs with sphere packing for $M = 2^k$ ($k = 1, 2, 3, \cdots$) transmit antennas. If the number of transmit antennas is not a power of two, the desired STBC can be obtained by deleting some columns from a larger STBC designed for a power of two antennas. The structure of the orthogonal designs will be exploited to achieve full diversity, and sphere packing will be used to maximize the diversity product (coding advantage).

A recursive expression for orthogonal designs was given in [7] as follows. Let $G_1(x_1) = x_1 I_1$, and

$$G_{2^k}(x_1, \cdots, x_{k+1}) = \begin{bmatrix} G_{2^{k-1}}(x_1, \cdots, x_k) & x_{k+1} I_{2^{k-1}} \\ -x_{k+1}^* I_{2^{k-1}} & G_{2^{k-1}}^{\mathcal{H}}(x_1, \cdots, x_k) \end{bmatrix}, \quad k = 1, 2, 3, \cdots. \tag{5}$$

Then, $G_{2^k}(x_1, x_2, \cdots, x_{k+1})$ is an orthogonal design of size $2^k \times 2^k$ with complex variables $x_1, x_2, \cdots, x_{k+1}$. The symbol rate of G_{2^k} is $(k+1)/2^k$, which is the maximum rate for orthogonal designs of square size ([5], [6], [7] and the references therein).

For $M = 2^k$ transmit antennas, STBCs can be constructed from the orthogonal design G_{2^k} as

$$C = \sqrt{2^k/(k+1)} \, G_{2^k}(x_1, x_2, \cdots, x_{k+1}), \tag{6}$$

with some specific choices of $x_1, x_2, \cdots, x_{k+1}$ such that $E(|x_1|^2 + |x_2|^2 + \cdots + |x_{k+1}|^2) = k+1$. For example, if the complex variables $x_1, x_2, \cdots, x_{k+1}$ are chosen independently from a QPSK constellation, there are totally $L = 4^{k+1}$ codewords to be transmitted. The normalization factor $\sqrt{2^k/(k+1)}$ in (6) ensures that C satisfies the energy constraint $E||C||_F^2 = MT = 2^{2k}$. In case of the conventional STBCs from orthogonal designs, the symbols in the orthogonal designs are chosen independently from PSK or QAM constellations. The basic idea of the new scheme is that we design these symbols jointly with sphere packing to further increase the coding advantage.

For two distinct codewords $C = \sqrt{\frac{2^k}{k+1}} \, G_{2^k}(x_1, x_2, \cdots, x_{k+1})$ and $\tilde{C} = \sqrt{\frac{2^k}{k+1}} \, G_{2^k}(\tilde{x}_1, \tilde{x}_2, \cdots, \tilde{x}_{k+1})$, the code difference matrix is given by

$$C - \tilde{C} = \sqrt{\frac{2^k}{k+1}} \, G_{2^k}(x_1 - \tilde{x}_1, x_2 - \tilde{x}_2, \cdots, x_{k+1} - \tilde{x}_{k+1}),$$

and

$$(C - \tilde{C})(C - \tilde{C})^{\mathcal{H}} = (C - \tilde{C})^{\mathcal{H}}(C - \tilde{C}) = \frac{2^k}{k+1} \left(\sum_{i=1}^{k+1} |x_i - \tilde{x}_i|^2 \right) I_{2^k}. \tag{7}$$

It is easy to see that for two different vectors $(x_1, x_2, \cdots, x_{k+1})$ and $(\tilde{x}_1, \tilde{x}_2, \cdots, \tilde{x}_{k+1})$, the full rank of the code difference matrix is guaranteed.

Having ensured that the STBCs achieve full diversity, the next step is to maximize the diversity product. From (4), the diversity product can be expressed as

$$\zeta = \frac{1}{2\sqrt{k+1}} \min_{(x_1,\cdots,x_{k+1}) \neq (\tilde{x}_1,\cdots,\tilde{x}_{k+1})} \left(\sum_{i=1}^{k+1} |x_i - \tilde{x}_i|^2 \right)^{\frac{1}{2}}. \tag{8}$$

As a consequence, the diversity product is determined by the minimum Euclidean distance of the set of $(k+1)$-dimensional complex vectors $\{(x_1, x_2, \cdots, x_{k+1})\}$ underlying the STBC. Therefore, sphere packing in the $2(k+1)$-dimensional real Euclidean space $R^{2(k+1)}$ [29] can be used to maximize the diversity product. More formally, assume that $S = \{[a_{l,1} \, a_{l,2} \cdots a_{l,2(k+1)}]^T \in R^{2(k+1)} : 0 \leq l \leq L-1\}$ is a set of L points from a $2(k+1)$-dimensional sphere packing with total energy $\sum_{l=0}^{L-1} \left(|a_{l,1}|^2 + |a_{l,2}|^2 + \cdots + |a_{l,2(k+1)}|^2 \right) \triangleq E$. For each vector in S, we can define a corresponding set of complex symbols $\{x_i\}_{1 \leq i \leq k+1}$ as: $x_i = a_{l,2i-1} + \mathbf{j} a_{l,2i}$, where $\mathbf{j} \triangleq \sqrt{-1}$. Then, the matrices

$$C_l = \sqrt{2^k L/E} \; G_{2^k}(x_1, x_2, \cdots, x_{k+1}), \quad l = 0, 1, \cdots, L-1, \tag{9}$$

form a set of space-time codewords whose diversity product is determined by the minimum Euclidean distance of S. The factor $\sqrt{2^k L/E}$ ensures that the resulting STBC satisfies the energy constraint (assuming that all codewords are equally likely to be transmitted).

In the following subsections, we will provide code design examples for 2 and 4 transmit antennas.

3.1   Code Design for 2 Transmit Antennas

For 2 transmit antennas, the 2×2 orthogonal design is

$$G_2(x_1, x_2) = \begin{bmatrix} x_1 & x_2 \\ -x_2^* & x_1^* \end{bmatrix}, \tag{10}$$

which was first used by Alamouti in space-time coding [3]. Later in [16], a similar structure with constraint $|x_1|^2 + |x_2|^2 = 1$ was used to build 2×2 unitary matrices, the so-called Hamiltonian constellation for differential modulation. This constraint is not necessary for the design of STBCs with coherent detection.

In the four-dimensional real Euclidean space R^4, the sphere packing with the best known minimum Euclidean distance is a lattice that consists of all points with integer coordinates (a_1, a_2, a_3, a_4) such that $a_1 + a_2 + a_3 + a_4$ is even [29], and it is usually denoted as D_4. Alternatively, D_4 may also be defined as the integer span of the vectors v_1, v_2, v_3, v_4 that form the rows of the *generator*

6

matrix

$$\begin{bmatrix} 2 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{bmatrix}.$$

We now combine the orthogonal design G_2 and the sphere packing D_4 to construct STBCs for 2 transmit antennas. Assume that $S = \{[a_{l,1} \, a_{l,2} \, a_{l,3} \, a_{l,4}]^T \in \mathrm{R}^4 \, : \, 0 \le l \le L-1\}$ is a set of L points from D_4 with total energy $\sum_{l=0}^{L-1} \left(|a_{l,1}|^2 + |a_{l,2}|^2 + |a_{l,3}|^2 + |a_{l,4}|^2\right) \triangleq E$. Let

$$C_l = \sqrt{2L/E} \; G_2(a_{l,1} + \mathbf{j}a_{l,2}, a_{l,3} + \mathbf{j}a_{l,4}), \quad l = 0, 1, \cdots, L-1. \tag{11}$$

Then, $\{C_l : 0 \le l \le L-1\}$ is a set of space-time codewords whose diversity product is determined by the minimum Euclidean distance of S.

