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# THE 'HALO' FAMILY OF 3-DIMENSIONAL PERIODIC ORBITS IN THE EARTH-MOON RESTRICTED 3-BODY PROBLEM

JOHN V. BREAKWELL

*Department of Aeronautics/Astronautics, Stanford University, 94305, U.S.A.*

JOHN V. BROWN

*Western Development Lab. Division, Ford Aerospace and Communication Corp., Palo Alto, Calif., U.S.A.*

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**Abstract.** The Halo orbits originating in the vicinities of both  $L_1$  and  $L_2$  grow larger, but shorter in period, as they shift towards the Moon. There is in each case a narrow band of stable orbits roughly half-way to the Moon. Nearer to the Moon, the orbits are fairly well-approximated by an 'almost rectilinear' analysis. The  $L_2$  family shrinks in size as it approaches the Moon, becoming stable again shortly before penetrating the lunar surface. The  $L_1$ -family becomes longer and thinner as it approaches the Moon, with a second narrow band of stable orbits with perilune, however, below the lunar surface.

## 1. Introduction

Farquhar and Kamel [1] computed a family of 'halo' orbits near the translunar  $L_2$  libration point but permanently 'visible' from Earth.

The existence of such orbits had been suspected for reasons set forth in the following two paragraphs:

The linearized motion near  $L_2$ , although unstable, includes a periodic motion in the  $xy$  plane, i.e., in the plane of the relative Earth-Moon motion, of frequency  $1.86n$ , where  $n$  is the angular rate of the Moon and the Earth. The instability manifests itself by real additional characteristic roots  $\pm 2.16n$ . The out-of-plane linearized  $z$ -motion is simple harmonic with frequency  $1.79n$ . If, as pointed out by Farquhar [2], initial conditions were to produce a periodic  $xy$ -motion and a  $z$ -motion, with phase initially  $90^\circ$  ahead or behind that of the  $xy$ -motion, the resulting orbit could remain continuously visible from Earth for many months, before the relative phase approaches  $0$  or  $180^\circ$ , thereby producing periodic occultation by the Moon.

If an orbit, however, is of a size comparable with the distance of  $L_2$  from the Moon, so that nonlinear terms become significant, nonlinear dependence of frequency on amplitude as well as nonlinear coupling might produce a purely periodic three-dimensional motion.

Farquhar and Kamel computed these orbits by a truncated Linstedt-Poincaré series. This series included, moreover, quasi-periodic additions due to solar gravity and lunar orbital eccentricity, which effects will naturally be ignored in this paper. Projections of typical periodic orbits obtained by the truncated series are shown in

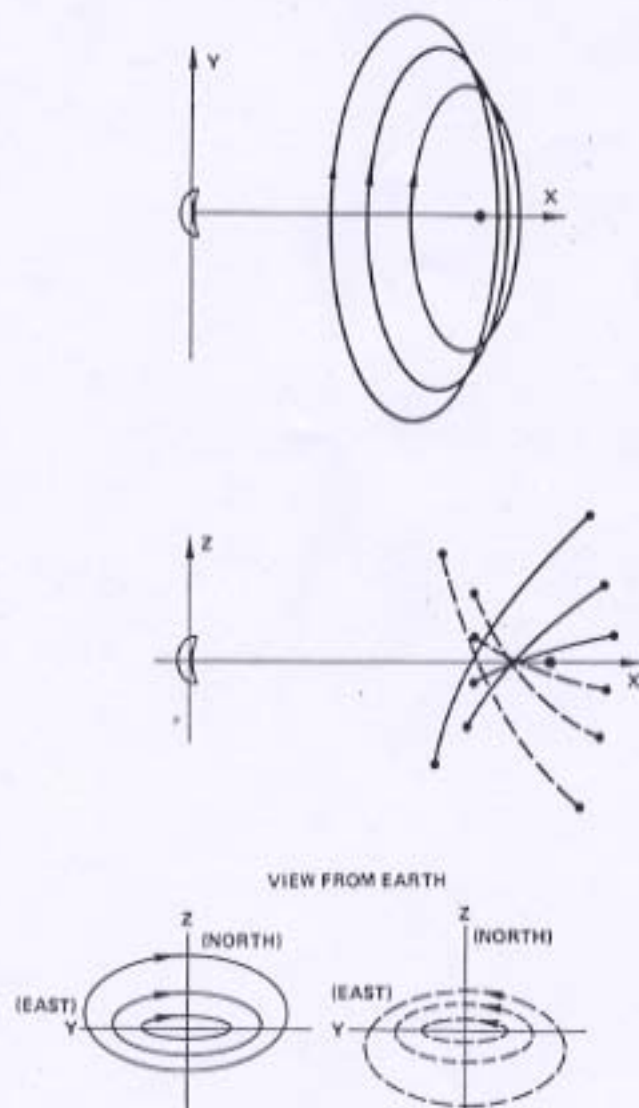
Fig. 1. Halo orbits near  $L_2$ .

Figure 1. The solid lines show the northern halo family for which  $z$  is in phase with  $x$  and  $90^\circ$  behind  $y$ ; the dotted lines show the mirror-image southern family obtained by reversing the sign of  $z$ . The convergence of the series deteriorates for orbits larger than those shown.

