EE 459/611: Smart Grid Economics, Policy, and Engineering

not dc power systems

Lecture 6: DC **Optimal** Power Flow

DC-OPF

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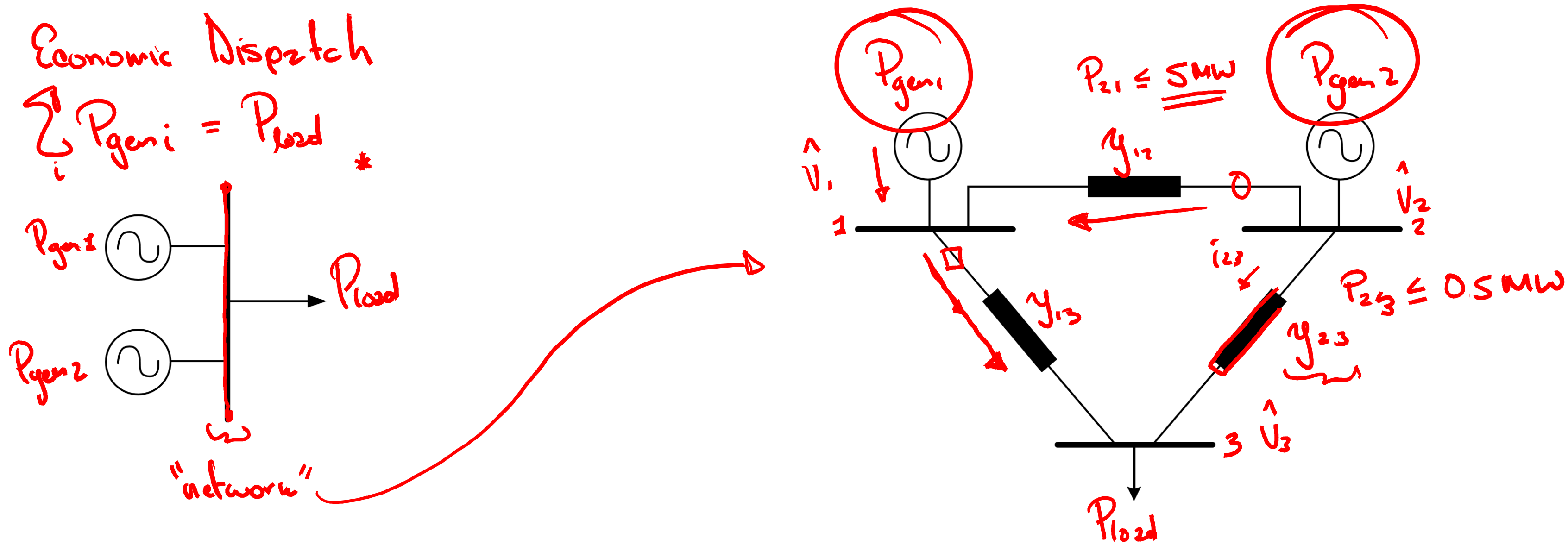
Topics that Will be Covered

- ***Nodal Analysis***
- *Power Flow Equations* (Nonlinear Power Flow)
- * *DC Power Flow - simplification*
- *DC Optimal Power Flow*
- *LP Formulation*
- *Locational Marginal Price* (LMP)

Economic Dispatch with Network Constraints – DC OPF

- In the previous lectures, we have not taken into account the network constraints



- **However,** it is important to take know the "power flows" throughout the system in order to:
  - **Not overload equipment/lines** — focus of DC OPF
  - Keep voltages within bounds

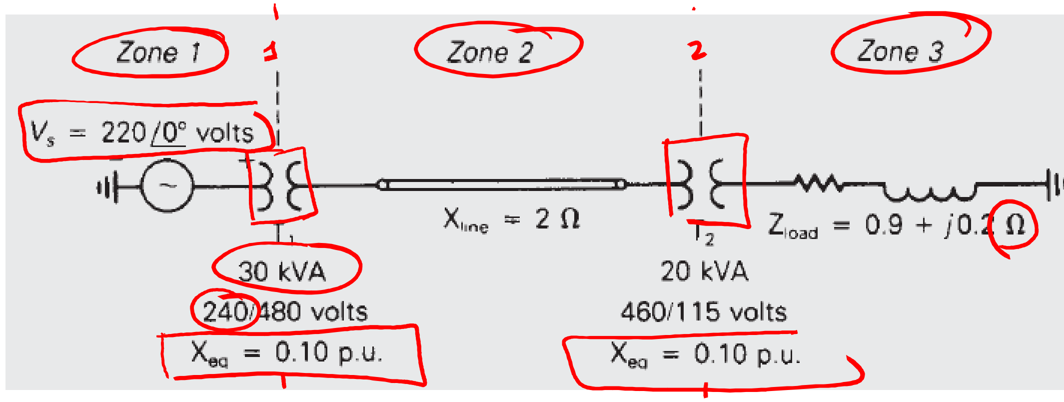
Review of Per Unit System

- The per unit system is helpful to analyze power systems with multiple areas!

Zone 1

$V_{base-1} = 240\ V$

$S_{base-1} = 30kVA$



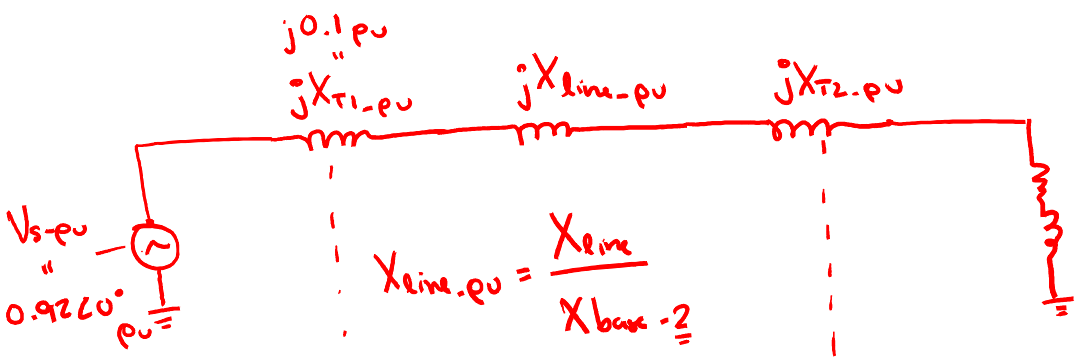
- Pick a base Power

$S_{base} = 30kVA$

- Having defined the bases for each area, the circuit becomes:

$V_{s\text{-}pu} = \dfrac{V_s}{V_{base\text{-}1}}$

$V_{s.pu} = \dfrac{220\angle 0° \cancel{V}}{240\ \cancel{V}}$



$X_{line.pu} = \dfrac{X_{line}}{X_{base\text{-}2}}$

$Z_{load\text{-}pu} = \dfrac{Z_{load}}{Z_{base\text{-}3}}$

pu circuit

No transformers

Review of Per Unit System

- Actual vs Base vs Per unit relationship $\boxed{\text{per unit} = \dfrac{\text{actual}}{\text{base value}}}$
- Per unit equations:

$$X_{pu} = \frac{X_{actual}}{X_{base\text{-}i}}$$

$\underset{\downarrow}{i}$ Zone

- $$S_{\text{base-}1\phi} = \frac{S_{\text{base-}3\phi}}{3}, \quad S_{\text{base-}3\phi} = P_{\text{base-}3\phi} = Q_{\text{base-}3\phi}$$

- $$V_{\text{baseLN}} = \frac{V_{\text{baseLL}}}{\sqrt{3}}$$

(apparent power)

$$S = P + jQ$$

P → active power

Q → Reactive Power

- $$I_{\text{base}} = \frac{S_{\text{base-}1\phi}}{V_{\text{baseLN}}} = \frac{S_{\text{base-}3\phi}}{\sqrt{3}V_{\text{baseLL}}}$$

- $$Z_{\text{base}} = \frac{V_{\text{baseLN}}}{I_{\text{base}}} = \frac{V^2_{\text{baseLN}}}{S_{\text{base-}1\phi}} = \frac{V^2_{\text{baseLL}}}{S_{\text{base-}3\phi}}$$

- $$R_{\text{base}} = Z_{\text{base}} = X_{\text{base}} = \frac{1}{Y_{\text{base}}}$$

Y_{base} → admittance base

$$Z = R + jX$$

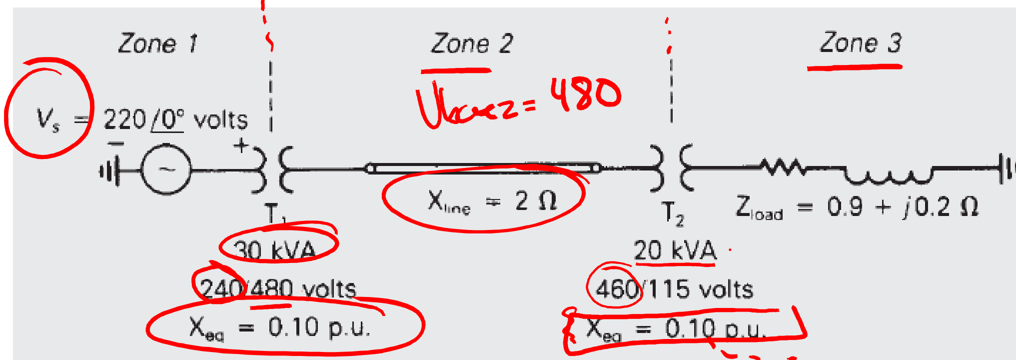
$\boxed{\text{Base values are REAL numbers}}$

Review of Per Unit System

- Convert the following circuit to per unit

Step 1: Define base Power

$S_{base} = 30\,kVA = P_{base} = Q_{base}$



Zone 1

$V_{base1} = 240V$

$S_{base1} = 30kVA$

$Z_{base1} = \frac{V_{base1}^2}{S_{base1}} = 1.92\Omega$

$I_{base1} = \frac{S_{base1}}{V_{base1}} = 125A$

Zone 2

$V_{base2} = 480V$

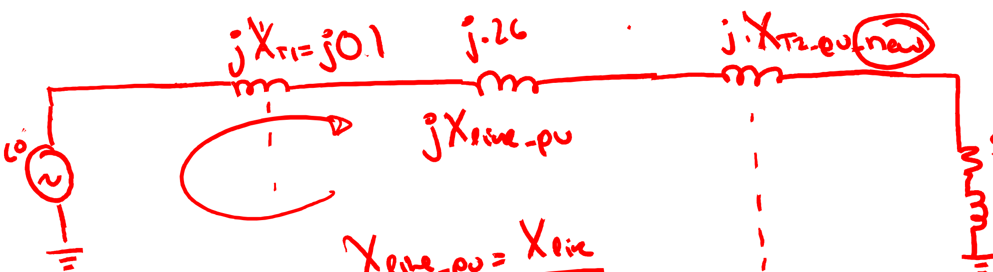
$S_{base2} = 30kVA$

$Z_{base2} = \frac{V_{base2}^2}{S_{base}} = 7.68\Omega$

Zone 3

$V_{base3} = 480 \cdot \frac{115}{460} = 120V$

$Z_{base3} = 0.48\Omega$



$V_{s,pu} = \frac{V_s}{V_{base1}} = \frac{220\angle 0V}{240V}$

$X_{line\_pu} = \frac{X_{line}}{X_{base2}} = \frac{2\Omega}{7.68\Omega} = .26pu$

$Z_{load,pu} = \frac{Z_{load}}{Z_{base3}} = \frac{0.9 + j0.2}{0.48}$

T_2:

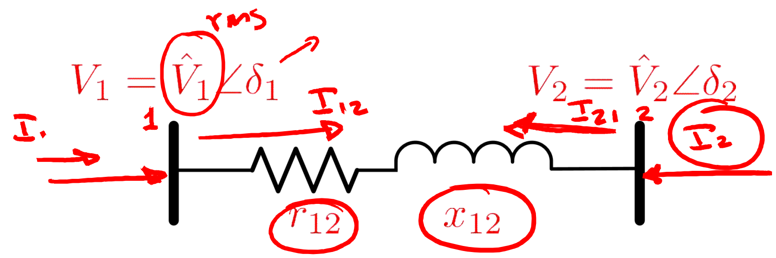
$X_{pu} = 0.1$

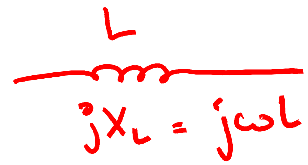
$V_{base} = 460$, $P_{base} = 20kVA$

$X_{T2,actual}$

Nodal Analysis – Admittance (KCL @ every bus)