We list the diversity products of the orthogonal designs with sphere packing (abbreviated as Orth. with S.P.) for 2 transmit antennas in Table 1, and compare them with those of the cyclic codes [15], the parametric codes [17], the diagonal algebraic codes [21], the high rate codes $B_2(\phi, \theta)$ [20], and the orthogonal designs with PSK or QAM constellations. Table 1 shows that the diversity product of the orthogonal designs with sphere packing is greater than that of the other schemes except the case of $L = 16$ and $L = 256$. The diversity product of the diagonal algebraic codes is the same as that of orthogonal designs with the same QAM constellations. Note that for $L = 4$, there are optimal space-time block codes [17, 18], in the sense that the diversity product achieves the upper bound $\sqrt{2/3}$, and they can be shown to be equivalent to the codes obtained by the proposed method.

3.2 Code Design for 4 Transmit Antennas

The 4×4 orthogonal design for 4 transmit antennas is given by

$$G_4(x_1, x_2, x_3) = \begin{bmatrix} x_1 & x_2 & x_3 & 0 \\ -x_2^* & x_1^* & 0 & x_3 \\ -x_3^* & 0 & x_1^* & -x_2 \\ 0 & -x_3^* & x_2^* & x_1 \end{bmatrix}. \tag{12}$$

The symbol rate of G_4 is 3/4. We consider sphere packing in six-dimensional real Euclidean space R^6 since there are three complex symbols x_1, x_2, x_3 in G_4.

In R^6, the sphere packing with the best known minimum Euclidean distance is the lattice E_6 [29]. E_6 has a simple description as a three-dimensional complex lattice over the Eisenstein

integers[2]. Precisely, E_6 is the Eisenstein integer span of the vectors e_1, e_2, e_3 that form the rows of the generator matrix $\begin{bmatrix} \theta & 0 & 0 \\ 0 & \theta & 0 \\ 1 & 1 & 1 \end{bmatrix}$, where $\theta = \mathbf{j}\sqrt{3}$.

We now combine the sphere packing E_6 and the orthogonal design G_4 to construct STBCs for 4 transmit antennas. Assume that $S = \{[a_{l,1}\,a_{l,2}\,\cdots\,a_{l,6}]^T \in \mathrm{R}^6 : 0 \le l \le L-1\}$ is a set of L points from E_6 with total energy $\sum_{l=0}^{L-1}\left(|a_{l,1}|^2 + |a_{l,2}|^2 + \cdots + |a_{l,6}|^2\right) \triangleq E$. Let

$$C_l = \sqrt{4L/E}\ G_4(a_{l,1} + \mathbf{j}a_{l,2}, a_{l,3} + \mathbf{j}a_{l,4}, a_{l,5} + \mathbf{j}a_{l,6}), \quad l = 0, 1, \cdots, L-1. \tag{13}$$

Then, $\{C_l : 0 \le l \le L-1\}$ is a set of space-time codewords whose diversity product is determined by the minimum Euclidean distance of S. We list the diversity products of the orthogonal designs with sphere packing for 4 transmit antennas in Table 2, and compare them with those of the cyclic codes [15], the orthogonal designs with PSK constellations, the diagonal algebraic codes [21], and the full-diversity block-orthogonal designs [12, 13]. We can see that the diversity products of the orthogonal designs with sphere packing are greater than those of other four schemes. Note that the diversity product of the diagonal algebraic codes is the same as that of the full-diversity block-orthogonal design in case of using the same QAM constellations. For $L = 4$, the diversity product of the proposed scheme achieves the upper bound 0.8165.

4   Block-Orthogonal Designs with Sphere Packing

In case of 4 or more transmit antennas, there are STBCs from block-orthogonal designs [9, 10] that can provide higher symbol transmission rate than those from orthogonal designs. In order to obtain higher coding advantage, we also consider STBC construction from block-orthogonal designs with sphere packing. First, we will provide the code design method for 4 transmit antennas, and then we will describe the general case for any power of two transmit antennas.

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[2]The set of *Eisenstein integers* is defined as ([29], p.52)

$$\mathcal{E} = \{k + \omega l \ : \ \text{both } k \text{ and } l \text{ are integers}\},$$

in which $\omega = (-1 + \mathbf{j}\sqrt{3})/2$. Note taht the hexagonal lattice modulation is corresponding to a one-dimensional lattice over the Eisenstein integers.

4.1 Code Design for 4 Transmit Antennas

For 4 transmit antennas, a STBC with symbol transmission rate 1 was constructed [10] from the 2×2 Alamouti scheme as follows:

$$Q_4(x_1, x_2, x_3, x_4) = \begin{bmatrix} A & B \\ B & A \end{bmatrix} = \begin{bmatrix} x_1 & x_2 & x_3 & x_4 \\ -x_2^* & x_1^* & -x_4^* & x_3^* \\ x_3 & x_4 & x_1 & x_2 \\ -x_4^* & x_3^* & -x_2^* & x_1^* \end{bmatrix}, \tag{14}$$

where $A = \begin{bmatrix} x_1 & x_2 \\ -x_2^* & x_1^* \end{bmatrix}, B = \begin{bmatrix} x_3 & x_4 \\ -x_4^* & x_3^* \end{bmatrix}$, and x_1, x_2, x_3, x_4 are chosen from some signal constellations. In general, this scheme is not guaranteed to achieve full diversity. However, by properly choosing the signal constellations, the full diversity can be achieved [12, 13]: x_1 and x_2 can be chosen from any constellation $\mathcal{A}$, and x_3 and x_4 can be chosen from the rotated constellation $e^{\mathbf{j}\phi}\mathcal{A}$.

We now construct block-orthogonal STBCs with sphere packing for 4 transmit antennas from the block-orthogonal design of (14), but the proposed method can be extended easily to other block-orthogonal structures. We combine the block-orthogonal design Q_4 and the sphere packing D_4 in the following way. Assume that $S = \{[a_{l,1}\, a_{l,2}\, a_{l,3}\, a_{l,4}] \in \mathrm{R}^4 : 0 \le l \le L_0 - 1\}$ is a set of L_0 points from D_4 with total energy $\sum_{l=0}^{L_0-1} \left(|a_{l,1}|^2 + |a_{l,2}|^2 + |a_{l,3}|^2 + |a_{l,4}|^2\right) \triangleq E$. Let

$$x_1 = a_{l,1} + \mathbf{j}a_{l,2}, \quad x_2 = a_{l,3} + \mathbf{j}a_{l,4}, \quad 0 \le l \le L_0 - 1,$$

$$x_3 = e^{\mathbf{j}\phi}(a_{l',1} + \mathbf{j}a_{l',2}), \quad x_4 = e^{\mathbf{j}\phi}(a_{l',3} + \mathbf{j}a_{l',4}), \quad 0 \le l' \le L_0 - 1.$$

If we define

$$C_{l,l'} = \sqrt{\frac{2L_0}{E}} Q_4(x_1, x_2, x_3, x_4), \tag{15}$$

then $\{C_{l,l'} : 0 \le l, l' \le L_0 - 1\}$ is a STBC with $L = L_0^2$ codewords. In fact, the symbols x_1 and x_2 are taken from the sphere packing S in the same way as described in Section 3.1, and the symbols x_3 and x_4 are taken from the sphere packing $S' \in \mathrm{R}^4$. The set of points S' is a rotated version of S since for each $[a_{l,1}\, a_{l,2}\, a_{l,3}\, a_{l,4}]^T \in S$, there is a corresponding $[a'_{l,1}\, a'_{l,2}\, a'_{l,3}\, a'_{l,4}]^T \in S'$ such that

$$\begin{bmatrix} a'_{l,1} \\ a'_{l,2} \\ a'_{l,3} \\ a'_{l,4} \end{bmatrix} = \begin{bmatrix} \cos\phi & -\sin\phi & 0 & 0 \\ \sin\phi & \cos\phi & 0 & 0 \\ 0 & 0 & \cos\phi & -\sin\phi \\ 0 & 0 & \sin\phi & \cos\phi \end{bmatrix} \begin{bmatrix} a_{l,1} \\ a_{l,2} \\ a_{l,3} \\ a_{l,4} \end{bmatrix}.$$