## 2. Analysis

If position is measured in the usual rotating frame with origin at the Earth-Moon barycenter, and if units of time and distance are chosen so that  $n = 1$  and Earth-

Moon distance =

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$$\bar{y} + 2$$

where

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condition:

$$\frac{d}{dt} \Phi =$$

where

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Moon distance = 1, the equations of motion are

$$\begin{aligned}\ddot{x} - 2\dot{y} &= \frac{\partial U}{\partial x} \\ \ddot{y} + 2\dot{x} &= \frac{\partial U}{\partial y} \\ \ddot{z} &= \frac{\partial U}{\partial z}\end{aligned}\quad (2.1)$$

where

$$U(x, y, z) = \frac{1}{2}(x^2 + y^2) + \frac{1-\mu}{d_{\oplus}(x, y, z)} + \frac{\mu}{d_{\ell}(x, y, z)}, \quad (2.2)$$

$\mu/(1-\mu)$  being the Moon-Earth mass-ratio, and  $d_{\oplus}$  and  $d_{\ell}$  denoting distances from Earth and Moon respectively.

A 1-parameter extension of the Farquhar-Kamel family of halo orbits is obtained by adjusting values of  $x_0, z_0, \dot{y}_0$  at a perpendicular crossing of the  $xz$  plane (i.e.,  $y_0 = \dot{x} = \dot{z} = 0$ ), so that the next crossing of this plane,  $y_1 = 0$  at  $t = t_1$ ,  $\text{sgn } \dot{y}_1 = -\text{sgn } \dot{y}_0$  is also perpendicular; i.e., the velocity components  $\dot{x}_1$  and  $\dot{z}_1$  are also zero. Because of the invariance of Equations (2.1) under the reflection  $t \rightarrow -t, y \rightarrow -y$ , it follows that such an orbit will necessarily repeat with period  $T = 2t_1$ . It will suffice to investigate the northern halo family.

The  $6 \times 6$  transition-matrix  $\Phi(t, 0)$  associated with (2.1) is the matrix of partial derivatives  $\partial X(t)/\partial X(0)$ , where  $X$  denotes a column-vector with entries  $x, y, z, \dot{x}, \dot{y}, \dot{z}$ .  $\Phi(t, 0)$  satisfies, of course, the matrix differential equation and initial condition:

$$\frac{d}{dt} \Phi(t, 0) = F(t) \Phi(t, 0), \quad \Phi(0, 0) = \text{identity matrix}, \quad (2.3)$$

where

$$F(t) = \begin{pmatrix} 0 & 1 & 0 \\ -U_{xx} & -1 & 0 \\ 0 & 0 & 2\Omega \end{pmatrix}, \quad (2.4)$$

in which  $I$  denotes the  $3 \times 3$  identity matrix,

$$\Omega = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad (2.5)$$

and  $U_{xx}(t)$  denotes the symmetric matrix of second partial derivatives of  $U$  w.r.t.  $x, y, z$ , evaluated along the orbit.

The transition-matrix  $\Phi(t_1, 0)$  at the end of a half-cycle of a nearly periodic orbit can be used to adjust the initial conditions so as to obtain periodicity. For example, if

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$x_0$  is held fixed,

$$\begin{pmatrix} \delta \dot{x}_1 \\ \delta \dot{z}_1 \end{pmatrix} = \begin{pmatrix} \Phi_{43} & \Phi_{45} \\ \Phi_{63} & \Phi_{65} \end{pmatrix} \begin{pmatrix} \delta z_0 \\ \delta y_0 \end{pmatrix} + \begin{pmatrix} \dot{x}_1 \\ \dot{z}_1 \end{pmatrix} \delta t,$$

where

$$0 = \delta y_1 = (\Phi_{23} \quad \Phi_{25}) \begin{pmatrix} \delta z_0 \\ \delta y_0 \end{pmatrix} + \dot{y}_1 \delta t,$$

and where  $\Phi_{ij}$  is a typical element of  $\Phi(t_1, 0)$ . Hence

$$\begin{pmatrix} \delta \dot{x}_1 \\ \delta \dot{z}_1 \end{pmatrix} = \left[ \begin{pmatrix} \Phi_{43} & \Phi_{45} \\ \Phi_{63} & \Phi_{65} \end{pmatrix} - \frac{1}{\dot{y}_1} \begin{pmatrix} \dot{x}_1 \\ \dot{z}_1 \end{pmatrix} (\Phi_{23} \quad \Phi_{25}) \right] \begin{pmatrix} \delta z_0 \\ \delta y_0 \end{pmatrix},$$

and a  $2 \times 2$  inversion yields  $\begin{pmatrix} \delta z_0 \\ \delta y_0 \end{pmatrix}$  so that  $\begin{pmatrix} \delta \dot{x}_1 \\ \delta \dot{z}_1 \end{pmatrix}$  cancels any previous  $\begin{pmatrix} \dot{x}_1 \\ \dot{z}_1 \end{pmatrix}$ , if sufficiently small. This provides an iterative calculation of  $x_0, y_0$ .

The transition-matrix at the end of a full cycle of the periodic orbit is

$$\Phi(T, 0) = \Phi(2t_1, 0).$$

Its eigen-values determine the stability of the orbit; two of the eigen-values are always unity and the other four eigen-values form a self-reciprocal set  $(\lambda_1, 1/\lambda_1, \lambda_2, 1/\lambda_2)$ , as follows from the invariance of (2.1) under  $t \rightarrow -t, y \rightarrow -y$ . The periodic orbit is thus unstable unless both  $\lambda_1$  and  $\lambda_2$  lie on the unit-circle.