- When we study **"ac"** systems, we need to consider the resistance and reactance of the line





Impedance $\begin{cases} z_{12} = r_{12} + jx_{12} & \text{(Rectangular)} \\ z_{12} = \hat{z}_{12}\angle\beta_{12} & \text{(Polar)} \end{cases}$

$\hat{z}_{12} = \sqrt{r_{12}^2 + x_{12}^2}$ $\quad \beta_{12} = \tan^{-1}\left(\frac{x_{12}}{r_{12}}\right)$

admittance $y_{12} = \dfrac{1}{z_{12}} = \hat{y}_{12}\angle\theta_{12}$

- Compute the **nodal equations** and define the **admittance**

(KCL @ every bus/node)

Bus 1: $I_1 = I_{12} = y_{12}(V_1 - V_2) = \hat{y}_{12}\angle\theta_{12}(\hat{V}_1\angle\delta_1 - \hat{V}_2\angle\delta_2)$

Bus 2: $I_2 = I_{21} = y_{12}(V_2 - V_1)$

$I_1 = y_{12}V_1 - y_{12}V_2$

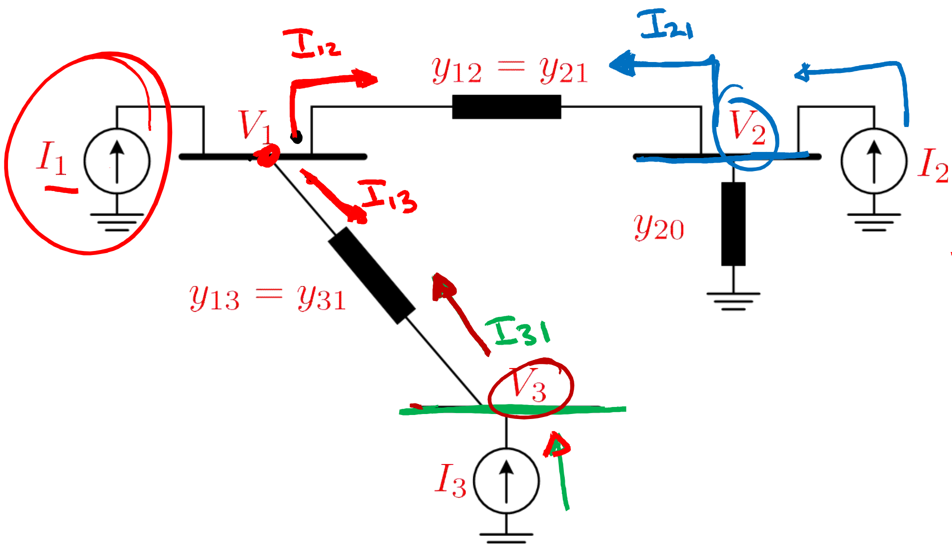
$I_2 = -y_{12}V_1 + y_{12}V_2$

Nodal Analysis – 3 Bus System

KCL @ every node/bus

- Compute the nodal equations at each bus in the following network:

y_{12} admittance of line 1-2



Bus 1

$I_1 = I_{12} + I_{13}$

$$\boxed{I_1 = \underbrace{y_{12}(V_1 - V_2)}_{I_{12}} + \underbrace{y_{13}(V_1 - V_3)}_{I_{13}}}$$

Bus 2

$I_2 = I_{21}$

$$\boxed{I_2 = y_{12}(V_2 - V_1)}$$

Bus 3

$I_3 = I_{31}$

$$\boxed{I_3 = y_{13}(V_3 - V_1)}$$

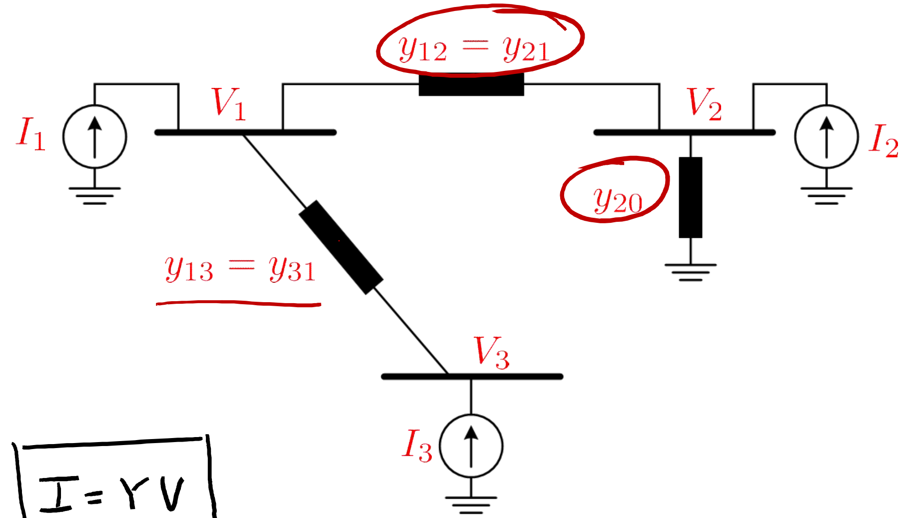
Suppose we know I_1, I_2, I_3

$\Rightarrow$ We can solve for V_1, V_2, V_3

3 eqns, 3 unknowns

Nodal Analysis - Admittance Matrix

- Compute the **nodal equations** at each bus in the following network:



$$
{}^{*}\left\{
\begin{aligned}
I_1 &= y_{12}\left(V_1 - V_2\right) + y_{13}\left(V_1 - V_3\right)\\
I_2 &= y_{21}\left(V_2 - V_1\right) + y_{20} V_2\\
I_3 &= y_{31}\left(V_3 - V_1\right)
\end{aligned}
\right.
$$

linear

- Rewrite the equations in matrix form: $\boxed{I = Y V}$

$$
\left\{
\begin{aligned}
I_1 &= V_1\left(y_{12} + y_{13}\right) + V_2\left(-y_{12}\right) + V_3\left(-y_{13}\right)\\
I_2 &= V_1\left(-y_{21}\right) + V_2\left(y_{21} + y_{20}\right) + V_3\left(0\right)\\
I_3 &= V_1\left(-y_{31}\right) + V_2\left(0\right) + V_3\left(y_{31}\right)
\end{aligned}
\right.
\Rightarrow
\underbrace{\begin{pmatrix} I_1 \\ I_2 \\ I_3 \end{pmatrix}}_{I}
=
\underbrace{\begin{pmatrix} y_{12}+y_{13} & -y_{12} & -y_{13} \\ -y_{12} & y_{12}+y_{20} & 0 \\ -y_{13} & 0 & y_{13} \end{pmatrix}}_{Y \text{ admittance matrix}^{*}}
\underbrace{\begin{pmatrix} V_1 \\ V_2 \\ V_3 \end{pmatrix}}_{V}
$$

In General Y

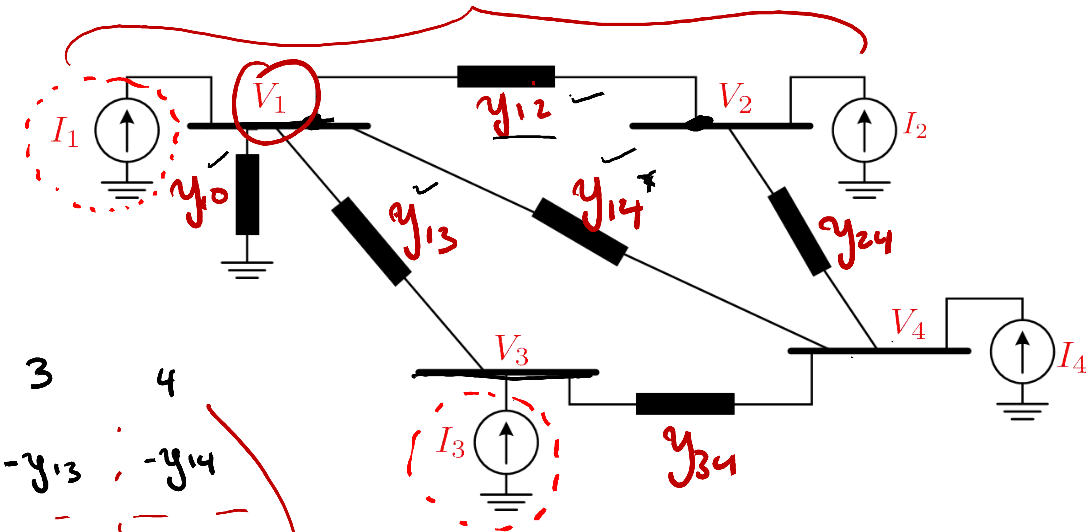
Diagonal terms $= \{Y_{11}, Y_{22}, Y_{33}\}$ = Sum of all admittances connected to bus i, Y_{ii}

Off-Diagonal $= Y_{ij}$ $i \neq j$ = negative of admittance between bus i and j $= -y_{ij}$

$Y_{ij} \neq y_{ij}$

Nodal Analysis – Example 2

- Write the admittance matrix



$I = YV$

$$\underset{4\times 4}{Y} = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \end{array} \begin{array}{c} \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \begin{pmatrix} \boxed{y_{10}+y_{12}+y_{13}+y_{14}} & -y_{12} & -y_{13} & -y_{14} \\ -y_{12} & \boxed{y_{12}+y_{24}} & 0 & -y_{24} \\ -y_{13} & 0 & \boxed{y_{13}+y_{34}} & -y_{34} \\ -y_{14} & -y_{24} & -y_{34} & \boxed{y_{14}+y_{24}+y_{34}} \end{pmatrix} \end{array}$$

$$\underbrace{\begin{pmatrix} I_1 \\ I_2 \\ I_3 \\ I_4 \end{pmatrix}}_{I} = Y \underbrace{\begin{pmatrix} V_1 \\ V_2 \\ V_3 \\ V_4 \end{pmatrix}}_{V} \Rightarrow V = Y^{-1} I$$

Topics that Will be Covered

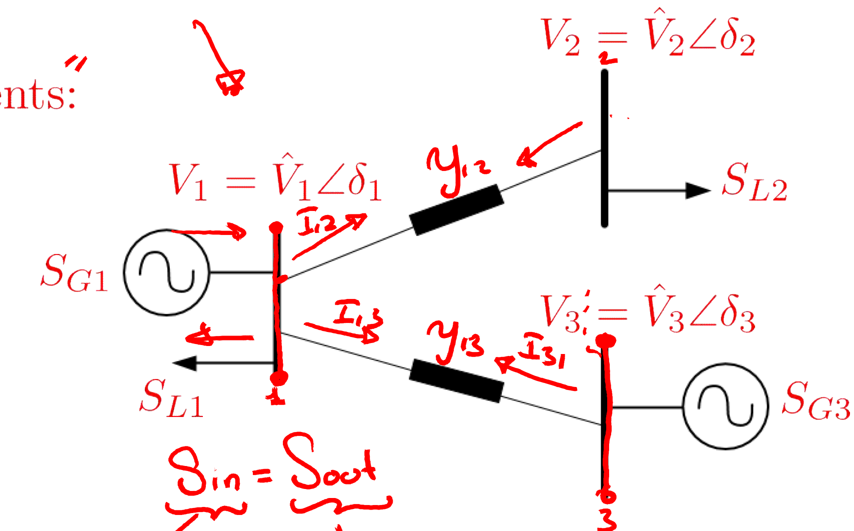
- *Nodal Analysis*
- ***Power Flow Equations*** (System of $\underbrace{\text{nonlinear}}$ equations) → Difficult to solve
- *DC Power Flow - simplification*
- *DC Optimal Power Flow*
- *LP Formulation*
- *Locational Marginal Price*

Power Balance at Every Node – Nodal Analysis

?