The factor $\sqrt{2L_0/E}$ in (15) ensures that the resulting STBC satisfies the energy constraint. The

diversity product can be calculated as [13]

$$\zeta = \frac{1}{2}\sqrt{\frac{L_0}{2E}}\min_{X \neq \tilde{X}}\left(\sum_{i=1}^{2}|(x_i - \tilde{x}_i) + (x_{2+i} - \tilde{x}_{2+i})|^2\right)^{1/4}\left(\sum_{i=1}^{2}|(x_i - \tilde{x}_i) - (x_{2+i} - \tilde{x}_{2+i})|^2\right)^{1/4},$$

(16)

where $X = (x_1, x_2, x_3, x_4)$ and $\tilde{X} = (\tilde{x}_1, \tilde{x}_2, \tilde{x}_3, \tilde{x}_4)$. Let

$$\begin{bmatrix} u_1 \\ u_3 \end{bmatrix} = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_3 \end{bmatrix}, \text{ and } \begin{bmatrix} u_2 \\ u_4 \end{bmatrix} = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} x_2 \\ x_4 \end{bmatrix},$$

then the diversity product in (16) becomes

$$\zeta = \frac{1}{2}\sqrt{\frac{L_0}{E}}\min_{U \neq \tilde{U}}(|u_1 - \tilde{u}_1|^2 + |u_2 - \tilde{u}_2|^2)^{1/4}(|u_3 - \tilde{u}_3|^2 + |u_4 - \tilde{u}_4|^2)^{1/4},$$

(17)

where $U = (u_1, u_2, u_3, u_4)$ and $\tilde{U} = (\tilde{u}_1, \tilde{u}_2, \tilde{u}_3, \tilde{u}_4)$. This form allows us to use the results of [19] directly to determine the value of the rotation angle ϕ. Since we choose (x_1, x_2) from the sphere packing S and choose (x_3, x_4) from a rotated version of S, it was shown in [19] that if the rotation angle is chosen as $\phi = \pi/4$, the diversity product in (16) or (17) is not zero, so the obtained STBC is guaranteed to achieve full diversity. Moreover, in case of the sphere packing D_4, the rotation angle $\phi = \pi/4$ is optimal in the sense that we cannot choose any other rotation angle to obtain a higher diversity product ([19], p.947). We list the diversity products of the block-orthogonal designs with sphere packing D_4 for 4 transmit antennas in Table 2. We can see that the diversity product can be increased by using block-orthogonal designs with sphere packing for large L, for example, $L = 4096$. However, there is little or no advantage by using block-orthogonal designs with sphere packing in case of $L = 16$ and 256.

4.2 Code Design for $M = 2^{k+1}\,(k \geq 1)$ Transmit Antennas

In this subsection, we consider the code design problem for $M = 2^{k+1}\,(k \geq 1)$ transmit antennas. In case of $M = 4\,(k = 1)$, the optimal rotation angle was obtained by taking advantage of the result in [19]. However, [19] does not offer a solution for the general case, so we have to develop a new method to determine the optimal value of ϕ.

In general, for any orthogonal design $G_{2^k}(x_1, x_2, \cdots, x_{k+1})$ given by (5), a block-orthogonal design $Q_{2^{k+1}}(x_1, x_2, \cdots, x_{2(k+1)})$ of size 2^{k+1} by 2^{k+1} can be constructed as follows [9, 10, 13]:

$$Q_{2^{k+1}}(x_1, x_2, \cdots, x_{2(k+1)}) = \begin{bmatrix} G_{2^k}(x_1, x_2, \cdots, x_{k+1}) & G_{2^k}(x_{k+2}, x_{k+3}, \cdots, x_{2(k+1)}) \\ G_{2^k}(x_{k+2}, x_{k+3}, \cdots, x_{2(k+1)}) & G_{2^k}(x_1, x_2, \cdots, x_{k+1}) \end{bmatrix}.$$ (18)

For any sphere packing $S = \{[a_{l,1}\, a_{l,2}\, \cdots\, a_{l,2(k+1)}]^T \in \mathrm{R}^{2(k+1)} : 0 \le l \le L_0 - 1\}$ with total energy $\sum_{l=0}^{L_0-1} \sum_{i=1}^{2(k+1)} |a_{l,i}|^2 \triangleq E$, we combine the block-orthogonal design $Q_{2^{k+1}}$ and the sphere packing S to construct STBCs for $M = 2^{k+1}$ transmit antennas. For each l $(0 \le l \le L_0 - 1)$, let

$$x_i = a_{l,2i-1} + \mathbf{j}a_{l,2i}, \quad 1 \le i \le k+1,$$

and for each l' $(0 \le l' \le L_0 - 1)$, let

$$x_{k+1+i} = a'_{l',2i-1} + \mathbf{j}a'_{l',2i}, \quad 1 \le i \le k+1,$$

where $S' = \{[a'_{l',1}\, a'_{l',2}\, \cdots\, a'_{l',2(k+1)}]^T \in \mathrm{R}^{2(k+1)} : 0 \le l' \le L_0 - 1\}$ is a rotated version of S, which is specified as: $a'_{l,2i-1} + \mathbf{j}a'_{l,2i} = e^{\mathbf{j}\phi}(a_{l,2i-1} + \mathbf{j}a_{l,2i})$ or equivalently

$$\begin{bmatrix} a'_{l,2i-1} \\ a'_{l,2i} \end{bmatrix} = \begin{bmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{bmatrix} \begin{bmatrix} a_{l,2i-1} \\ a_{l,2i} \end{bmatrix} \tag{19}$$

for $1 \le i \le k+1$ and $0 \le l \le L_0 - 1$. The rotation angle ϕ will be specified later, and it depends on the sphere packing S. If we define

$$C_{l,l'} = \sqrt{\frac{2^k L_0}{E}} Q_{2^{k+1}}(x_1, x_2, \cdots, x_{2(k+1)}), \tag{20}$$

then $\{C_{l,l'} : 0 \le l, l' \le L_0 - 1\}$ is a STBC with $L = L_0^2$ codewords for $M = 2^{k+1}$ transmit antennas. The factor $\sqrt{2^k L_0 / E}$ ensures that the resulting STBC satisfies the energy constraint. The diversity product can be calculated as [13]

$$\zeta = \frac{1}{2}\sqrt{\frac{L_0}{2E}} \min_{X \ne \tilde{X}} \left(\sum_{i=1}^{k+1} |(x_i - \tilde{x}_i) + (x_{i+k+1} - \tilde{x}_{i+k+1})|^2 \right)^{\frac{1}{4}} \left(\sum_{i=1}^{k+1} |(x_i - \tilde{x}_i) - (x_{i+k+1} - \tilde{x}_{i+k+1})|^2 \right)^{\frac{1}{4}}, \tag{21}$$

where $X = (x_1, x_2, \cdots, x_{2(k+1)})$ and $\tilde{X} = (\tilde{x}_1, \tilde{x}_2, \cdots, \tilde{x}_{2(k+1)})$. Obviously, we have

$$\begin{aligned} \zeta &\le \frac{1}{2}\sqrt{\frac{L_0}{2E}} \min_{X_1 \ne \tilde{X}_1} \left(\sum_{i=1}^{k+1} |x_i - \tilde{x}_i|^2 \right)^{1/2} \\ &= \frac{1}{2}\sqrt{\frac{L_0}{2E}} \min_{0 \le l \ne l' \le L_0-1} \left(\sum_{i=1}^{2(k+1)} |a_{l,i} - a_{l',i}|^2 \right)^{1/2} \\ &= \frac{1}{2}\sqrt{\frac{L_0}{2E}}\, d_{min}(S), \end{aligned} \tag{22}$$

where $X_1 = (x_1, x_2, \cdots, x_{k+1})$ and $\tilde{X}_1 = (\tilde{x}_1, \tilde{x}_2, \cdots, \tilde{x}_{k+1})$, and $d_{min}(S)$ denotes the minimum Euclidean distance of the sphere packing S. We observe that the diversity product is upper bounded by the normalized minimum Euclidean distance of the sphere packing S.