In the interest of numerical accuracy, the full-cycle transition-matrix  $\Phi(2t_1, 0)$  can be obtained from the half-cycle transition-matrix  $\Phi(t_1, 0)$  without the necessity of numerical integration beyond time  $t = t_1$ . This is possible because, if  $\Phi^*(t, 0)$  denotes

$$\left( \begin{array}{c|c} -I & 0 \\ \hline -\Omega & I \end{array} \right) \Phi(t, 0) \left( \begin{array}{c|c} -I & 0 \\ \hline -\Omega & I \end{array} \right), \quad \Phi^* \text{ is 'symplectic',}$$

i.e.,

$$\Phi^{*T} \left( \begin{array}{c|c} 0 & I \\ \hline -I & 0 \end{array} \right) \Phi^* = \left( \begin{array}{c|c} 0 & I \\ \hline -I & 0 \end{array} \right),$$

as follows rather easily from the Hamiltonian form of the equations of motion if, instead of velocity components  $\dot{x}, \dot{y}, \dot{z}$ , we make use of generalized momenta:  $p_x = \dot{x} - y, p_y = \dot{y} + x, p_z = \dot{z}$ . The symplectic property of  $\Phi^*$ , together with the invariance of (2.1) under  $t \rightarrow -t, y \rightarrow -y$ , leads to:

$$\Phi(0, -t_1) = A \Phi(0, t_1) A = A \left( \begin{array}{c|c} 0 & I \\ \hline -I & 2\Omega \end{array} \right) \Phi^T(t_1, 0) \left( \begin{array}{c|c} 2\Omega & -I \\ \hline I & 0 \end{array} \right) A, \quad (2.6)$$

where  $A$  denotes

$$\begin{pmatrix} 1 & & & & & \\ & -1 & & & & \\ & & 1 & & & \\ & & & -1 & & \\ 0 & & & & 1 & \\ & & & & & -1 \end{pmatrix}$$

Finally

$$\Phi(2t_1, 0) = \Phi(2t_1, t_1)\Phi(t_1, 0) = \Phi(0, -t_1)\Phi(t_1, 0). \tag{2.7}$$

The four critical eigen-values ( $\lambda_1, 1/\lambda_1, \lambda_2, 1/\lambda_2$ ) can be obtained with greater numerical accuracy as the eigen-values of a  $4 \times 4$  matrix  $M$  derivable from  $\Phi(2t_1, 0)$ . This derivation makes use of the well-known constancy, along any solution of (2.1), of Jacobi's constant:

$$C = 2U(x, y, z) - (\dot{x}^2 + \dot{y}^2 + \dot{z}^2). \tag{2.8}$$

Suppose, then, that  $x_0, z_0, \dot{x}_0, \dot{z}_0$  (but not  $y_0$ ) are varied in such a way that  $C$  is unchanged, so that

$$\delta y_0 = -\frac{1}{C_{y_0}}(C_{x_0}\delta x_0 + C_{z_0}\delta z_0), \tag{2.9}$$

where use has been made of the vanishing of the partial derivatives  $C_{x_0}$  and  $C_{z_0}$  when  $\dot{x}_0 = \dot{z}_0 = 0$ . Furthermore, the correction  $\delta T$  to the time of the second crossing of the  $xz$  plane after time zero is given by

$$\delta y_2 = \dot{y}(T)\delta T + (\partial y/\partial x_0, \partial y/\partial z_0, \partial y/\partial \dot{x}_0, \partial y/\partial \dot{y}_0, \partial y/\partial \dot{z}_0) \cdot$$

$$\begin{pmatrix} \delta x_0 \\ \delta z_0 \\ \delta \dot{x}_0 \\ \delta \dot{y}_0 \\ \delta \dot{z}_0 \end{pmatrix} = 0 \tag{2.10}$$

in which  $\dot{y}(T) = \dot{y}_0$ . It follows easily from (2.9) and (2.10) that

$$\begin{pmatrix} dx_2 \\ dz_2 \\ d\dot{x}_2 \\ d\dot{z}_2 \end{pmatrix} = M \begin{pmatrix} \delta x_0 \\ \delta z_0 \\ \delta \dot{x}_0 \\ \delta \dot{z}_0 \end{pmatrix}, \tag{2.11}$$

previous  $\begin{pmatrix} \dot{x}_1 \\ \dot{z}_1 \end{pmatrix}$ , if

orbit is

the eigen-values are self-reciprocal set  $t \rightarrow -t, y \rightarrow -y$ . The unit-circle.

matrix  $\Phi(2t_1, 0)$  can omit the necessity of  $\dot{z}_0$ , if  $\Phi^*(t, 0)$  denotes

ations of motion if, generalized momenta: together with the

$$\frac{2\Omega}{I} \begin{pmatrix} -I \\ 0 \end{pmatrix} A, \tag{2.6}$$

where

$$M = \Phi^{\dagger} - \frac{1}{C_{\dot{y}_0}} \Phi^{\dagger}_5 (C_{x_0}, C_{\dot{z}_0}, 0, 0) - \frac{1}{\dot{y}_0} \begin{pmatrix} 0 \\ 0 \\ \ddot{x}_0 \\ \ddot{z}_0 \end{pmatrix} \left[ \Phi^{\dagger}_2 - \frac{1}{C_{\dot{y}_0}} \Phi_{25} (C_{x_0}, C_{\dot{z}_0}, 0, 0) \right], \quad (2.12)$$

in which  $\Phi^{\dagger}$  denotes the result of deleting the second and fifth rows and columns of  $\Phi(2t_1, 0)$ ,  $\Phi^{\dagger}_5$  the fifth column of  $\Phi(2t_1, 0)$  without rows 2 and 5, and  $\Phi^{\dagger}_2$  the second row without columns 2 and 5.