- Use nodal analysis to compute the "bus currents":

$$\left(I_{G1}-I_{L1}\right) = \begin{cases} I_1 = I_{12} + I_{13} = \dots \\ I_2 = I_{21} = y_{12}\,(V_1 - V_2) \\ I_3 = I_{31} = y_{13}\,(V_1 - V_3) \end{cases}$$



$\underbrace{S_{in}} = \underbrace{S_{out}}$

$S_{Gi} - S_{Li}$ = Power leaving bus through lines

- Write the admittance matrix:

$$\begin{pmatrix} I_1 \\ I_2 \\ I_3 \end{pmatrix} \overset{?}{=} \underbrace{\begin{array}{c} \begin{array}{ccc} 1 & 2 & 3 \end{array} \\ \begin{pmatrix} y_{12}+y_{13} & -y_{12} & -y_{13} \\ -y_{12} & y_{12} & 0 \\ -y_{13} & 0 & y_{13} \end{pmatrix} \end{array}}_{Y_{22}} \begin{pmatrix} V_1 \\ V_2 \\ V_3 \end{pmatrix}$$

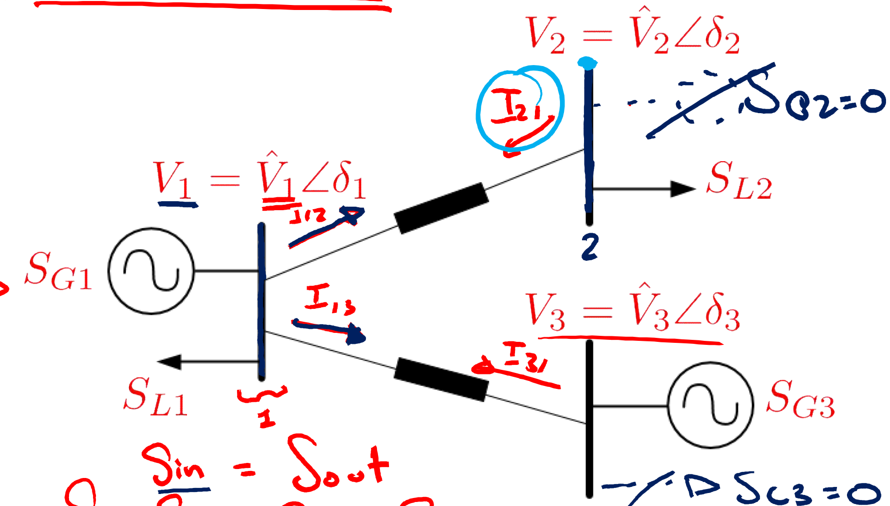
$$y_{ij} \neq Y_{ij}$$

$$\begin{pmatrix} I_1 \\ I_2 \\ I_3 \end{pmatrix} = \begin{pmatrix} \hat{Y}_{11}\angle\phi_{11} & \hat{Y}_{12}\angle\phi_{12} & \hat{Y}_{13}\angle\phi_{13} \\ \hat{Y}_{21}\angle\phi_{21} & \hat{Y}_{22}\angle\phi_{22} & \hat{Y}_{23}\angle\phi_{23} \\ \hat{Y}_{31}\angle\phi_{31} & \hat{Y}_{32}\angle\phi_{32} & \hat{Y}_{33}\angle\phi_{33} \end{pmatrix} \begin{pmatrix} V_1 \\ V_2 \\ V_3 \end{pmatrix}$$

$$I = Y \quad V$$

Power Balance at Every Node

- At every bus in a power system, power has to be balanced



○ What are the currents in the network:

Recall: $S = P + jQ = VI^*$ $\qquad I = \hat{I}\angle\theta$, $\quad I^* = \hat{I}\angle-\theta$

$S_{in} = S_{out}$

$\underbrace{S_{G1} - S_{L1}}_{} = \underbrace{S_{12} + S_{13}}_{}$

○ At node 1:

$$\underbrace{S_{G1} - S_{L1}}_{S_{in}} = \underbrace{V_1 I_{12}^*}_{S_{12}} + \underbrace{V_1 I_{13}^*}_{S_{13}} = V_1(\underbrace{I_{12} + I_{13}}_{})^* = \boxed{\underbrace{V_1 I_1^*}_{S_{out}}}$$

$I_1 = I_{12} + I_3$

○ At node 2 and 3:

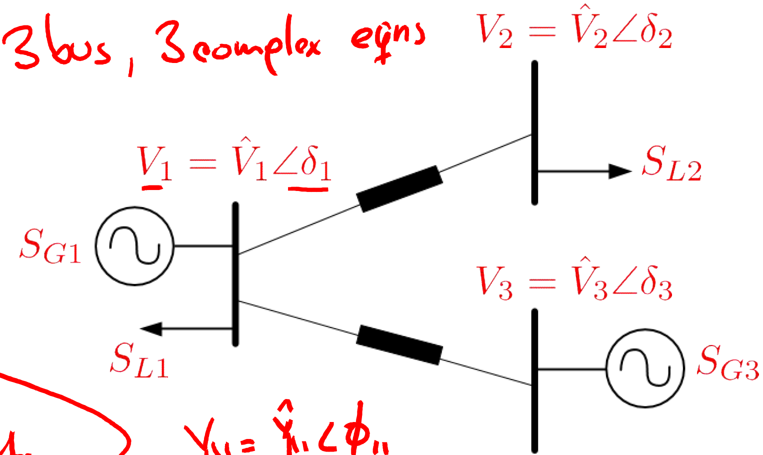
2: $\underset{0}{\cancel{S_{G2}}} - S_{L2} = \boxed{V_2 I_2^*} = V_2(I_{21}^*)$

3: $\underbrace{S_{G3} - \underset{0}{\cancel{S_{L3}}}}_{} = \boxed{V_3 I_3^*} = V_3(I_{31}^*)$

$$\begin{pmatrix} I_1 \\ I_2 \\ I_3 \end{pmatrix} = \begin{pmatrix} \hat{Y}_{11}\angle\phi_{11} & \hat{Y}_{12}\angle\phi_{12} & \hat{Y}_{13}\angle\phi_{13} \\ \hat{Y}_{21}\angle\phi_{21} & \hat{Y}_{22}\angle\phi_{22} & \hat{Y}_{23}\angle\phi_{23} \\ \hat{Y}_{31}\angle\phi_{31} & \hat{Y}_{32}\angle\phi_{32} & \hat{Y}_{33}\angle\phi_{33} \end{pmatrix} \begin{pmatrix} V_1 \\ V_2 \\ V_3 \end{pmatrix}$$

Power Flow Equations

3 bus, 3 complex eqns



- Let's go back to the different nodes:

1: $S_{G1} - S_{L1} = \overbrace{V_1 I_1^*}^{S_{out}}$

2: $-S_{L2} = V_2 I_2^*$

3: $S_{G3} = V_3 I_3^*$

Next Step: Decompose into their real and imaginary parts.

$Y_{11} = \hat{Y}_{11}\angle\phi_{11}$

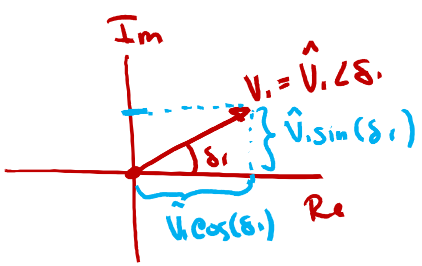
$$\begin{pmatrix} I_1 \\ I_2 \\ I_3 \end{pmatrix} = \begin{pmatrix} \hat{Y}_{11}\angle\phi_{11} & \hat{Y}_{12}\angle\phi_{12} & \hat{Y}_{13}\angle\phi_{13} \\ \hat{Y}_{21}\angle\phi_{21} & \hat{Y}_{22}\angle\phi_{22} & \hat{Y}_{23}\angle\phi_{23} \\ \hat{Y}_{31}\angle\phi_{31} & \hat{Y}_{32}\angle\phi_{32} & \hat{Y}_{33}\angle\phi_{33} \end{pmatrix} \begin{pmatrix} V_1 \\ V_2 \\ V_3 \end{pmatrix}$$

1: $\{S_{G1} - S_{L1} = V_1 I_1^* = V_1 \left(Y_{11} V_1 + Y_{12} V_2 + Y_{13} V_3\right)^*$

$S_{G1} - S_{L1} = V_1 Y_{11}^* V_1^* + V_1 Y_{12}^* V_2^* + V_1 Y_{13}^* V_3^* \}$

1:

- $Re\{S_{G1} - S_{L1}\} = P_{G1} - P_{L1} = \hat{V}_1 \hat{Y}_{11} \hat{V}_1 \cos(\delta_1 - \phi_{11} - \delta_1) + \hat{V}_1 \hat{Y}_{12} \hat{V}_2 \cos(\delta_1 - \delta_2 - \phi_{12}) + \hat{V}_1 \hat{Y}_{13} \hat{V}_3 \cos(\delta_1 - \delta_3 - \phi_{13})$
- $Im\{S_{G1} - S_{L1}\} = Q_{G1} - Q_{L1} =$ " sin( ) + " sin(") + " sin(")



$S_i = \underbrace{P_i}_{Re} + j\underbrace{Q_i}_{Im}$

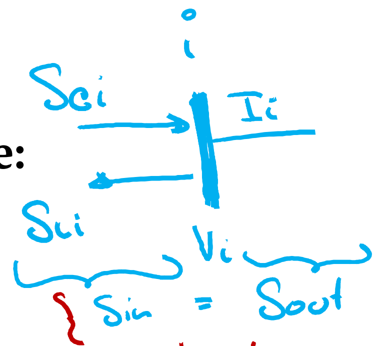
$V_1 = \hat{V}_1 \angle\delta_1$

$Y_{ij} = \hat{Y}_{ij}\angle\phi_{ij}$

$Re\{V_1\} = \hat{V}_1 \cos(\delta_1)$

Power Flow Equations General Formula

- For a general power network the **power flow equations are:**



$$\begin{pmatrix} I_1 \\ I_2 \\ I_3 \end{pmatrix} = \begin{pmatrix} \hat{Y}_{11}\angle\phi_{11} & \hat{Y}_{12}\angle\phi_{12} & \hat{Y}_{13}\angle\phi_{13} \\ \hat{Y}_{21}\angle\phi_{21} & \hat{Y}_{22}\angle\phi_{22} & \hat{Y}_{23}\angle\phi_{23} \\ \hat{Y}_{31}\angle\phi_{31} & \hat{Y}_{32}\angle\phi_{32} & \hat{Y}_{33}\angle\phi_{33} \end{pmatrix} \begin{pmatrix} V_1 \\ V_2 \\ V_3 \end{pmatrix}$$

N N

$V_i = \hat{V}_i \angle \delta_i$

\# of buses

Volt angle at bus k

i bus

- $$\underbrace{\mathbf{P_i = P_{Gi} - P_{Li}}}_{\text{in}} = \underbrace{\mathbf{\hat{V}_i \sum_{k=1}^{N} \hat{Y}_{ik} \hat{V}_k} \cos(\delta_\mathbf{i} - \delta_\mathbf{k} - \phi_\mathbf{ik})}_{\text{out}}$$

volt. angle at bus i

$Y_{ik} = \hat{Y}_{ik} \angle \phi_{ik}$

- $$\mathbf{Q_i = Q_{Gi} - Q_{Li} = \hat{V}_i \sum_{k=1}^{N} \hat{Y}_{ik} \hat{V}_k} \sin(\delta_\mathbf{i} - \delta_\mathbf{k} - \phi_\mathbf{ik})$$

N buses $\Longrightarrow$ 2N equations $\Longrightarrow$ 2N unknowns

{ N active power eqns.
{ N Reactive " "

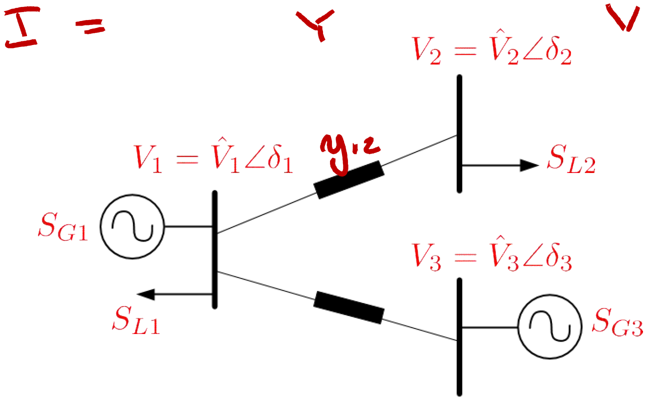
Power Flow Equations – Caution!