11

For a fixed sphere packing S, our objective is to find an optimal rotation angle ϕ to maximize the diversity product ζ in (21). From the result in the previous subsection, we know that in case of the sphere packing from the 4-dimensional lattice D_4, $\phi = \pi/4$ is an optimal rotation angle. The following theorem generalizes this result by showing that for any sphere packing from the $2(k+1)$-dimensional lattice $D_{2(k+1)}$, which is defined as ([29], p.117)

$$D_{2(k+1)} \triangleq \left\{ [n_1 \ n_2 \ \cdots \ n_{2(k+1)}] \in \mathbf{Z}^{2(k+1)} : n_1 + n_2 + \cdots + n_{2(k+1)} \ is \ even \right\},$$

the rotation angle $\phi = \pi/4$ is optimal in the sense that the upper bound in (22) is achieved.

Theorem 1 *For any sphere packing S from the $2(k+1)$-dimensional lattice $D_{2(k+1)}$, if the rotation angle in (19) is chosen as $\phi = \pi/4$, the diversity product of the STBC defined in (20) by combining the sphere packing S and the block-orthogonal design $Q_{2^{k+1}}$ is*

$$\zeta = \frac{1}{2} \sqrt{\frac{L_0}{2E}} \, d_{min}(S). \tag{23}$$

A proof of the theorem can be found in Appendix. Note that the minimum Euclidean distance of the sphere packing $D_{2(k+1)}$ is $\sqrt{2}$ ([29], p.117), i.e., $d_{min}(S) = \sqrt{2}$, thus the diversity product is $\zeta = \frac{1}{2}\sqrt{\frac{L_0}{E}}$. From (22) and (23), we observe that in case of using the sphere packing $D_{2(k+1)}$, $\phi = \pi/4$ is an optimal rotation angle. For $M = 4$ transmit antennas ($k = 1$), we showed this in the previous subsection using the results of [19] which was obtained via analytical tools from algebraic number theory. Theorem 1 provides an alternative way to determine the optimal rotation angle.

5   Simulation Results

We present some simulation results in this section. All of the simulated communication systems had 1 receive antenna. Assuming that there are L codewords in a STBC, and each codeword is transmitted over $T = M$ channel uses, the rate of the STBC is $(\log_2 L)/M$ bits per channel use, which corresponds to $(\log_2 L)/M$ bits/s/Hz spectral efficiency. We assumed that the channel state information is known exactly at the receiver and the ML decoding method was used. We present block error rate (bler) versus average SNR curves.

5.1   2 Transmit Antennas

First, we compare the proposed scheme, orthogonal design with sphere packing D_4, with other three schemes: the conventional orthogonal design, the parametric code [17], and the cyclic code

[15]. Figure 1 shows the simulation results for $L = 64$ case, resulting in a spectral efficiency of 3 bits/s/Hz. In order to maintain the same spectral efficiency, we simulated the conventional orthogonal design with 8PSK and 8QAM modulations, respectively. The curves demonstrate that the proposed method has the best performance. For example, at a bler of 10^{-2}, the orthogonal design with sphere packing has an improvement of about 0.5 dB over the conventional orthogonal design with 8QAM, 1.5 dB over the orthogonal design with 8PSK, 2 dB over the parametric code, and 4 dB over the cyclic code. We observe that all the bler curves have approximately the same asymptotic slope, suggesting that all these schemes achieve the same diversity order.

Figure 2 depicts the simulation results for the $L = 256$ case (spectral efficiency of 4 bits/s/Hz). Our scheme is compared with the diagonal algebraic code [21], the high rate code $B_2(\phi, \theta)$ [20], the Golden code [22], and the conventional orthogonal design with 16-QAM. We chose 16-QAM constellation for the diagonal algebraic code, and the corresponding unitary rotation matrix was $\frac{1}{\sqrt{2}} \begin{bmatrix} 1 & e^{\mathbf{j}\pi/4} \\ 1 & -e^{\mathbf{j}\pi/4} \end{bmatrix}$. We used QPSK constellation for the high rate code $B_2(\phi, \theta)$ with the optimum parameters $\phi = e^{\mathbf{j}\pi/4}$ and $\theta = e^{2\mathbf{j}\pi/6}$ (see [20], section II.D). The figure shows that the orthogonal design with sphere packing has an improvement of about 0.75 dB over the conventional orthogonal design, 1.5 dB over the Golden code, 2 dB over the diagonal algebraic code, and about 2.5 dB over the high rate code $B_2(\phi, \theta)$ at a bler of 10^{-2}.

We also observe that in the $L = 64$ case, even though the diversity product of the parametric code ($\zeta = 0.3070$) is larger than that of the orthogonal design with 8PSK ($\zeta = 0.2706$), the simulated bler curves of Figure 1 show that the the latter outperforms the former. A similar phenomenon can be seen in Figure 2 for $L = 256$: the orthogonal design with 16-QAM ($\zeta = 0.2237$) has a better performance than the diagonal algebraic code ($\zeta = 0.2237$), the high rate code $B_2(\phi, \theta)$ ($\zeta = 0.2544$), and the Golden code ($\zeta = 0.3344$). The reason for this is that the diversity product can only characterize the worst-case pair-wise error probability, while the actual performance is governed by the whole spectrum of the determinants of the code difference matrices $C - \tilde{C}$ [27].

5.2   4 Transmit Antennas

For 4 transmit antennas ($M = 4$), we compare both orthogonal designs with sphere packing and block-orthogonal design with sphere packing to some existing approaches. The performance of the STBCs designed for $L = 64$ (spectral efficiency of 1.5 b/s/Hz) is shown in Figure 3. The figure depicts the bler curves of the orthogonal design with sphere packing, the conventional orthogonal

design with QPSK, and the cyclic code. The curves show that the proposed scheme outperforms the other two schemes. For example, at a bler of 10^{-3}, we observe a performance improvement of about 1 dB compared to the conventional orthogonal design with QPSK, and about 2.5 dB compared to the cyclic code.

Figure 4 contains the simulation results for the $L = 256$ case (spectral efficiency of 2 bits/s/Hz). Our scheme is compared with the diagonal algebraic code [21], and the full-diversity block-orthogonal design [12, 13]. We used QPSK constellation for the diagonal algebraic code, and the corresponding unitary rotation matrix was

$$\frac{1}{2} \begin{bmatrix} 1 & \theta & \theta^2 & \theta^3 \\ 1 & -\theta & \theta^2 & -\theta^3 \\ 1 & \mathbf{j}\theta & -\theta^2 & -\mathbf{j}\theta^3 \\ 1 & -\mathbf{j}\theta & -\theta^2 & \mathbf{j}\theta^3 \end{bmatrix}, \quad \theta = e^{\mathbf{j}\pi/8}.$$

We also chose QPSK constellation for the full-diversity block-orthogonal design to maintain the same spectral efficiency. We can see that at a bler of 10^{-4}, the orthogonal design with sphere packing outperforms the block-orthogonal design about 0.5 dB, and outperforms the diagonal algebraic code about 1.75 dB.