### 3. Numerical Results

The eigen-values ( $\lambda_1, 1/\lambda_1, \lambda_2, 1/\lambda_2$ ) of  $M$  have been computed, along with the periodic orbits themselves, at frequent intervals of  $x_0 = x_{\max}$  (now measured from the Moon, away from the Earth); the  $xz$  projections of the orbits are shown in Figure 2. The eigen-values turn out either to be real or to lie on the unit-circle. Stability is thus conveniently decided by the values of two stability indices:  $\nu_i = \frac{1}{2}[\lambda_i + (1/\lambda_i)]$  ( $i = 1, 2$ ): the orbit is stable if both  $|\nu_1|$  and  $|\nu_2| < 1$ , unstable if either

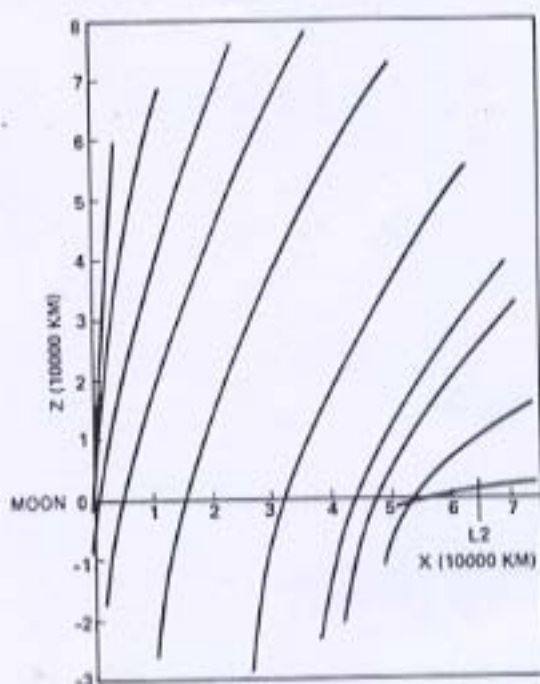
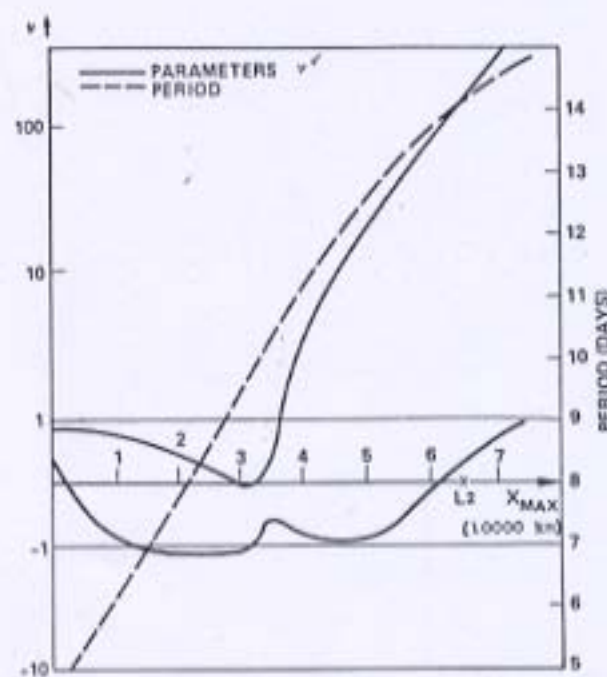


Fig. 2.  $X$ - $Z$  projection of the  $L_2$  Halo family.

(2.12)

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and 5, and  $\Phi_2^1$  theed, along with the  
measured from the  
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ity indices:  $\nu_1 =$   
, unstable if eitherFig. 3. Stability parameters  $\nu_1$  and period.

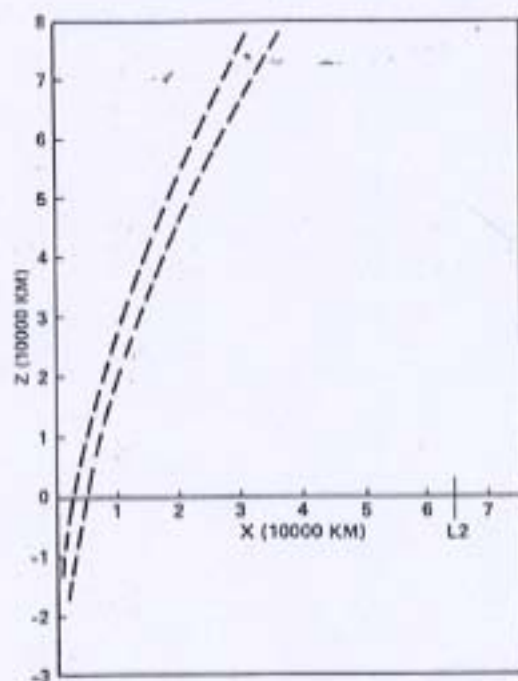
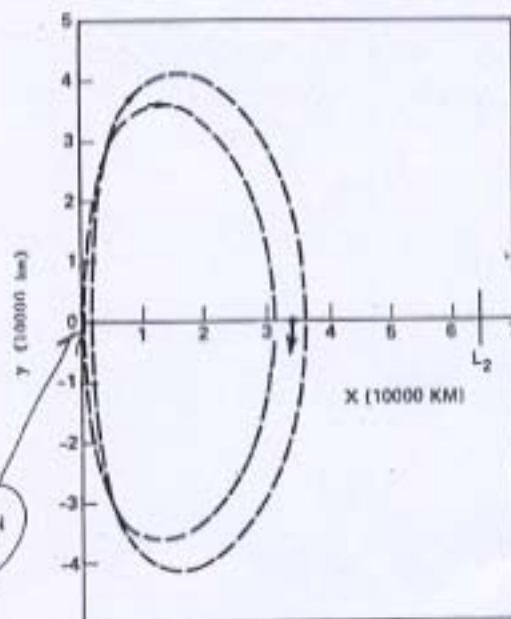
$|\nu_1| > 1$ . The indices  $\nu_1, \nu_2$  are plotted in Figure 3 vs  $x_{max}$ , along with the orbital period. The drastic instability of the family close to  $L_2$  diminishes as the orbits grow nearer to the Moon and shorter in period. There is actually a narrow band of stable orbits for  $x_{max}$  between 36 500 km (where one pair of  $\lambda$ 's coincides at +1 and the Jacobi constant reaches a minimum) and 31 300 km (where the other pair of  $\lambda$ 's coincides at -1). Projections of this band of stable orbits are shown in Figures 4a-c.