- **Be careful with the notation**
- Keep track of the **right angles** and **admittances** to use

$$\mathbf{P_i = P_{Gi} - P_{Li} = \hat{V}_i \sum_{k=1}^{N} \hat{Y}_{ik} \hat{V}_k} \cos(\delta_\mathbf{i} - \delta_\mathbf{k} - \phi_\mathbf{ik})$$

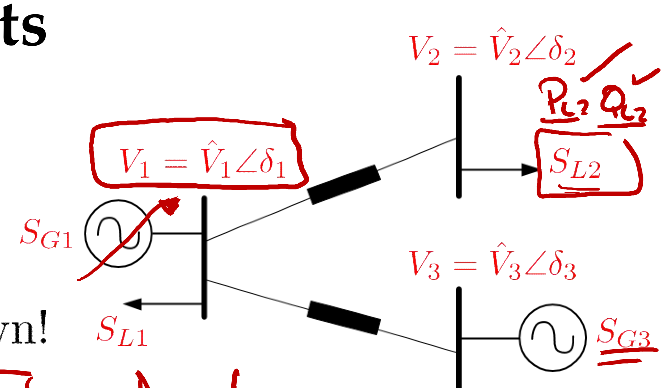
$$\mathbf{Q_i = Q_{Gi} - Q_{Li} = \hat{V}_i \sum_{k=1}^{N} \hat{Y}_{ik} \hat{V}_k} \sin(\delta_\mathbf{i} - \delta_\mathbf{k} - \phi_\mathbf{ik})$$

$$\underbrace{\begin{pmatrix} I_1 \\ I_2 \\ I_3 \end{pmatrix}}_{I} = \underbrace{\overbrace{\begin{pmatrix} \hat{Y}_{11}\angle\phi_{11} & \hat{Y}_{12}\angle\phi_{12} & \hat{Y}_{13}\angle\phi_{13} \\ \hat{Y}_{21}\angle\phi_{21} & \hat{Y}_{22}\angle\phi_{22} & \hat{Y}_{23}\angle\phi_{23} \\ \hat{Y}_{31}\angle\phi_{31} & \hat{Y}_{32}\angle\phi_{32} & \hat{Y}_{33}\angle\phi_{33} \end{pmatrix}}^{\text{admittance matrix}}}_{Y} \underbrace{\begin{pmatrix} V_1 \\ V_2 \\ V_3 \end{pmatrix}}_{V}$$



$$\underbrace{y_{12} = \hat{y}_{12}\angle\theta_{12}}_{\text{line admittance 1-2}} \quad \textbf{vs.} \quad \neq \quad \underbrace{Y_{12} = \hat{Y}_{12}\angle\phi_{12}}_{\text{Adm. Matrix element (1,2)}}$$

Power Flow Comments



- At every bus, the variables are: $\underbrace{P_i,\ Q_i,\ \hat{V}_i,\ \delta_i}$
- Generally, two variables are know and two are unknown! → Depends on the type of elements connected to the bus
- The goal of the power flow is to solve for those unknown

Typically is bus 1

$\hat{V}_1 = 1$ pu
$\delta_1 = 0$

TABLE 6.1 Bus Types for Power Flow Formulation

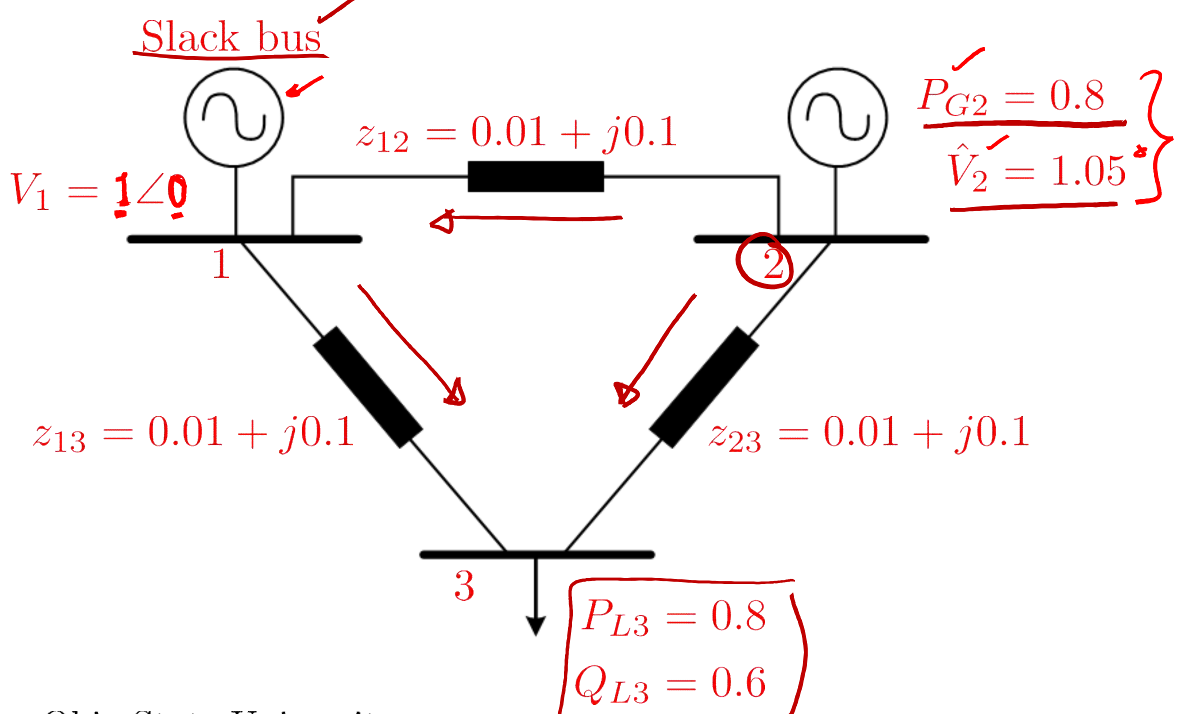
Bus Type	Code	Known	Unknowns
Slack generator (Reference bus)	Vδ	V_i, δ_i	P_{Gi}, Q_{Gi}
Slack demand or tie	Qδ	Q_{Gi}, δ_i	P_{Gi}, V_i
Demand (Load)	PQ	P_{Gi}, Q_{Gi}	V_i, δ_i
Generator	PV	P_{Gi}, V_i	Q_{Gi}, δ_i
Controlled voltage magnitude	CV	P_{Gi}, V_i	δ_i, α

Known 2

Unknowns 2

Power Flow Example

- The variables are given in PU, $V_{\text{base}} = 138 \text{ kV}_{LL}$, $S_{\text{base}3\phi} = 100$ MVA
- **Impedance** of every line is $0.01 + j0.1$
- Bus 1 is a **Slack** bus, i.e. $V = 1\angle 0$ $\Rightarrow$ Known: $\hat{V}_1 = 1$, $\delta_1 = 0$, Unknown: P_{g1}, Q_{g1}
- Bus 2 is a **PV** bus, i.e. $P_{G2} = 0.8$, $\hat{V}_2 = 1.05$: Known, Unknown: $\boxed{Q_{g2}, \delta_2}$
- Bus 3 is a **PQ** bus, i.e. $P_{L3} = 0.8$, $Q_{L3} = 0.6$: Known, Unknown: $\hat{V}_3, \delta_3$



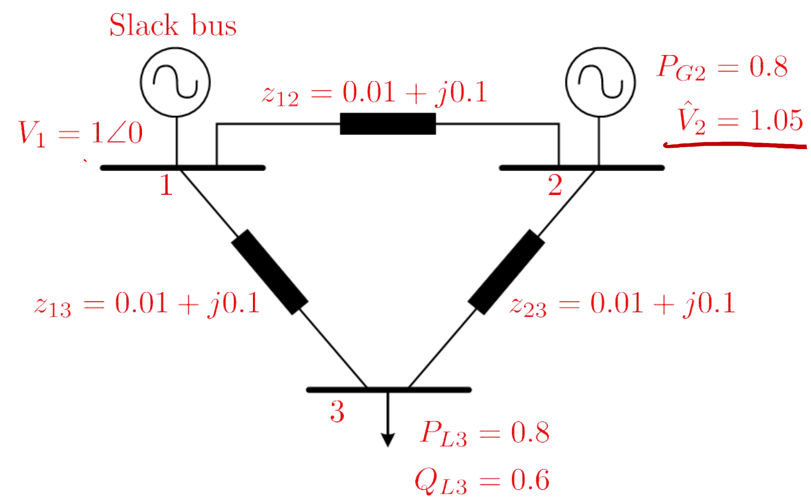
*Reference: A. Conejo, Ohio State University

Power Flow Example

- Compute the admittance matrix

$$\underline{Y} = \begin{pmatrix} 19.90\angle -1.47 & 9.95\angle 1.67 & 9.95\angle 1.67 \\ 9.95\angle 1.67 & 19.90\angle -1.47 & 9.95\angle 1.67 \\ 9.95\angle 1.67 & 9.95\angle 1.67 & 19.90\angle -1.47 \end{pmatrix}$$

$$Y = \begin{pmatrix} Y_{11} & Y_{12} & Y_{13} \\ \vdots & \ddots & \vdots \\ \cdot & \cdot & \cdot \end{pmatrix}$$



- Compute the powerflow equations

$$\mathbf{P_i = P_{Gi} - P_{Li} = \hat{V}_i \sum_{k=1}^{N} \hat{Y}_{ik}\hat{V}_k \cos(\delta_i - \delta_k - \phi_{ik})}$$

$$\mathbf{Q_i = Q_{Gi} - Q_{Li} = \hat{V}_i \sum_{k=1}^{N} \hat{Y}_{ik}\hat{V}_k \sin(\delta_i - \delta_k - \phi_{ik})}$$

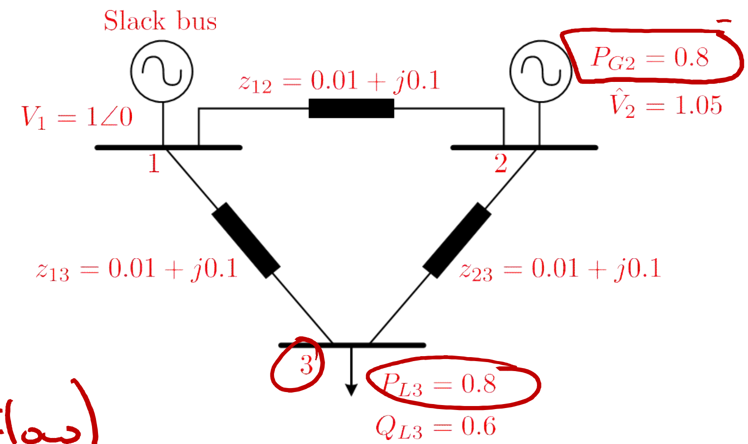
Bus	Known	Unknown
1	$\hat{V}_1 = 1,\ \delta_1 = 0$	$P_1,\ Q_1$
2	$P_2 = 0.8,\ \hat{V}_2 = 1.05$	$Q_2,\ \delta_2$
3	$P_3 = -0.8,\ Q_3 = -0.6$	$\hat{V}_3,\ \delta_3$

Bus 1:

$$P_{g1} - \cancel{P_{L1}}^{0} = \underset{\hat{V}_1}{(1)} \sum_{k=1}^{3} \hat{Y}_{1k}\hat{V}_k \cos(\underset{\delta_1}{(0)} - \delta_k - \phi_{1k})$$

1: $$\begin{cases} P_{g1} = \overbrace{(1)(19.9)(1)\cos(0-0+1.47)}^{k=1} + \overbrace{(1)(9.95)(1.05)\cos(0-\delta_2-1.67)}^{k=2} + \overbrace{(1)(9.95)(\hat{V}_3)\cos(0-\delta_3-1.67)}^{K=3} \\ Q_{g1} = \text{''} \sin(\text{''}) + \text{''} \sin(\text{''}) + \text{''} \sin(\text{''}) \end{cases}$$

Power Flow Example



- Compute the admittance matrix

$$Y = \begin{pmatrix} 19.90\angle{-1.47} & 9.95\angle 1.67 & 9.95\angle 1.67 \\ 9.95\angle 1.67 & 19.90\angle{-1.47} & 9.95\angle 1.67 \\ 9.95\angle 1.67 & 9.95\angle 1.67 & 19.90\angle{-1.47} \end{pmatrix}$$

- Compute the powerflow equations (Active Power Flow)

1: $$P_1 = 19.09\cos(1.471) + 9.95(1.05)\cos(-\delta_2 - 1.67) + 9.95(\hat{V}_3)\cos(-\delta_3 - 1.67)$$