Finally, in Figure 5, we show that the performance can be further improved if we construct STBCs from block-orthogonal designs with sphere packing. Figure 5 provides the simulation results for the $L = 4096$ case, giving a spectral efficiency of 3 bits/s/Hz. We compared four schemes: the conventional orthogonal design with 16-QAM, the orthogonal design with sphere packing, the block-orthogonal design with 8-QAM, and the block-orthogonal design with sphere packing. We constructed the STBC from orthogonal design with sphere packing by taking $L = 4096$ points from E_6, and the STBC from block-orthogonal design with sphere packing by taking $L_0 = 64$ points from D_4. From the simulation results we observe that at a bler of 10^{-4}, the orthogonal design with sphere packing is better than the conventional orthogonal design about 1.5 dB, and the block-orthogonal design with sphere packing has an improvement of about 0.75 dB over orthogonal design with sphere packing, and 1 dB over the block-orthogonal design with 8-QAM. Moreover, we can see that the performance curve of the block-orthogonal design with sphere packing has almost the same asymptotic slope as the other two schemes. This confirms that the STBC constructed from the block-orthogonal design with the sphere packing D_4 (with $\phi = \pi/4$) achieves full diversity.

6 Conclusion

In this correspondence, we focused on the problem of designing STBCs from orthogonal and block-orthogonal designs. We proposed a code construction method by combining orthogonal and block-orthogonal designs with sphere packing. We constructed codes for 2 and 4 transmit antennas and 1.5, 2, 3 and 4 bits/s/Hz spectral efficiencies. Both the theoretical diversity product values and the simulation results demonstrate the superior performance of the proposed method. In case of 2 transmit and 1 receive antennas and a spectral efficiency of 4 bits/s/Hz, we observed a coding gain of about 0.75 dB over the conventional orthogonal design, 1.5 dB over the Golden code, 2 dB over the diagonal algebraic code, and about 2.5 dB over the high rate code $B_2(\phi, \theta)$ at a bler of 10^{-2}. By jointly designing the symbols in the orthogonal and block-orthogonal designs, we exploited the additional degrees of freedom to further improve the performance at the expense of having to decode the symbols jointly. However, the decoding complexity can be significantly reduced if sphere decoding algorithms [25, 26] are used in which searching radius can be adjusted to reduce the decoding complexity and there is tradeoff between the decoding complexity and performance degradation. Finally, we would like to mention that the code designs discussed in this paper were optimized based on pairwise bler performance. It would be interesting to optimize the code designs based on bit-error-rate performance and investigate optimal bit labeling in future work.

Appendix: Proof of Theorem 1

Since the diversity product is upper bounded by $\frac{1}{2}\sqrt{\frac{L_0}{2E}}\, d_{min}(S)$, and $d_{min}(S) = \sqrt{2}$ in this case, in order to prove the theorem, it is sufficient to show that

$$\zeta \geq \frac{1}{2}\sqrt{\frac{L_0}{E}}. \tag{24}$$

By Cauchy's inequality, the diversity product in (21) can be lower bounded as follows:

$$\zeta \geq \frac{1}{2}\sqrt{\frac{L_0}{2E}} \min_{X \neq \tilde{X}} \left(\sum_{i=1}^{k+1} \left| (x_i - \tilde{x}_i)^2 - (x_{i+k+1} - \tilde{x}_{i+k+1})^2 \right| \right)^{1/2}, \tag{25}$$

where $X = (x_1, x_2, \cdots, x_{2(k+1)})$ and $\tilde{X} = (\tilde{x}_1, \tilde{x}_2, \cdots, \tilde{x}_{2(k+1)})$. Defining d as

$$d \triangleq \left(\sum_{i=1}^{k+1} \left| (x_i - \tilde{x}_i)^2 - (x_{i+k+1} - \tilde{x}_{i+k+1})^2 \right| \right)^{1/2}, \tag{26}$$

in order to show $\zeta \geq \frac{1}{2}\sqrt{\frac{L_0}{E}}$, it is sufficient to prove that $d \geq \sqrt{2}$ for any $X \neq \tilde{X}$.

15

For any pair of vectors $X_1 = (x_1, x_2, \cdots, x_{k+1})$ and $\tilde{X}_1 = (\tilde{x}_1, \tilde{x}_2, \cdots, \tilde{x}_{k+1})$ from the $2(k+1)$-dimensional lattice $D_{2(k+1)}$, the difference vector $X_1 - \tilde{X}_1$ can be represented as

$$X_1 - \tilde{X}_1 = (n_1 + \mathbf{j}n_2, \ n_3 + \mathbf{j}n_4, \ \cdots, \ n_{2k+1} + \mathbf{j}n_{2k+2}) \tag{27}$$

for some integers $n_1, n_2, \cdots, n_{2(k+1)}$. According to the definition of the lattice $D_{2(k+1)}$, we know that $n_1 + n_2 + \cdots + n_{2(k+1)}$ is even. Similarly, for any pair of vectors $X_2 = (x_{k+2}, x_{k+3}, \cdots, x_{2(k+1)})$ and $\tilde{X}_2 = (\tilde{x}_{k+2}, \tilde{x}_{k+3}, \cdots, \tilde{x}_{2(k+1)})$ from the rotated version of the lattice $D_{2(k+1)}$ with a rotation angle $\phi = \pi/4$ as defined in (19), the difference vector $X_2 - \tilde{X}_2$ can be represented as

$$X_2 - \tilde{X}_2 = e^{\mathbf{j}\pi/4}(m_1 + \mathbf{j}m_2, \ m_3 + \mathbf{j}m_4, \ \cdots, \ m_{2k+1} + \mathbf{j}m_{2k+2}) \tag{28}$$

for some integers $m_1, m_2, \cdots, m_{2(k+1)}$. The sum of these integers $m_1, m_2, \cdots, m_{2(k+1)}$ is also even. Substituting (27) and (28) into (26), we have

$$
\begin{aligned}
d^2 &= \sum_{i=1}^{k+1} \left| (n_{2i-1} + \mathbf{j}n_{2i})^2 - e^{\mathbf{j}\pi/2}(m_{2i-1} + \mathbf{j}m_{2i})^2 \right| \\
&= \sum_{i=1}^{k+1} \left[(n_{2i-1}^2 - n_{2i}^2 + 2m_{2i-1}m_{2i})^2 + (m_{2i}^2 - m_{2i-1}^2 + 2n_{2i-1}n_{2i})^2 \right]^{1/2} \triangleq \sum_{i=1}^{k+1} A_i.
\end{aligned}
$$

Since all of $n_1, n_2, \cdots, n_{2(k+1)}$ and $m_1, m_2, \cdots, m_{2(k+1)}$ are integers, so if there are two indices i_1 and i_2 ($i_1 \neq i_2$) such that $A_{i_1} \neq 0$ and $A_{i_2} \neq 0$, then $d^2 \geq A_{i_1} + A_{i_2} \geq 2$, which is the desired result.

In the following, we will prove that it is impossible to have the case that all of A_i ($1 \leq i \leq k+1$) are zeros under the constraint that $X \neq \tilde{X}$, i.e., $n_1, n_2, \cdots, n_{2(k+1)}$ and $m_1, m_2, \cdots, m_{2(k+1)}$ cannot be zero at the same time. Finally, if there is only one non-zero A_{i_0}, i.e., $A_{i_0} \neq 0$ and $A_i = 0$ for all $i \neq i_0$, we will show that $A_{i_0} \geq 2$, which also implies that $d^2 \geq 2$. The rest of proof is divided into three steps.

Step 1: First, we show that for any $i, 1 \leq i \leq k+1$, if $A_i = 0$, then $n_{2i-1} = n_{2i} = 0$ and $m_{2i-1} = m_{2i} = 0$.