The existence, but not the stability, of such orbits can be understood by regarding them as orbits of the Moon with a substantial Earth perturbation. Thus the tilt of the  $xz$  projection away from the Earth as  $x$  increases (Figure 4a) provides a torque, due to the Earth's gravity-gradient, about the positive  $y$ -axis, causing the orbital angular momentum vector  $\mathbf{h}/\Omega$ , relative to the Moon, to precess and keep the orbit fixed in the rotating frame.

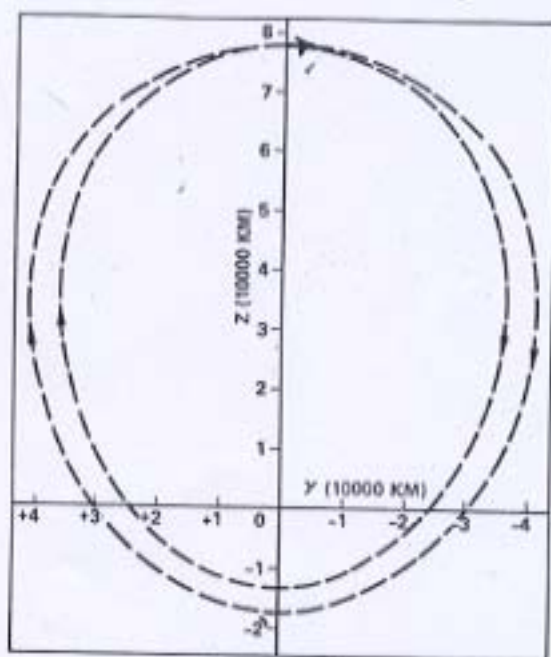
As  $x_{max}$  decreases further, the orbits grow mildly unstable and then stable again at  $x_{max} = 28\,500$  km,  $(d_c)_{min} \cong 4000$  km, and for  $x_{max}$  less than 3000 km, the maximum  $z$  is less than 64 000 km,  $(d_c)_{min}$  falls below 1740 km and the orbits intersect the lunar surface.

#### 4. The Almost Rectilinear Orbits

The almost rectilinear orbits towards which the halo family tends as  $x_{max}$  decreases are amenable to an analytical approximation used in reference [3] in a different

Fig. 4a. Stable orbit  $X$ - $Z$  position bounds,  $L_2$  branch.Fig. 4b. Stable orbit  $X$ - $Y$  position bounds,  $L_2$  branch.



Fig. 4c. Stable orbit Y-Z position bounds,  $L_2$  branch.

AS WE  
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context: The Sun-Earth system. If position is measured from the Moon and if we start by supposing that  $x, y \ll z \ll 1$ , the equations of motion are (approximately):

$$\begin{aligned} \ddot{x} - 2\dot{y} &= 3x + \frac{1}{2}z^2 - \frac{\mu x}{z^3} \\ \ddot{y} + 2\dot{x} &= -\frac{\mu y}{z^3} \end{aligned} \quad (4.1)$$

$$\ddot{z} = -z - \frac{\mu}{z^3}$$

in which the only important second-order term in the expansion of  $\nabla(1/d_{\oplus})$  about the Moon is the term  $\frac{1}{2}z^2$  in the  $x$ -component.

The third equation, (4.1)<sub>3</sub>, leads to an obvious integral:

$$\frac{\dot{z}^2}{2} + \frac{1}{2}z^2 - \frac{\mu}{z} = \mathcal{E}, \quad \text{a constant.} \quad (4.2)$$

We may introduce the dimensionless parameter  $\eta$  ( $-\infty < \eta < 1$ ) related to  $\mathcal{E}$  and  $z_{\max}$  by:

$$\eta = \frac{2\mathcal{E}}{z_{\max}^2} = 1 - \frac{2\mu}{z_{\max}^3}, \quad (4.3)$$

and related in turn to the Jacobi constant  $C$  by:

$$C = 3 - 4\mu + \mu^2 - \eta \left( \frac{2\mu}{1-\eta} \right)^{2/3} \quad (4.4)$$

Equation (4.2) leads to a quadrature for the time from  $z = 0$ ,

$$t = \int_0^{z/z_{\max}} \frac{\sqrt{\zeta} d\zeta}{\sqrt{(1-\zeta)(1-\eta+\zeta+\zeta^2)}}, \quad (4.5)$$

which is an elliptic integral of the third kind.

The relatively small deviations  $x, y$  from the  $z$ -axis satisfy the first two equations (4.1)<sub>1</sub> and (4.1)<sub>2</sub>, which are non-homogeneous linear with time-varying coefficients and forcing term. The assumption  $x, y \ll z$  eliminates, of course, the brief period of time when the orbit is closest to the Moon and when the Earth's effect on the orbit is small. If the variables  $x, y, \dot{x}, \dot{y}$ , are replaced by certain osculating Moon-centered orbit elements, the resulting differential equations will be valid throughout the orbit. Specifically we chose,

$$P_1 = \left( \frac{1-\eta}{2} \right)^{1/3} \frac{y\dot{z} - z\dot{y}}{z_{\max}^2 \left( \frac{1-\eta}{2} + \zeta^3 \right)}, \quad P_3 = \left( \frac{1-\eta}{2} \right)^{1/3} \frac{x\dot{z} - \dot{z}x}{z_{\max}^2 \left( \frac{1-\eta}{2} + \zeta^3 \right)},$$

$$P_2 = \frac{(z^2 + z\dot{z})y - z\dot{z}y}{z^2 \dot{z}}, \quad P_4 = \frac{(z^2 + z\dot{z})x - z\dot{z}x}{z^2 \dot{z}} \quad (4.6)$$