2: $$\underline{\underline{0.8}} = 9.95(1.05)\cos(\delta_2 - 1.67) + 19.90(1.05)^2\cos(1.47) + 9.95(1.05)(\hat{V}_3)\cos(\delta_2 - \delta_3 - 1.67)$$

3: $$-0.8 = 9.95(\hat{V}_3)\cos(\delta_3 - 1.67) + 9.95(\hat{V}_3)(1.05)\cos(\delta_3 - \delta_2 - 1.67) + 19.90(\hat{V}_3)^2\cos(1.47)$$

Bus	Known	Unknown
1	$\hat{V}_1 = 1,\ \delta_1 = 0$	$P_1,\ Q_1$
2	$P_2 = 0.8,\ \hat{V}_2 = 1.05$	$Q_2,\ \delta_2$
3	$P_3 = -0.8,\ Q_3 = -0.6$	$\hat{V}_3,\ \delta_3$

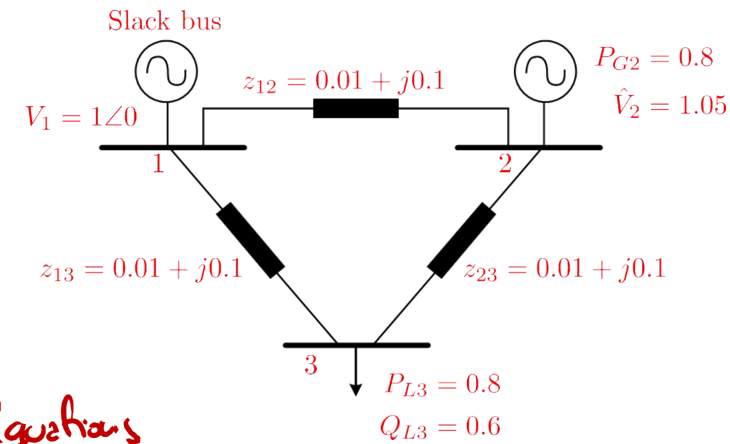
$$\mathbf{P_i = P_{Gi} - P_{Li} = \hat{V}_i \sum_{k=1}^{N} \hat{Y}_{ik}\hat{V}_k}\cos(\delta_i - \delta_k - \phi_{ik})$$

$$\mathbf{Q_i = Q_{Gi} - Q_{Li} = \hat{V}_i \sum_{k=1}^{N} \hat{Y}_{ik}\hat{V}_k}\sin(\delta_i - \delta_k - \phi_{ik})$$

Power Flow Example

- Compute the admittance matrix

$$Y = \begin{pmatrix} 19.90\angle{-1.47} & 9.95\angle 1.67 & 9.95\angle 1.67 \\ 9.95\angle 1.67 & 19.90\angle{-1.47} & 9.95\angle 1.67 \\ 9.95\angle 1.67 & 9.95\angle 1.67 & 19.90\angle{-1.47} \end{pmatrix}$$



- Compute the powerflow equations Reactive Power Equations

1: $Q_1 = 19.09\sin(1.471) + 9.95(1.05)\sin(-\delta_2 - 1.67) + 9.95(\hat{V}_3)\sin(-\delta_3 - 1.67)$

2: $Q_2 = 9.95(1.05)\sin(\delta_2 - 1.67) + 19.90(1.05)^2\sin(1.47) + 9.95(1.05)(\hat{V}_3)\sin(\delta_2 - \delta_3 - 1.67)$

3: $-0.6 = 9.95(\hat{V}_3)\sin(\delta_3 - 1.67) + 9.95(\hat{V}_3)(1.05)\sin(\delta_3 - \delta_2 - 1.67) + 19.90(\hat{V}_3)^2\sin(1.47)$

Bus	Known	Unknown
1	$\hat{V}_1 = 1,\ \delta_1 = 0$	$P_1,\ Q_1$
2	$P_2 = 0.8,\ \hat{V}_2 = 1.05$	$Q_2,\ \delta_2$
3	$P_3 = -0.8,\ Q_3 = -0.6$	$\hat{V}_3,\ \delta_3$

$$\mathbf{P_i = P_{Gi} - P_{Li} = \hat{V}_i \sum_{k=1}^{N} \hat{Y}_{ik}\hat{V}_k} \cos(\delta_i - \delta_k - \phi_{ik})$$

$$\mathbf{Q_i = Q_{Gi} - Q_{Li} = \hat{V}_i \sum_{k=1}^{N} \hat{Y}_{ik}\hat{V}_k} \sin(\delta_i - \delta_k - \phi_{ik})$$

Power Flow Example

- Write the power flow equations 3 buses ⇒ 6 equations ⇒ 6 unknowns

1: $$P_1 = 19.09\cos(1.471) + 9.95(1.05)\cos(-\delta_2 - 1.67) + 9.95(\hat{V}_3)\cos(-\delta_3 - 1.67)$$

2: $$0.8 = 9.95(1.05)\cos(\delta_2 - 1.67) + 19.90(1.05)^2\cos(1.47) + 9.95(1.05)(\hat{V}_3)\cos(\delta_2 - \delta_3 - 1.67)$$

3: $$-0.8 = 9.95(\hat{V}_3)\cos(\delta_3 - 1.67) + 9.95(\hat{V}_3)(1.05)\cos(\delta_3 - \delta_2 - 1.67) + 19.90(\hat{V}_3)^2\cos(1.47)$$

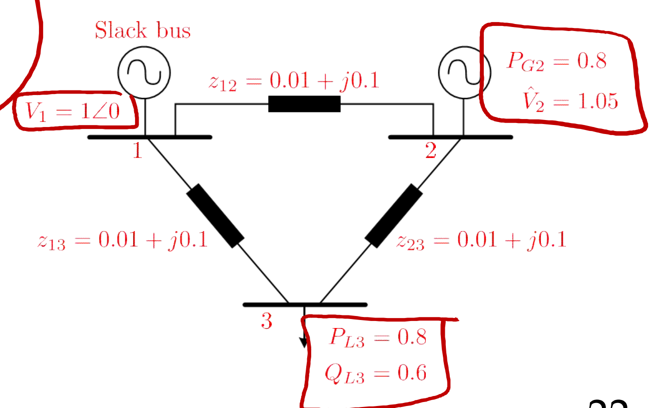
1: $$Q_1 = 19.09\sin(1.471) + 9.95(1.05)\sin(-\delta_2 - 1.67) + 9.95(\hat{V}_3)\sin(-\delta_3 - 1.67)$$

2: $$Q_2 = 9.95(1.05)\sin(\delta_2 - 1.67) + 19.90(1.05)^2\sin(1.47) + 9.95(1.05)(\hat{V}_3)\sin(\delta_2 - \delta_3 - 1.67)$$

3: $$-0.6 = 9.95(\hat{V}_3)\sin(\delta_3 - 1.67) + 9.95(\hat{V}_3)(1.05)\sin(\delta_3 - \delta_2 - 1.67) + 19.90(\hat{V}_3)^2\sin(1.47)$$

- How to solve these equations for the unknowns?

$$\underbrace{P_{g1},\ Q_{g1},\ Q_{g2},\ \delta_2,\ \hat{V}_3,\ \delta_3}_{6\text{ unknowns}}$$



Solving Systems of Equations

- Remember, solving system of linear equations

  Solve for x_1 Gaussian elimination

  - $10x_1 + 2x_2 = 10$
  - $1x_1 - 3x_2 = 4$

  $x_2 = (\quad)$ solve for x_2

  , $Ax = b$, $x = A^{-1}b$

  x ↓ $\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$

- Nonlinear equations do not generally have a similar procedure

- Therefore, we look for numerical methods that $\underbrace{\text{iteratively}}$ help us find x

- **Problem Statement:** Given a set of nonlinear equations $F(x)$
  - find x to satisfy $F(x) = 0$

  $x \in \mathbb{R}^N$

$$F(x) = \begin{pmatrix} f_1(x_1, \ ..., \ x_N) \\ f_2(x_1, \ ..., \ x_N) \\ \vdots \\ f_N(x_1, \ ..., \ x_N) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

Numerical Methods:

- Newton Raphson ✓
- Gauss-Seidel ✓

Power Flow Example Numerical Solution

- Write the power flow equations with **everything on one side** $F(x) = 0$

Bus

1: $$0 = -P_1 + 19.09\cos(1.471) + 9.95(1.05)\cos(-\delta_2 - 1.67) + 9.95(\hat{V}_3)\cos(-\delta_3 - 1.67)$$

2 $$0 = -0.8 + 9.95(1.05)\cos(\delta_2 - 1.67) + 19.90(1.05)^2\cos(1.47) + 9.95(1.05)(\hat{V}_3)\cos(\delta_2 - \delta_3 - 1.67)$$

3 $$0 = 0.8 + 9.95(\hat{V}_3)\cos(\delta_3 - 1.67) + 9.95(\hat{V}_3)(1.05)\cos(\delta_3 - \delta_2 - 1.67) + 19.90(\hat{V}_3)^2\cos(1.47)$$

1 $$0 = -Q_1 + 19.09\sin(1.471) + 9.95(1.05)\sin(-\delta_2 - 1.67) + 9.95(\hat{V}_3)\sin(-\delta_3 - 1.67)$$

2 $$0 = -Q_2 + 9.95(1.05)\sin(\delta_2 - 1.67) + 19.90(1.05)^2\sin(1.47) + 9.95(1.05)(\hat{V}_3)\sin(\delta_2 - \delta_3 - 1.67)$$

3 $$0 = 0.6 + 9.95(\hat{V}_3)\sin(\delta_3 - 1.67) + 9.95(\hat{V}_3)(1.05)\sin(\delta_3 - \delta_2 - 1.67) + 19.90(\hat{V}_3)^2\sin(1.47)$$

- Identify the unknowns and create a vector x

$$x = [\underbrace{\hat{V}_3}_{x_1}\ \underbrace{\delta_2}_{x_2}\ \delta_3\ \cdots\ P_{g1}\ Q_{g1}\ \underbrace{Q_{g2}}_{x_6}]^T$$

Bus	Known	Unknown
1	$\hat{V}_1 = 1,\ \delta_1 = 0$	$P_1,\ Q_1$
2	$P_2 = 0.8,\ \hat{V}_2 = 1.05$	$Q_2,\ \delta_2$
3	$P_3 = -0.8,\ Q_3 = -0.6$	$\hat{V}_3,\ \delta_3$

- Re-write equations based on $x_1,\ x_2,\ \ldots,\ x_N$

Power Flow Example Numerical Solution – Matlab Code

- Create a function which outputs the powerflow equations evaluated at different x

```
function F = pflowfun(x)

F(1)  = -x(4) +(19.900744)*(1.000000)*(1.000000)*cos((0.000000)-(0.000000)-(-1.471128))  ...
              +(9.950372)*(1.000000)*(1.050000)*cos((0.000000)-x(2)-(1.670465))          ...
              +(9.950372)*(1.000000)*x(1)*cos((0.000000)-x(3)-(1.670465));

F(2)  = -(0.800000)  +(9.950372)*(1.050000)*(1.000000)*cos(x(2)-(0.000000)-(1.670465))...
                     +(19.900744)*(1.050000)*(1.050000)*cos(x(2)-x(2)-(-1.471128))     ...
                     +(9.950372)*(1.050000)*x(1)*cos(x(2)-x(3)-(1.670465));

F(3)  = -(-0.800000)  +(9.950372)*x(1)*(1.000000)*cos(x(3)-(0.000000)-(1.670465))...
                      +(9.950372)*x(1)*(1.050000)*cos(x(3)-x(2)-(1.670465))      ...
                      +(19.900744)*x(1)*x(1)*cos(x(3)-x(3)-(-1.471128));

F(4)  = -x(5)   +(19.900744)*(1.000000)*(1.000000)*sin((0.000000)-(0.000000)-(-1.471128))...
                +(9.950372)*(1.000000)*(1.050000)*sin((0.000000)-x(2)-(1.670465))        ...
                +(9.950372)*(1.000000)*x(1)*sin((0.000000)-x(3)-(1.670465));

F(5)  = -x(6)   +(9.950372)*(1.050000)*(1.000000)*sin(x(2)-(0.000000)-(1.670465))  ...
                +(19.900744)*(1.050000)*(1.050000)*sin(x(2)-x(2)-(-1.471128))      ...
                +(9.950372)*(1.050000)*x(1)*sin(x(2)-x(3)-(1.670465));

F(6)  = -(-0.600000)   +(9.950372)*x(1)*(1.000000)*sin(x(3)-(0.000000)-(1.670465))...
                       +(9.950372)*x(1)*(1.050000)*sin(x(3)-x(2)-(1.670465))    ...
                       +(19.900744)*x(1)*x(1)*sin(x(3)-x(3)-(-1.471128));
```