Without loss of generality, we assume that $A_1 = 0$. The proof for $A_i = 0$ ($2 \leq i \leq k+1$) is similar. With the assumption that $A_1 = 0$, we know that both $n_1^2 - n_2^2 + 2m_1m_2$ and $m_2^2 - m_1^2 + 2n_1n_2$ are zeros, i.e.,

$$n_2^2 - n_1^2 = 2m_1m_2, \tag{29}$$

$$m_1^2 - m_2^2 = 2n_1n_2. \tag{30}$$

16

The equation in (29) implies that either both n_1 and n_2 are odd, or both n_1 and n_2 are even. If both n_1 and n_2 are odd, denoted as $n_1 = 2\tilde{n}_1 + 1$ and $n_2 = 2\tilde{n}_2 + 1$, then substituting them into (29), we have $4(\tilde{n}_2^2 + \tilde{n}_2 - \tilde{n}_1^2 - \tilde{n}_1) = 2m_1 m_2$. It follows that one of m_1 and m_2 is even. On the other hand, from (30) we know that either both m_1 and m_2 are odd, or both m_1 and m_2 are even. Thus we conclude that both m_1 and m_2 are even, denoted as $m_1 = 2\tilde{m}_1$ and $m_2 = 2\tilde{m}_2$. Substituting them into (30), we have $4(\tilde{m}_1^2 - \tilde{m}_2^2) = 2n_1 n_2$, which is contradictory to the assumption that both n_1 and n_2 are odd. Thus, both n_1 and n_2 must be even, denoted as $n_1 = 2\tilde{n}_1$ and $n_2 = 2\tilde{n}_2$

Similarly, we can prove that both m_1 and m_2 must be even, denoted as $m_1 = 2\tilde{m}_1$ and $m_2 = 2\tilde{m}_2$. Substituting $n_1 = 2\tilde{n}_1$, $n_2 = 2\tilde{n}_2$, $m_1 = 2\tilde{m}_1$ and $m_2 = 2\tilde{m}_2$ into (29) and (30), we obtain

$$\tilde{n}_2^2 - \tilde{n}_1^2 = 2\tilde{m}_1 \tilde{m}_2, \tag{31}$$

$$\tilde{m}_1^2 - \tilde{m}_2^2 = 2\tilde{n}_1 \tilde{n}_2. \tag{32}$$

Repeating the above argument, we can prove that all of $\tilde{n}_1, \tilde{n}_2, \tilde{m}_1$ and $\tilde{m}_2$ are even. We can continue this process repetitively. Since all of n_1, n_2, m_1 and m_2 are finite integers, we conclude that some of n_1, n_2, m_1 and m_2 must be zero.

If $n_1 = 0$, from (30) we have $m_1^2 = m_2^2$, i.e., $m_1 = \pm m_2$. Substituting $n_1 = 0$ and $m_1 = \pm m_2$ into (29), we arrive at $n_1 = 2m_1 m_2 = \pm 2m_2^2$. It follows that $n_2 = 0$ and $m_2 = 0$, since both n_2 and m_2 are integers. Thus, all of n_1, n_2, m_1 and m_2 are zeros. If $n_2 = 0$ or $m_1 = 0$ or $m_2 = 0$, we can prove that all of n_1, n_2, m_1 and m_2 are zeros in a similar way.

Step 2: In this step, we show that under the constraint that $X \neq \tilde{X}$, one of A_i $(1 \leq i \leq k+1)$ must be non-zero. According to the result in Step 1, we know that for any $i, 1 \leq i \leq k+1$, if $A_i = 0$, then $n_{2i-1} = n_{2i} = 0$ and $m_{2i-1} = m_{2i} = 0$. If all of A_i $(1 \leq i \leq k+1)$ are zeros, then all of $n_1, n_2, \cdots, n_{2(k+1)}$ and $m_1, m_2, \cdots, m_{2(k+1)}$ are zeros, which is contradictory to the constraint that $X \neq \tilde{X}$. As a result, for two distinct vectors X and $\tilde{X}$, at least one of the A_i's must be non-zero. Therefore, the only remaining case we have to consider is when exactly one A_i is non-zero.

Step 3: Finally, we show that if there is one index i_0 such that $A_{i_0} \neq 0$ and $A_i = 0$ for all $i \neq i_0$, then $A_{i_0} \geq 2$. Without loss of generality, we assume that $A_1 \neq 0$ and $A_i = 0$ for all $2 \leq i \leq k+1$. From the result of Step 1, we know that $n_{2i-1} = n_{2i} = 0$ and $m_{2i-1} = m_{2i} = 0$ for all $2 \leq i \leq k+1$. Since all of $n_1, n_2, \cdots, n_{2(k+1)}$ and $m_1, m_2, \cdots, m_{2(k+1)}$ cannot be zero at the same time, one of n_1, n_2, m_1, m_2 must be non-zero. Moreover, since both $n_1 + n_2 + \cdots + n_{2(k+1)}$ and $m_1 + m_2 + \cdots + m_{2(k+1)}$ are even, both $n_1 + n_2$ and $m_1 + m_2$ are even in this case.

17

From the assumption that $A_1 \neq 0$, i.e., $A_1 = \left[(n_1^2 - n_2^2 + 2m_1m_2)^2 + (m_2^2 - m_1^2 + 2n_1n_2)^2\right]^{1/2} \neq 0$, we know that at least one of $n_1^2 - n_2^2 + 2m_1m_2$ and $m_2^2 - m_1^2 + 2n_1n_2$ must be non-zero. Since

$$n_1^2 - n_2^2 + 2m_1m_2 = (n_1 + n_2)(n_1 - n_2) + 2m_1m_2,$$
$$m_2^2 - m_1^2 + 2n_1n_2 = (m_2 + m_1)(m_2 - m_1) + 2n_1n_2,$$

and both $n_1 + n_2$ and $m_1 + m_2$ are even, both $n_1^2 - n_2^2 + 2m_1m_2$ and $m_2^2 - m_1^2 + 2n_1n_2$ are even. If $n_1^2 - n_2^2 + 2m_1m_2 \neq 0$, then $|n_1^2 - n_2^2 + 2m_1m_2| \geq 2$. If $m_2^2 - m_1^2 + 2n_1n_2 \neq 0$, then $|m_2^2 - m_1^2 + 2n_1n_2| \geq 2$. Therefore, we conclude that $A_1 \geq 2$. Thus, we have proved the theorem completely. $\square$

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Biography of Weifeng Su:

Weifeng Su (M03) received the Ph.D. degree in electrical engineering from the University of Delaware, Newark in 2002. He received his B.S. and Ph.D. degrees in applied mathematics from Nankai University, Tianjin, China, in 1994 and 1999, respectively. His research interests span a broad range of areas from signal processing to wireless communications and networking, including space-time coding and modulation for MIMO wireless communications, MIMO-OFDM systems, cooperative communications for wireless networks, and ultra-wideband (UWB) communications.

Dr. Su has been an Assistant Professor at the Department of Electrical Engineering, State University of New York (SUNY) at Buffalo since March 2005. From June 2002 to March 2005, he was a Postdoctoral Research Associate with the Department of Electrical and Computer Engineering and the Institute for Systems Research (ISR), University of Maryland, College Park. Dr. Su received the Signal Processing and Communications Faculty Award from the University of Delaware in 2002 as an outstanding graduate student in the field of signal processing and communications. In 2005, he received the Invention of the Year Award from the University of Maryland. Dr. Su has been an Associate Editor of IEEE Transactions on Vehicular Technology, IEEE Signal processing Letters, and a Guest Editor of special issue on cooperative communications and networking of IEEE journal of Selected Areas in Communications.