$\zeta$  denotes  $z/z_{\max}$ ;  $P_1, P_3$  are proportional to the orbital angular momentum  $\mathbf{r} \times \mathbf{v}$  (in the rotating frame); while  $P_2, P_4$  are the small angular deviations from the  $z$ -axis of the apolune direction; i.e.,  $P_2, P_4$  are certain linear combinations [3] of  $\pi/2 - i$  and  $3\pi/2 - \omega$  where  $i, \omega$  are the usual orbital elements, referred to the  $xy$  plane. The  $P$ 's satisfy the linear non-homogeneous equation:

$$\frac{d}{ds} \begin{pmatrix} P_1 \\ P_2 \\ P_3 \\ P_4 \end{pmatrix} = \frac{4}{1-\eta+2\zeta^3} \begin{pmatrix} \sqrt{\zeta} & 0 \\ sR/\xi & 0 \\ 0 & \sqrt{\zeta} \\ 0 & sR/\xi \end{pmatrix} \times$$

$$\times \left\{ \begin{pmatrix} P_1 \\ P_2 \\ P_3 \\ P_4 \end{pmatrix} - \frac{\frac{2}{3}\mu^{1/3}\zeta^3}{R} \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\} \quad (4.7)$$

where  $R$  denotes  $\sqrt{1-\eta+\zeta+\zeta^2}$ ,  $W$  denotes  $1-\eta+2\eta\zeta-4\zeta^3$ ,  $\xi$  denotes  $[(1-\eta)/2]^{1/3}$  and where the independent variable  $s$  is defined by:

$$s = \begin{cases} -\sqrt{1-\zeta}, & z > 0, \\ +\sqrt{1-\zeta}, & z < 0. \end{cases} \quad (4.8)$$

(4.4)

(4.5)

This independent variable has several advantages; in the first place it is not necessary to invert (4.5) and express  $\zeta$  as a function of  $t$ , and the infinite value of  $\dot{z}$  at  $z = 0$  is not a problem; secondly, the vanishing of  $\dot{z}$  at  $z = z_{\max}(\zeta = 1)$  makes  $s$  preferable to  $z$  or  $\zeta$ .

The linearity of Equation (4.7) leads to a simple form for the  $P$ 's at  $s = +1$  (return) in terms of the  $P$ 's at  $s = -1$  (departure):

$$\begin{pmatrix} P_1(+1) \\ P_2(+1) \\ P_3(+1) \\ P_4(+1) \end{pmatrix} = T(\eta) \begin{pmatrix} P_1(-1) \\ P_2(-1) \\ P_3(-1) \\ P_4(-1) \end{pmatrix} + \mu^{1/3} A(\eta), \quad (4.9)$$

where  $T(\eta)$  is the solution (at  $s = 1$ ) of the homogeneous matrix equation associated with (4.7), the initial matrix being the identity (at  $s = -1$ ), and  $\mu^{1/3} A(\eta)$  is the solution of (4.7) with the zero initial  $P$ 's ( $P_i(-1) = 0$ ).

The desired periodic orbits are given by:

$$[\theta_4 - T(\eta)] \begin{pmatrix} P_1(-1) \\ P_2(-1) \\ P_3(-1) \\ P_4(-1) \end{pmatrix} = \mu^{1/3} A(\eta), \quad (4.10)$$

where  $\theta_4$  denotes the  $4 \times 4$  identity matrix, and they are unstable if the eigen-values of  $T(\eta)$  lie outside the unit-circle. These periodic orbits have been computed for various  $\eta$ ; in every case  $P_2(-1)$  and  $P_3(-1)$  are zero, indicating perpendicular crossing of the  $xz$  plane. The eigen-values of  $T(\eta)$ , moreover, are in quite good agreement, at least if  $\eta < -3.5$  ( $z_{\max} < 68\,000$  km), with the eigen-values of  $M$  associated with the exact, numerically integrated, orbits. The agreement of the orbit  $x(z)$ ,  $y(z)$  is moderately good and improves, as expected, when  $z_{\max}$  decreases, as is evident in Figure 5, where  $z_{\max}$  is plotted vs  $x_{\max}$ . The stability of the periodic solutions of (4.7) changes with increasing  $\eta$  at  $\eta = -3.543$  ( $z_{\max} = 67\,300$  km) when one pair of eigen-values of  $T(\eta)$  leaves the unit-circle at  $-1$ . At  $\eta = -2.471$  ( $z_{\max} = 73\,600$  km), moreover, the other pair of eigen-values leaves the unit-circle at  $+1$ . The periodic solution, given by (4.10) leads to infinitely large positive  $x$  as  $\eta \rightarrow -2.471$  from below (see Figure 5).

### 5. A Second Halo Family Emanating from Near $L_1$

For  $\eta > -2.471$ , the solution of (4.10) reappears from infinity with opposite sign. It was therefore conjectured that the corresponding  $x(z)$ ,  $y(z)$  approximate a portion of a second family of halo periodic orbits, in this case emanating from the vicinity of  $L_1$ . This conjecture has been verified and the values  $|x_{\max}|$ ,  $z_{\max}$  are shown for both families in Figure 5, together with the approximations (dotted lines) furnished by the almost rectilinear theory.