Power Flow Example Numerical Solution – Matlab Code

- Create a function which outputs the powerflow equations evaluated at different x
- Then, make a **realistic guess** for the initial point x_0

$$x = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{pmatrix} = \begin{pmatrix} \hat{V}_3 \\ \delta_2 \\ \delta_3 \\ P_1 \\ Q_1 \\ Q_2 \end{pmatrix}$$

- Use the command **fsolve** to obtain solution

$F(x) = 0$

```
%% Solve the power flow equations
x0 = [1 0 0 0.5 0.5 0.5]';   initial condition

options = optimoptions('fsolve','Display','iter');
x = fsolve(@pflowfun,x0,options)
```

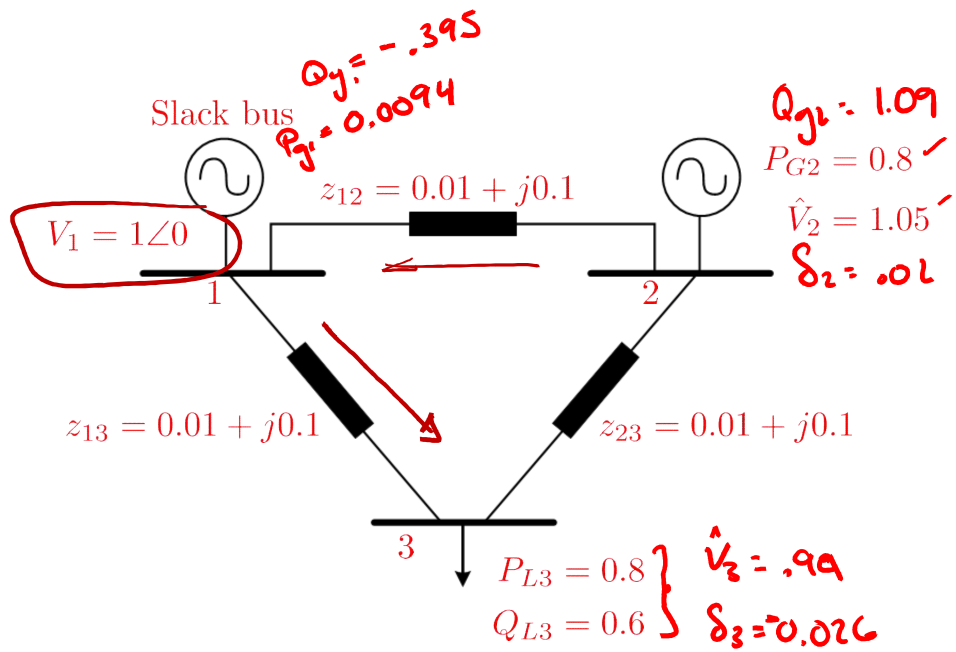
```
x =

    0.9899 = V̂3
    0.0200 = δ2
   -0.0262 = δ3
    0.0094 = P1
   -0.3946 = Q1
    1.0891 = Q2
```

- The initial point x_0 is important, as sometimes the algorithm converges to a solution that does not make sense!

Power Flow Example - Results

- View of the solution

$$x = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{pmatrix} = \begin{pmatrix} \hat{V}_3 \\ \delta_2 \\ \delta_3 \\ P_1 \\ Q_1 \\ Q_2 \end{pmatrix} = \begin{pmatrix} 0.9899 \\ 0.02 \\ -0.0262 \\ 0.0094 \\ -0.3946 \\ 1.0891 \end{pmatrix}$$



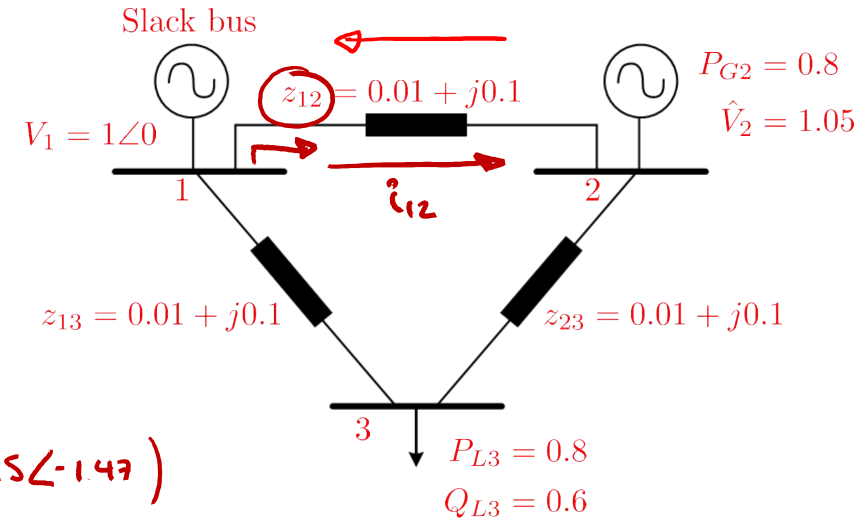
Bus	Known	Unknown
1	$\hat{V}_1 = 1,\ \delta_1 = 0$	$P_1 = 0.0094,\ Q_1 = -0.395$
2	$P_2 = 0.8,\ \hat{V}_2 = 1.05$	$Q_2 = 1.09,\ \delta_2 = 0.02$
3	$P_3 = -0.8,\ Q_3 = -0.6$	$\hat{V}_3 = .99,\ \delta_3 = -0.026$

$.9 \le \hat{V}_i \le 1.1$

Power Flow Example - Results

- Compute the current through the lines

Bus	Known	Unknown
1	$\hat{V}_1 = 1,\ \delta_1 = 0$	$P_1 = 0.0094,\ Q_1 = -0.395$
2	$P_2 = 0.8,\ \hat{V}_2 = 1.05$	$Q_2 = 1.09,\ \delta_2 = 0.02$
3	$P_3 = -0.8,\ Q_3 = -0.6$	$\hat{V}_3 = .99,\ \delta_3 = -0.026$



- $i_{ij} = (V_i - V_j)y_{ij}, \quad \forall\, i \neq j$

- $i_{12} \triangleq (V_1 - V_2)y_{12} = (1\angle 0 - 1.05\angle 0.02)\frac{1}{z_{12}}$

$(9.95\angle -1.47) = \frac{1}{z_{12}}$

$$i_{12} = \underbrace{0.538\angle 2.071}\ \text{rad}$$

- $S_{12} \triangleq V_1 i_{12}^* = (1\angle 0)(0.538\angle -2.071) = \underbrace{0.538\angle -2.071 \text{ pu}}$

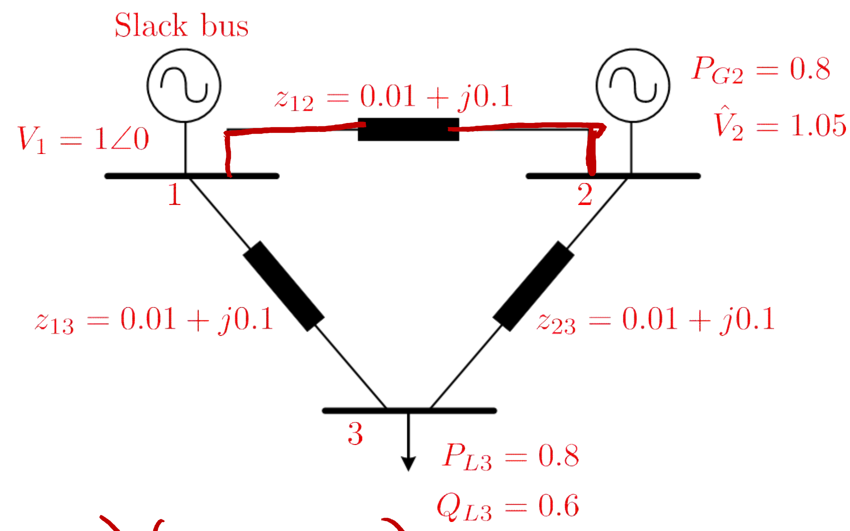
power flowing from bus 1 to 2

$$\begin{cases} P_{12} = -0.257 \text{ pu} \\ Q_{12} = -0.48 \text{ pu} \end{cases}$$

Power Flow Example - Results

- Compute the power losses at the lines

Bus	Known	Unknown
1	$\hat{V}_1 = 1,\ \delta_1 = 0$	$P_1 = 0.0094,\ Q_1 = -0.395$
2	$P_2 = 0.8,\ \hat{V}_2 = \underline{1.05}$	$Q_2 = 1.09,\ \underline{\delta_2 = 0.02}$
3	$P_3 = -0.8,\ Q_3 = -0.6$	$\hat{V}_3 = .99,\ \delta_3 = -0.026$



$$\cancel{P_{\text{loss}-ij} = (V(i) - V(j))i_{ij}^*}$$

$$S_{loss\,12} \hat{=} (V_1 - V_2)\, i_{12}^* = (1\angle 0 - 1.05\angle .02)(.538\angle -2.071)$$

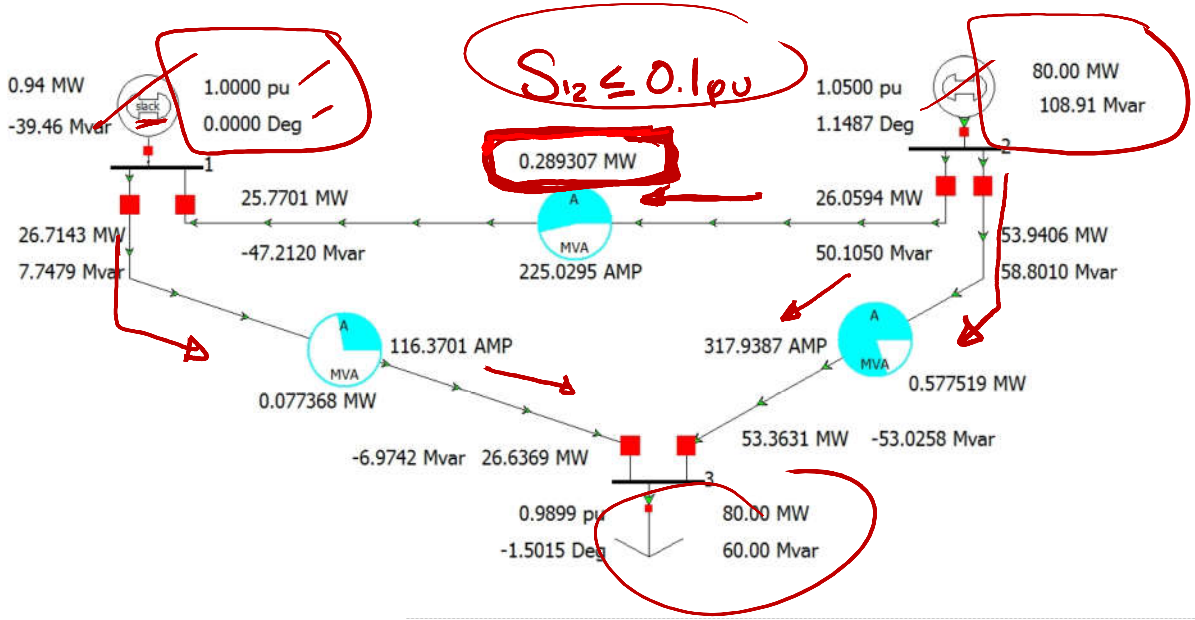
$$\Rightarrow S_{loss\,12} = \underbrace{0.0029}_{P_{loss\,12}} + j\underbrace{(.0289)}_{Q_{loss\,1-2}} \quad pu$$