Biography of Zoltan Safar:

Zoltan Safar (M'04) received the University Diploma in electrical engineering from the Technical University of Budapest, Budapest, Hungary in 1996, and the M.S. and Ph.D. degrees in electrical and computer engineering from the University of Maryland, College Park, MD, USA in 2001 and 2003, respectively. His research interests include wireless communications and multimedia signal processing, with particular focus on indoor and outdoor mobile positioning systems and algorithms, MIMO communication systems, space-time and space-frequency coding.

Currently, Dr. Safar is a senior system engineer at Samsung Electro-Mechanics R&D Center, Atlanta, GA, USA. From 2003 to 2005, he was an assistant professor at the Department of Innovation, IT University of Copenhagen, Copenhagen, Denmark. Dr. Safar was the recipient of the Outstanding Systems Engineering Graduate Student Award from the Institute for Systems Research, University of Maryland in 2003.

Biography of K. J. Ray Liu:

K. J. Ray Liu (F03) received the B.S. degree from the National Taiwan University and the Ph.D. degree from UCLA, both in electrical engineering. He is Professor and Associate Chair, Graduate Studies and Research, and Director of Communications and Signal Processing Laboratory of Electrical and Computer Engineering Department, University of Maryland, College Park. He leads the Maryland Signals and Information Group conducting research encompassing broad aspects of information technology including signal processing, communications, networking, information forensics and security, biomedical and bio-informatics.

Dr. Liu is the recipient of numerous honors and awards including best paper awards from IEEE Signal Processing Society (twice), IEEE Vehicular Technology Society, and EURASIP; IEEE Signal Processing Society Distinguished Lecturer, EURASIP Meritorious Service Award, and National Science Foundation Young Investigator Award. He also received various teaching and research recognitions from University of Maryland including university level Distinguished Scholar-Teacher Award, Invention of the Year Award, and college-level Poole and Kent Company Senior Faculty Teaching Award.

Dr. Liu is Vice President - Publications and on the Board of Governor of IEEE Signal Processing Society. He was the Editor-in-Chief of IEEE Signal Processing Magazine and the founding Editor-in-Chief of EURASIP Journal on Applied Signal Processing.
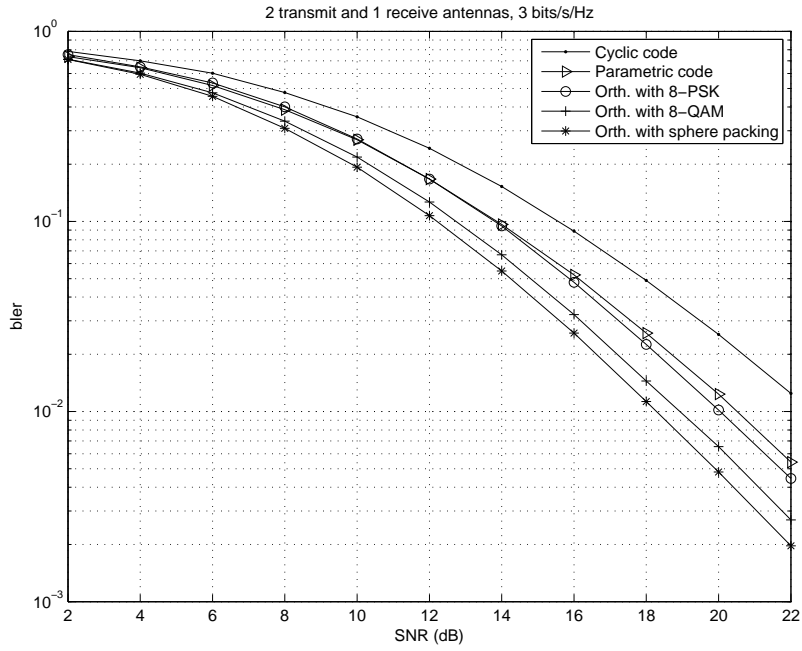
Figure 1: Block error rate performance of cyclic code '·', parametric code '▷', orth. with 8PSK '○', orth. with 8QAM '+', and orth. with sphere packing '∗'. $L = 64$.
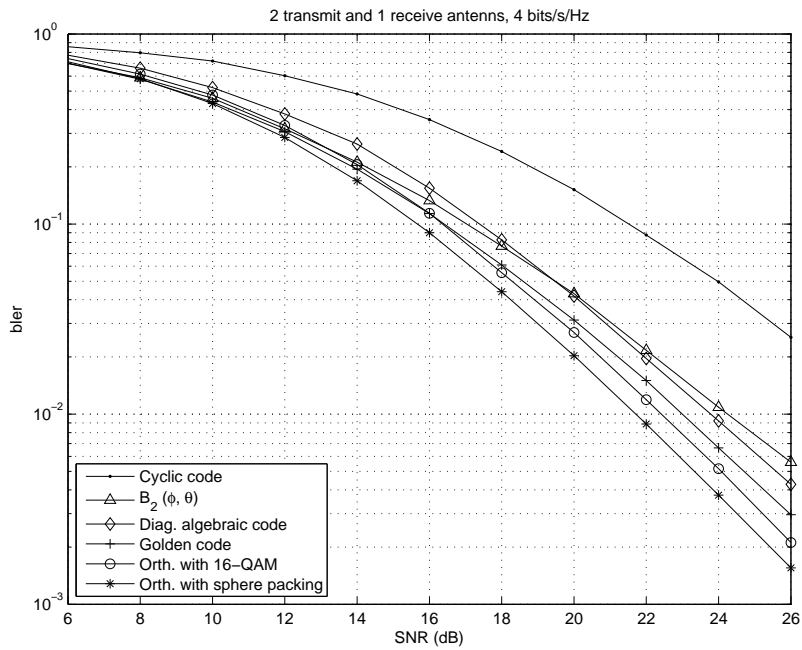

Figure 2: Block error rate performance of cyclic code '·', diagonal algebraic code '◇', $B_2(\phi, \theta)$ [20] '△', the Golden code '+', orth. with 16-QAM '○', and orth. with sphere packing '∗'. $L = 256$.
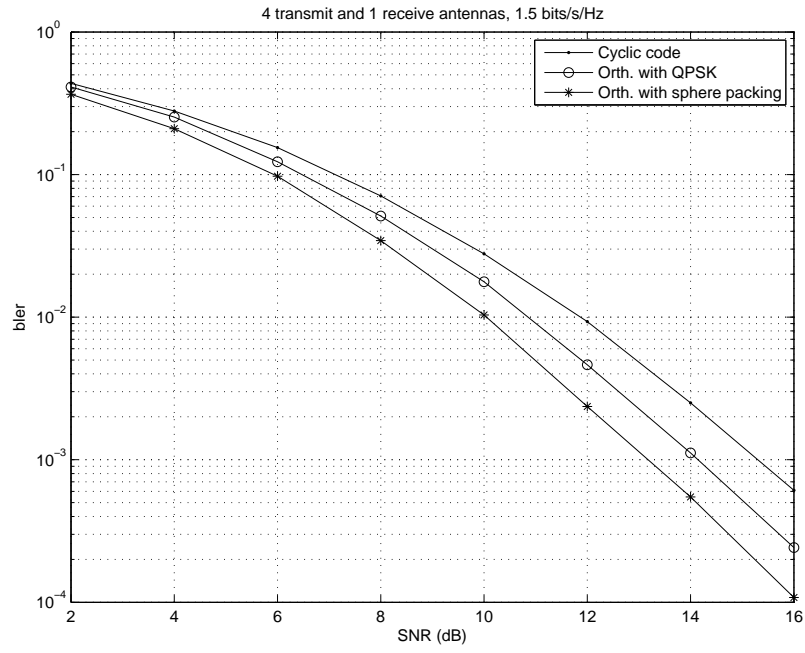
Figure 3: Block error rate performance of cyclic codes ’·’, orth. with QPSK ’◦’, and orth. with sphere packing ’∗’. $L = 64$.
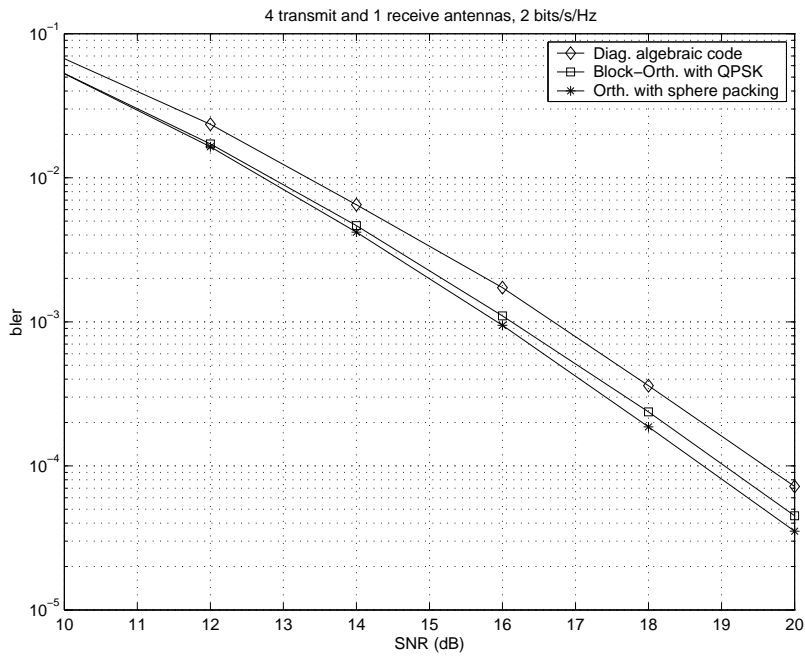