The first two equations have time-varying coefficients and, the brief period of the effect on the orbit is negligible. The Moon-centered orbit is constant throughout the orbit.

$$\frac{x\dot{z} - \dot{z}x}{z_{\max}^2 \left( \frac{1-\eta}{2} + \zeta^3 \right)} - z\dot{z}\dot{x} \quad (4.6)$$

The momentum  $\mathbf{r} \times \mathbf{v}$  (in the  $xy$  plane) is from the  $z$ -axis of the  $xy$  plane. The  $P$ 's

$$\begin{pmatrix} P_1 \\ P_2 \\ P_3 \\ P_4 \end{pmatrix} = \frac{3\mu^{1/3}\zeta^3}{R} \begin{pmatrix} 0 \\ 0 \\ 1 \\ 1 \end{pmatrix} \quad (4.7)$$

$-4\zeta^3$ ,  $\xi$  denotes  $y$ :

(4.8)

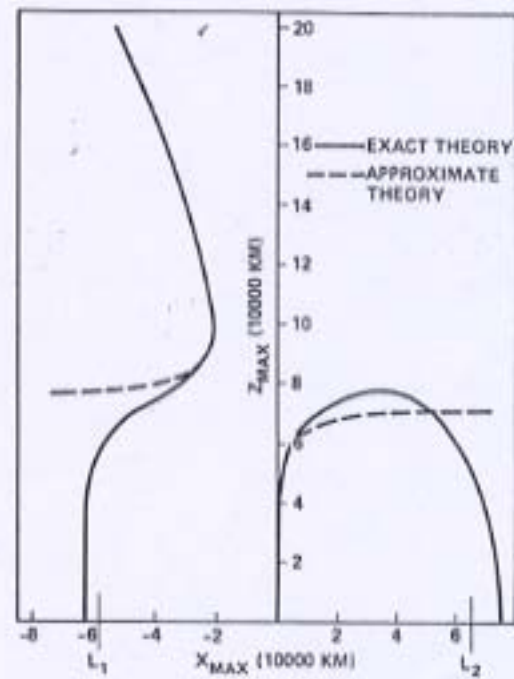
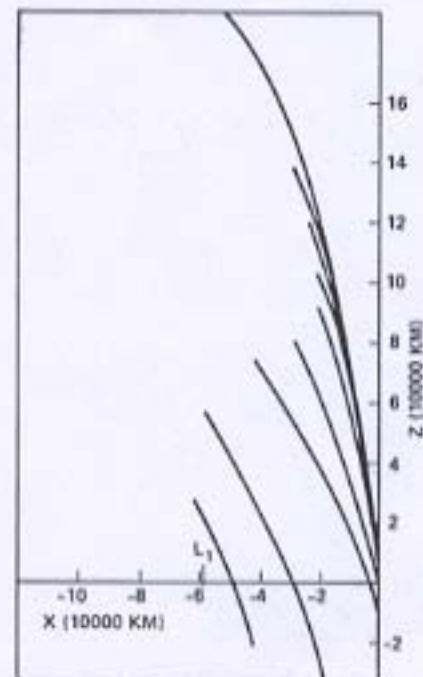


Fig. 5. The Halo family apolune points.

Fig. 6.  $X$ - $Z$  projection of the  $L_1$  halo family.

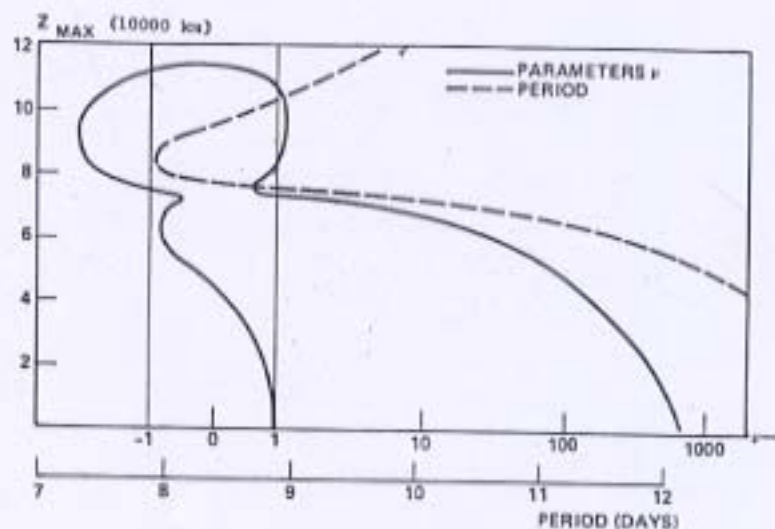


Fig. 7. Stability parameters  $\nu_1$  and period ( $L_1$  branch).

The  $xz$  projections of the new family are shown in Figure 6, and the values of the stability indices  $\nu_1$ ,  $\nu_2$  are plotted vs  $z_{max}$ , along with the period in Figure 7 up to  $z_{max} = 113\ 000$  km. Again there is a narrow band of stable orbits, for  $73\ 000\ \text{km} < z_{max} < 74\ 500\ \text{km}$ , shown in projections in Figures 8. The angular momentum vector  $\mathbf{h}/\zeta$  and the torque due to the Earth's gravity gradient are opposite to those for the  $L_2$  family, again causing the orbit to precess and remain fixed in the rotating frame. As

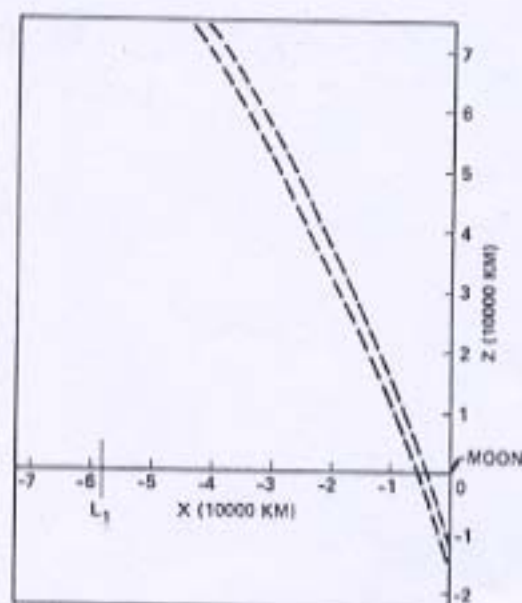


Fig. 8a. Stable orbit  $X$ - $Z$  position bounds,  $L_1$  branch.