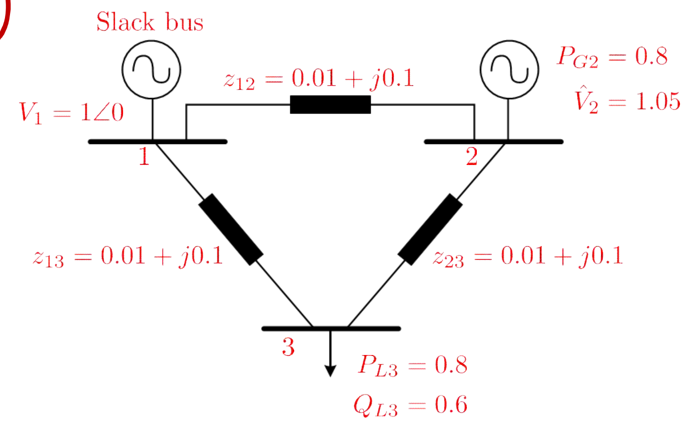
$$S_{base\,1\phi} = 100MVA = P_{base} = Q_{base}$$

$$P_{loss\,1-2} = (P_{loss}\,pu)(\underline{P_{base}}) = \boxed{.29MW} = 290kW$$

Power Flow Example - PowerWorld

- Using powerworld we can obtain the same results, $S_{\text{base}} = 100$ MVA





Bus	Known	Unknown
1	$\hat{V}_1 = 1,\ \delta_1 = 0$	$P_1 = 0.0094,\ Q_1 = -0.395$
2	$P_2 = 0.8,\ \hat{V}_2 = 1.05$	$Q_2 = 1.09,\ \delta_2 = 0.02$
3	$P_3 = -0.8,\ Q_3 = -0.6$	$\hat{V}_3 = .99,\ \delta_3 = -0.026$

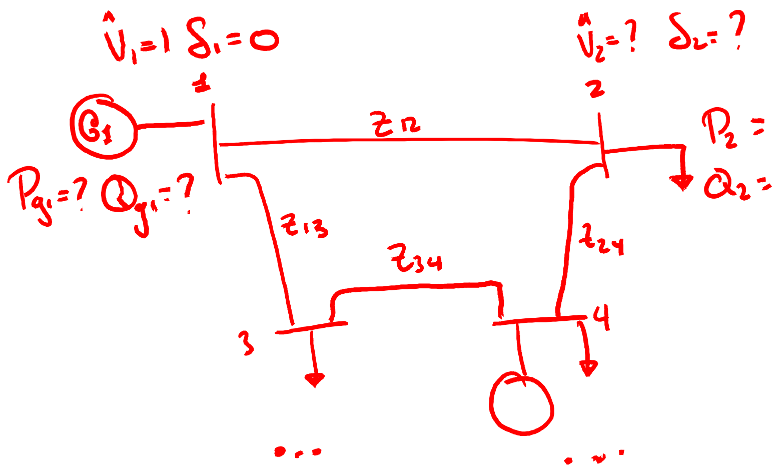
∗Reference: A. Conejo, Ohio State University

Power Flow Summary

- Numerical method to analyze the way in which power flows in a electrical network

- At every bus 4 variables: $\hat{V}_i, \delta_i, P_i, Q_i \quad \forall i \in \mathcal{N}$ → Set of buses

  2 known, 2 unknown



$$P_{gi} - P_{Li} = \hat{V}_i \sum_{k=1}^{N} \hat{V}_k \hat{Y}_{ik} \cos(\cdots)$$

$$Q_{gi} - Q_{Li} = \ \text{''} \quad \text{''} \ \sin(\cdots)$$

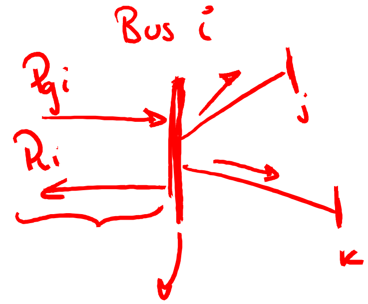
Topics that Will be Covered

- *Nodal Analysis* ✓
- *Power Flow Equations* ✓
- * ***DC Power Flow - simplification***

not direct current

- *Economic Dispatch with Network Constraints – DC OPF*
- *LP Formulation*
- * *Locational Marginal Price* (LMP)

Power Flow Equations



- Recall the general power flow equations:

Active Power

$$P_i = \underbrace{P_{Gi} - P_{Li}}_{} = \hat{V}_i \sum_{k=1}^{N} \hat{Y}_{ik} \hat{V}_k \cos(\delta_i - \delta_k - \phi_{ik})$$

Reactive Power

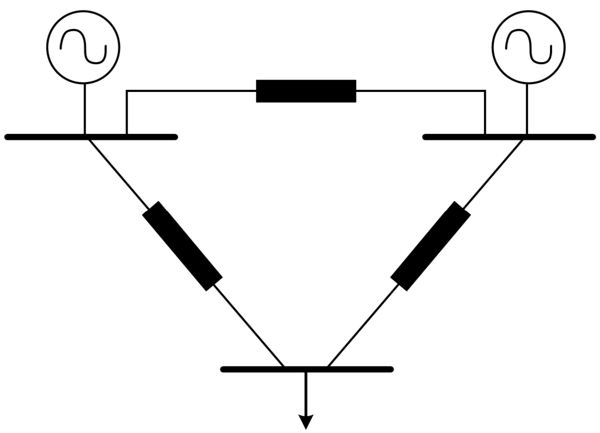
$$Q_i = Q_{Gi} - Q_{Li} = \hat{V}_i \sum_{k=1}^{N} \hat{Y}_{ik} \hat{V}_k \sin(\delta_i - \delta_k - \phi_{ik})$$

$V_i = \hat{V}_i \angle \delta_i$

$S_i = P_i + jQ_i$

- The nonlinearity of these equations make them difficult to solve

- For the economic dispatch problem, we are mainly interested in active power flow



Remember: Cost of each generator is a function of

* active power only*

$\left(\frac{\$}{MWh}\right) f_i(P_i)$

$\cancel{Q}$

Power Flow Equations – Assumptions

- Recall the general power flow equations:

$$\begin{cases} \mathbf{P_i = P_{Gi} - P_{Li} = \hat{V}_i \sum_{k=1}^{N} \hat{Y}_{ik} \hat{V}_k \cos(\delta_i - \delta_k - \phi_{ik})} \\ \cancel{\mathbf{Q_i = Q_{Gi} - Q_{Li} = \hat{V}_i \sum_{k=1}^{N} \hat{Y}_{ik} \hat{V}_k \sin(\delta_i - \delta_k - \phi_{ik})}} \end{cases}$$

- To simplify these equations, we can make the following assumptions:

Assumption 1: Voltage magnitudes are close to 1 in per unit

$\hat{V}_i \approx 1$ pu $\forall i \in \mathcal{N}$ → set of buses $\mathcal{N} = \{1, 2, 3\}$

Assumption 2: Resistances are much smaller than reactances

$\forall (i,j) \in \mathcal{E}$ $r_{ij} << x_{ij} \Rightarrow z_{ij} \approx j x_{ij}$ (ignore the resistance)

$\mathcal{E}$: set of lines

Assumption 3: Phase differences across lines are small

Voltage

$\sin(\delta_i - \delta_j) \approx \delta_i - \delta_j$ (Small angle approx)

$\delta_i - \delta_j$: voltage angles

Power Flow Equations – Assumption 1

- Recall the general power flow equations:

$$\mathbf{P_i = P_{Gi} - P_{Li} = \underline{\hat{V}_i} \sum_{k=1}^{N} \hat{Y}_{ik} \underline{\hat{V}_k} \cos(\delta_i - \delta_k - \phi_{ik})}$$

$$\cancel{\mathbf{Q_i = Q_{Gi} - Q_{Li} = \hat{V}_i \sum_{k=1}^{N} \hat{Y}_{ik} \hat{V}_k \sin(\delta_i - \delta_k - \phi_{ik})}}$$

Assumption 1: Voltage magnitudes are close to $\underline{1}$ in per unit

$$\Rightarrow \hat{V}_i = 1 \quad \forall i \in N$$

$$P_i = P_{gi} - P_{Li} = (1) \sum_{k=1}^{N} Y_{ik} (1) \cos(\delta_i - \delta_k - \phi_{ik})$$

$\Downarrow$

Net power

$$P_{gi} - P_{Li} = \sum_{k=1}^{N} Y_{ik} \underline{\underline{\cos}}(\underline{\delta_i} - \underline{\delta_k} - \phi_{ik})$$

Power Flow Equations – Assumption 2

- After applying assumption 1:

$$P_i = P_{Gi} - P_{Li} \approx \sum_{k=1}^{N} \hat{Y}_{ik} \cos(\delta_i - \delta_k - \phi_{ik})$$

$$\cancel{Q_i = Q_{Gi} - Q_{Li} \approx \sum_{k=1}^{N} \hat{Y}_{ik} \sin(\delta_i - \delta_k - \phi_{ik})}$$

Assumption 2: Resistances are much smaller than reactances

$Z_{ik} = \underbrace{\cancel{r_{ik}}}_{\approx 0} + jX_{ik}$ $\quad$ $Z_{ik} = jX_{ik}$

- Find admittance of line ik $\Rightarrow$ "y_{ik}" $= \frac{1}{Z_{ik}} = \frac{1}{jX_{ik}} = \underbrace{\frac{1}{X_{ik}}}_{\text{"Susceptance"}} \angle -\frac{\pi}{2} = b_{ik} \angle -\pi/2$

- $\underset{\substack{\downarrow \\ \text{element of } Y \text{ (adm. matrix)}}}{Y_{ik}} = -y_{ik} = b_{ik} \angle \pi/2$ $\quad$ $i \neq k$ (off-diagonal)

$\cos(\theta - \pi/2) = \sin(\theta)$

$$P_i = P_{Gi} - P_{Li} = \sum_{k=1}^{N} b_{ik} \cos\left(\delta_i - \delta_k - \frac{\pi}{2}\right) = \underbrace{\sum_{k=1}^{N} b_{ik} \sin(\delta_i - \delta_k)}$$

Power Flow Equations – Assumptions 3

- After applying assumption 2:

$$\mathbf{P_i = P_{Gi} - P_{Li}} \approx \sum_{\mathbf{k=1}}^{\mathbf{N}} \overbrace{\mathrm{Im}\{\mathbf{Y_{ik}}\}}^{b_{ik}} \sin(\delta_\mathbf{i} - \delta_\mathbf{k})$$

$$\cancel{\mathbf{Q_i = Q_{Gi} - Q_{Li}} \approx \sum_{\mathbf{k=1}}^{\mathbf{N}} \mathrm{Im}\{\mathbf{Y_{ik}}\} \cos(\delta_\mathbf{i} - \delta_\mathbf{k})}$$

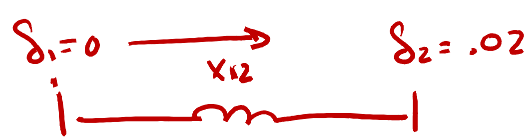
Assumption 3: Phase differences across lines are small

$$\Rightarrow \sin(\delta_i - \delta_k) \approx \delta_i - \delta_k$$

$$\boxed{P_i = P_{gi} - P_{Li} = \sum_{k=1}^{N} b_{ik}(\delta_i - \delta_k)}$$

linear (DC) power flow equations

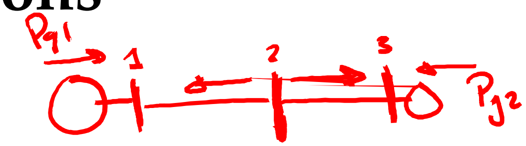
variables P_i, δ_i



$$b_{12} = \frac{1}{x_{12}}$$

$$P_{12} \approx b_{12}(\delta_1 - \delta_2)$$

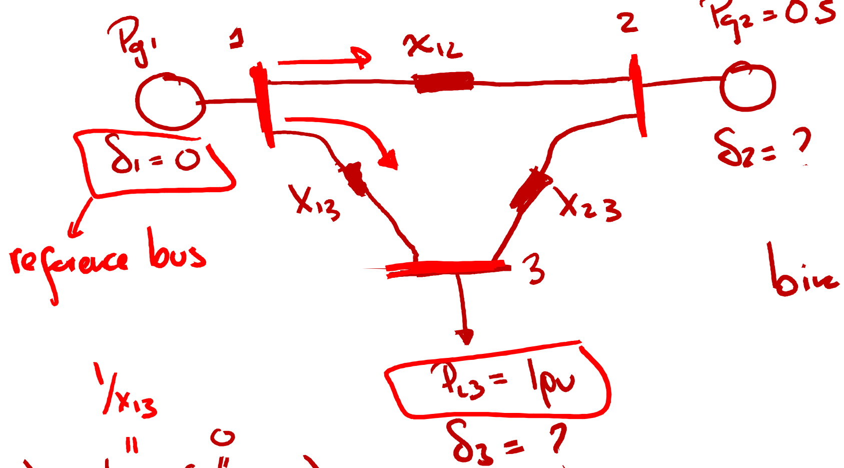
DC Power Flow Equations



- Finally, the DC power flow equations are as follows:

$$\underline{P_{gi}-P_{ci}} = \overset{\text{Net}}{``P_i"} \approx \sum_{k=1}^{N} \mathrm{Im}\{\mathbf{Y_{ik}}\}\,(\delta_i-\delta_k) \triangleq \sum_{k=1}^{N} \overset{1/x_{ik}}{\mathbf{b_{ik}}}\,(\delta_i-\delta_k) \qquad \forall \mathbf{i} = \mathbf{1},\ \mathbf{2},\ \ldots,\ \mathbf{N}$$