Figure 4: Block error rate performance of diagonal algebraic code ’⋄’, block-orth. with QPSK ’□’, and orth. with sphere packing ’∗’. $L = 256$.
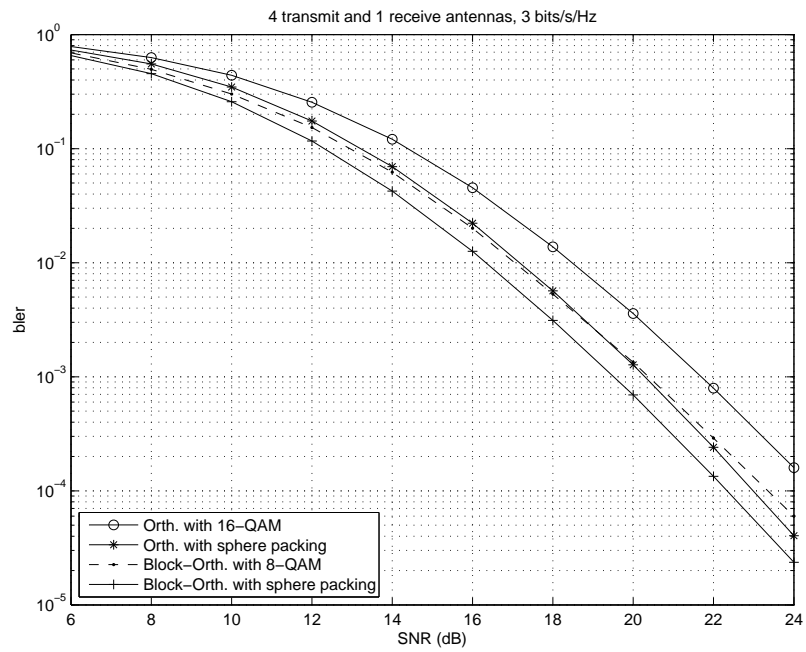
Figure 5: Block error rate performance of orth. with 16-QAM 'o', orth. with sphere packing '*', block-orth. with 8QAM '·', and block-orth. with sphere packing '+'. $L = 4096$.

Table 1: Comparison of diversity product for 2 transmit antennas. Note that we abbreviate orthogonal designs with sphere packing as Orth. with S.P..

Size L	Rate (bits/s/Hz)	Upper bound $\sqrt{\frac{L}{2(L-1)}}$	Diversity product ζ	Comments
4	1	0.8165	0.7071	Cyclic code
			0.7071	Diag. algebr. code
			0.7071	Orth. with BPSK
			0.8165	Orth. with S.P.
8	1.5	0.7559	0.5946	Cyclic code
			0.7071	Parametric code
			0.7071	Orth. with S.P.
16	2	0.7303	0.3827	Cyclic code
			0.5946	Parametric code
			0.5000	Diag. algebr. code
			0.5000	Orth. with QPSK
			0.4729	Golden code with BPSK
			0.5535	Orth. with S.P.
32	2.5	0.7184	0.2494	Cyclic code
			0.3827	Parametric code
			0.4658	Orth. with S.P.
64	3	0.7127	0.1985	Cyclic code
			0.3070	Parametric code
			0.2706	Orth. with 8PSK
			0.3162	Orth. with 8QAM
			0.3860	Orth. with S.P.
128	3.5	0.7099	0.1498	Cyclic code
			0.2606	Parametric code
			0.3226	Orth. with S.P.
256	4	0.7085	0.0988	Cyclic code
			0.2237	Diag. algebr. code
			0.2544	$B_2(\phi, \theta)$ with QPSK
			0.3344	Golden code with 4QAM
			0.2237	Orth. with 16QAM
			0.2722	Orth. with S.P.
512	4.5	0.7078	0.0697	Cyclic code
			0.2285	Orth. with S.P.
1024	5	0.7075	0.0481	Cyclic code
			0.1581	Diag. algebr. code
			0.1581	Orth. with 32QAM
			0.1918	Orth. with S.P.

Table 2: Comparison of diversity product for 4 transmit antennas. Note that we abbreviate orthogonal designs with sphere packing as Orth. with S.P., and block-orthogonal designs with sphere packing as Block-Orth. with S.P..

Size L	Rate (bits/s/Hz)	Upper bound $\sqrt{\frac{L}{2(L-1)}}$	Diversity product ζ	Comments
4	0.5	0.8165	0.7071	Cyclic code
			0.8165	Orth. with S.P.
8	0.75	0.7559	0.5946	Cyclic code
			0.5774	Orth. with BPSK
			0.7184	Orth. with S.P.
16	1	0.7303	0.5453	Cyclic code
			0.5000	Diag. algebr. code
			0.6426	Orth. with S.P.
			0.5000	Block-Orth. with BPSK
			0.5774	Block-Orth. with S.P.
32	1.25	0.7184	0.3827	Cyclic code
			0.5611	Orth. with S.P.
64	1.5	0.7127	0.3399	Cyclic code
			0.4083	Orth. with QPSK
			0.5062	Orth. with S.P.
128	1.75	0.7099	0.2594	Cyclic code
			0.4390	Orth. with S.P.
256	2	0.7085	0.2208	Cyclic code
			0.3536	Diag. algebr. code
			0.3884	Orth. with S.P.
			0.3536	Block-Orth. with QPSK
			0.3885	Block-Orth. with S.P.
512	2.25	0.7078	0.1940	Cyclic code
			0.2210	Orth. with 8PSK
			0.3467	Orth. with S.P.
1024	2.5	0.7075	0.1532	Cyclic code
			0.3105	Orth. with S.P.
4096	3	0.7072	0.2390	Diag. algebr. code
			0.1826	Orth. with 16QAM
			0.2454	Orth. with S.P.
			0.2236	Block-Orth. with 8QAM
			0.2729	Block-Orth. with S.P.