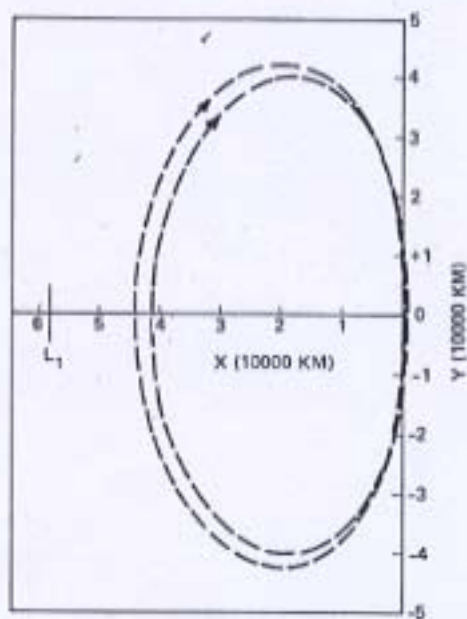


Fig. 8b. Stable orbit X-Y position bounds,  $L_2$  branch.

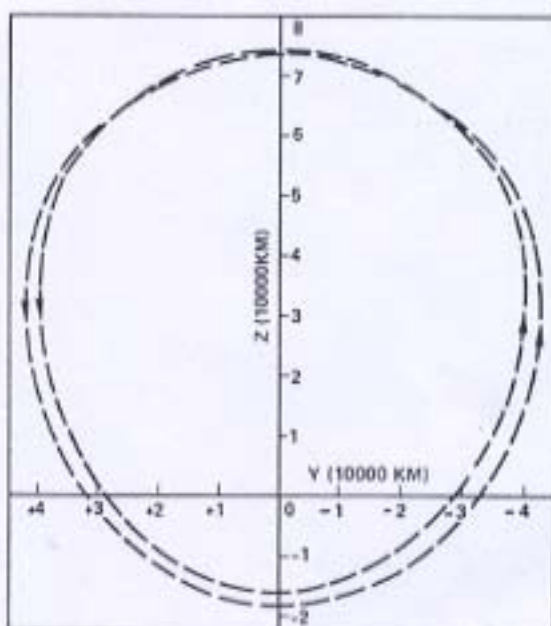


Fig. 8c. Stable orbit Y-Z position bounds,  $L_1$  branch.

$z_{\max}$  increases further we encounter a second narrow band of stable orbits, for  $112\,000\text{ km} < z_{\max} < 113\,000\text{ km}$ . This band has perilune below the lunar surface. At  $z_{\max} = 113\,000\text{ km}$  the two  $\nu_i$ 's coincide, and for larger  $z_{\max}$  we obtain instability due to a pair of complex conjugate eigen-values  $\lambda$  outside the unit-circle, their reciprocals lying inside the unit-circle. Figure 9 shows the behavior of those  $\lambda$ 's having positive imaginary part as  $z_{\max}$  increases beyond  $113\,000\text{ km}$ . The perilune distance, meanwhile, continues to decrease until  $z_{\max}$  reaches  $130\,000\text{ km}$ .

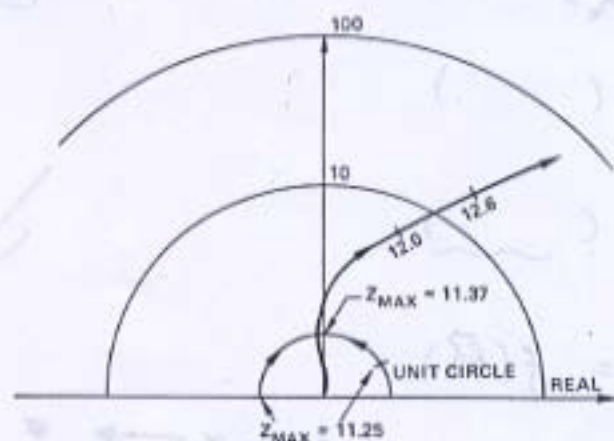


Fig. 9. Eigenvalues for higher  $z_{\max}$ .

## 6. Conclusions and Recommendations

The Halo orbits originating in the vicinities of both  $L_1$  and  $L_2$  grow larger, but shorter in period, as they shift towards the moon. There is in each case a narrow band of stable orbits roughly half-way to the Moon. Nearer to the Moon, the orbits are fairly well-approximated by an 'almost rectilinear' analysis described in [3]. The  $L_2$  family shrinks in size as it approaches the Moon, becoming stable again shortly before penetrating the lunar surface. The  $L_1$ -family becomes longer and thinner as it approaches the Moon, with a second narrow band of stable orbits with perilune, however, below the lunar surface.

It would be interesting to pursue the  $L_1$ -family for even greater  $z_{\max}$ . It would also be interesting to examine the neighborhoods of that member of the  $L_2$  family, and of the two members of the  $L_1$  family, where one  $\nu_i = +1$  and the Jacobi constant has an extreme value, to see if there are any bifurcations onto new branches of periodic orbit families.

The stable bands, roughly half-way from the Moon to  $L_1$  and  $L_2$ , are of special possible significance in view of recent interest in a space colony built of lunar material [4]. It is desirable to investigate the influence on their stability of both lunar orbit eccentricity and solar gravity.

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$$\left. \begin{aligned} (0) &= x \\ (1) &= x + \alpha A \\ (2) &= x \\ (3) &= x \\ (4) &= x + \beta A \end{aligned} \right\}$$

John V. Breakwell  
 John V. Brown  
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