- Let's look at an example for $\underline{N = 3}$



$$b_{ik} = \frac{1}{x_{ik}}$$

$i = 1$

$$P_1 = P_{g1} - \overset{0}{\cancel{P_{c1}}} = \sum_{k=1}^{3} b_{ik}(\delta_1 - \delta_k)$$

$$\underline{1:}\quad \underline{P_{g1}} = \underbrace{\cancel{b_{11}(\overset{0}{\delta_1}-\delta_1)}}_{} + \underbrace{\overset{1/x_{12}}{b_{12}}(\overset{0}{\delta_1}-\delta_2)}_{P_{12}} + \underbrace{\overset{1/x_{13}}{b_{13}}(\overset{0}{\delta_1}-\delta_3)}_{P_{13}}$$

$$2:\quad P_{g2} = 0.5 = b_{21}(\delta_2 - \underset{0}{\delta_1}) + \cancel{b_{22}(\overset{0}{\delta_2-\delta_2})} + b_{23}(\delta_2-\delta_3)$$

$$3:\quad \underset{-P_{c3}}{P_3} = -1 = b_{31}(\delta_3-\delta_1) + b_{32}(\delta_3-\delta_2) + \underset{0}{\cancel{b_{33}(\delta_3-\delta_3)}}$$

3 eqns 3 unknowns
P_{g1}
δ_2
δ_3

$\Rightarrow$ linear! $\Rightarrow$ Matrix Form $\Rightarrow$ $\underline{P} = B\underline{\delta}$ $\quad P = \begin{pmatrix} P_1 \\ P_2 \\ P_3 \end{pmatrix} \quad \delta = \begin{pmatrix} \delta_1 \\ \delta_2 \\ \delta_3 \end{pmatrix}$

DC Power Flow Equations – <u>Final Form</u>

- Or, written in different form:

$$\boxed{\mathbf{P_i} \approx \sum_{\mathbf{k=1}}^{\mathbf{N}} \mathbf{B_{ik}} \delta_\mathbf{k} \qquad \forall \mathbf{i} = \mathbf{1},\ \mathbf{2},\ \ldots,\ \mathbf{N}}$$

- Where $\tilde{b}_{ik}$ can be considered an admittance matrix corresponding to the imaginary part

$$\mathbf{P} = \mathbf{B}\delta$$

Previous slide $N=3$

$$\underbrace{P = \begin{pmatrix} P_1 \\ P_2 \\ P_3 \end{pmatrix}}_{\text{known}} = \underbrace{\begin{pmatrix} b_{12}+b_{13} & -b_{12} & -b_{13} \\ -b_{12} & b_{12}+b_{23} & -b_{23} \\ -b_{13} & -b_{23} & b_{13}+b_{23} \end{pmatrix}}_{B} \begin{pmatrix} \delta_1 \\ \delta_2 \\ \delta_3 \end{pmatrix}$$

unknowns

$P = B\delta$

$\Rightarrow \delta = B^{-1} P$

can be invertible?

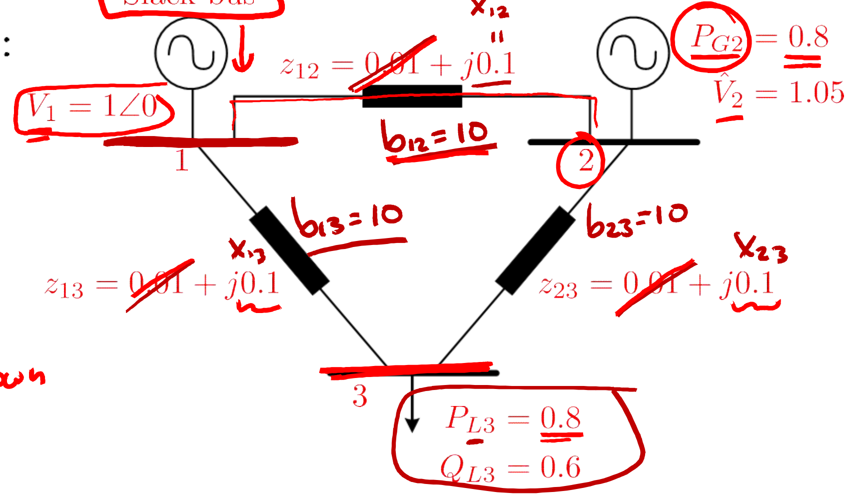
<u>Susceptance Matrix</u> (Real Matrix)

(very similar to <u>adm.</u> matrix)

Y (complex matrix)

DC Power Flow Example

References   Some power system $b_{ij} \hat{=} \frac{1}{x_{ij}}$



- Find the dc power flow for the following system:
- Compute the B matrix: $(P = B\delta)$

Unknown $(b_{12}+b_{13})$ $\delta_1 = 0$

$$\begin{pmatrix} P_1 \\ 0.8 \\ -0.8 \end{pmatrix} = \begin{pmatrix} P_1 \\ P_2 \\ P_3 \end{pmatrix} = \underbrace{\begin{pmatrix} 20 & -10 & -10 \\ -10 & 20 & -10 \\ -10 & -10 & 20 \end{pmatrix}}_{B} \begin{pmatrix} \delta_1 \\ \delta_2 \\ \delta_3 \end{pmatrix}$$

$\delta_2, \delta_3 \rightarrow$ Unknown

$B \longrightarrow Bx = 0 \quad x \neq 0$

$x = \begin{pmatrix} \vdots \end{pmatrix} \quad \lambda_1 = 0$

$X_{12} = X_{23} = X_{13} = 0.1$

$b_{12} = b_{13} = b_{23} = \frac{1}{x_{12}} = 10$

$P = B\delta \Rightarrow \delta = B^{-1}P$

- Matrix B is not invertible! $\Rightarrow B^{-1}$ doesn't exists

$\Rightarrow$ we cannot $\boxed{\cancel{\delta = B^{-1}P}}$

$\Rightarrow \det(B) = 0 = \lambda_1\lambda_2\lambda_3 \Rightarrow$ at least one eigenvalue is equal to 0

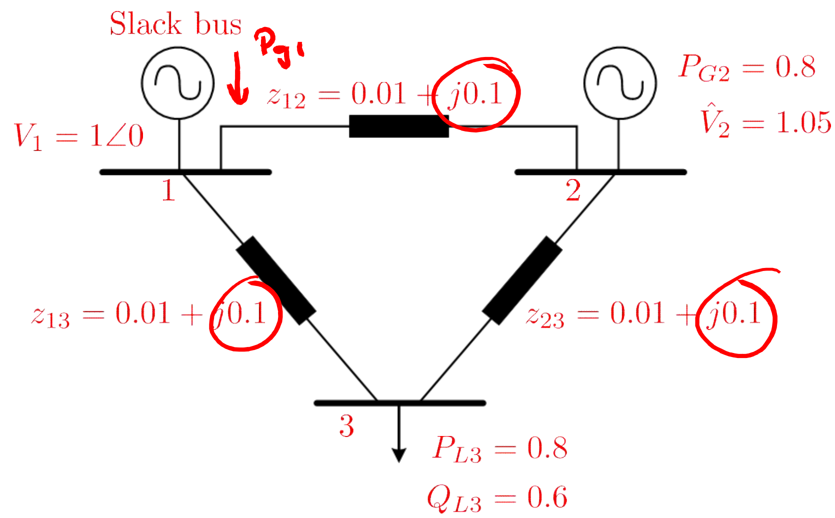
$\Rightarrow \lambda_1 = 0 \Rightarrow Bx_1 = \lambda x_1 \Rightarrow Bx_1 = 0$ for some $x_1 \neq 0$

Bus	Known	Unknown
1	$\delta_1 = 0$	P_1
2	$P_2 = 0.8$	δ_2
3	$P_3 = -0.8$	δ_3

P_i, δ_i

DC Power Flow Example

- Find the dc power flow for the following system:
- Place in Matrix form:



$$\begin{pmatrix} P_{g1} \\ 0.8 \\ -0.8 \end{pmatrix} = \begin{pmatrix} 20 & -10 & -10 \\ -10 & 20 & -10 \\ -10 & -10 & 20 \end{pmatrix} \begin{pmatrix} 0 \\ \delta_2 \\ \delta_3 \end{pmatrix}$$

3 eqns 3 unknowns $P_{g1}, \delta_2, \delta_3$

write bottom two equations

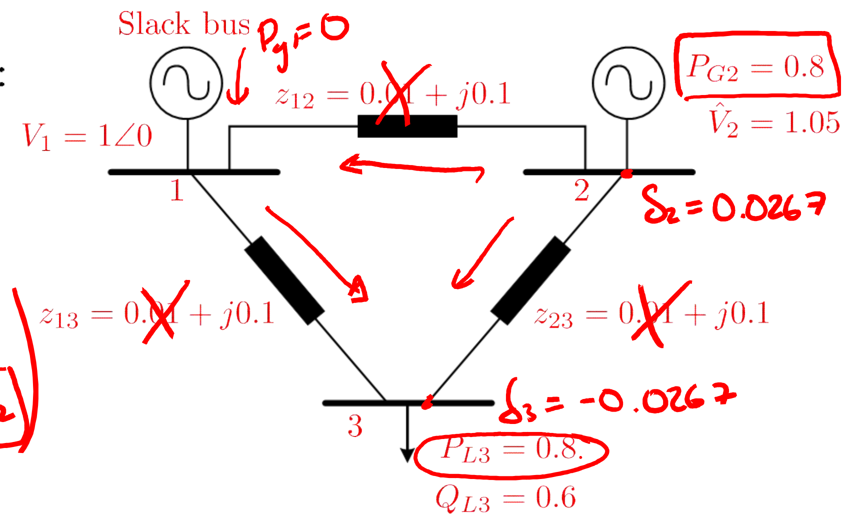
$$\begin{pmatrix} 0.8 \\ -0.8 \end{pmatrix} = \begin{pmatrix} 20 & -10 \\ -10 & 20 \end{pmatrix} \begin{pmatrix} \delta_2 \\ \delta_3 \end{pmatrix} \rightarrow \text{Now we can solve for } \delta_2, \delta_3$$

Solve for P_{g1}

$$P_{g1} = -10\delta_2 - 10\delta_3$$

DC Power Flow Example

- Find the dc power flow for the following system:



- Place in Matrix form:

Submatrix of B

$$B = \begin{pmatrix} B_{11} & B_{12} \\ B_{21} & \boxed{B_{22}} \end{pmatrix}$$

$$\begin{bmatrix} 0.8 \\ -0.8 \end{bmatrix} = \begin{bmatrix} 20 & -10 \\ -10 & 20 \end{bmatrix} \begin{bmatrix} \delta_2 \\ \delta_3 \end{bmatrix}$$

- The solution is:

$$\begin{bmatrix} \delta_2 \\ \delta_3 \end{bmatrix} = \left(\begin{bmatrix} 20 & -10 \\ -10 & 20 \end{bmatrix} \right)^{-1} \begin{bmatrix} 0.8 \\ -0.8 \end{bmatrix} = \begin{bmatrix} 0.0267 \\ -0.0267 \end{bmatrix} \text{rad} = \begin{bmatrix} 1.53^o \\ -1.53^o \end{bmatrix} \qquad \begin{bmatrix} \delta_2 \\ \delta_3 \end{bmatrix} = \underbrace{\begin{bmatrix} 1.53^o \\ -1.53^o \end{bmatrix}}_{\text{degrees}}$$

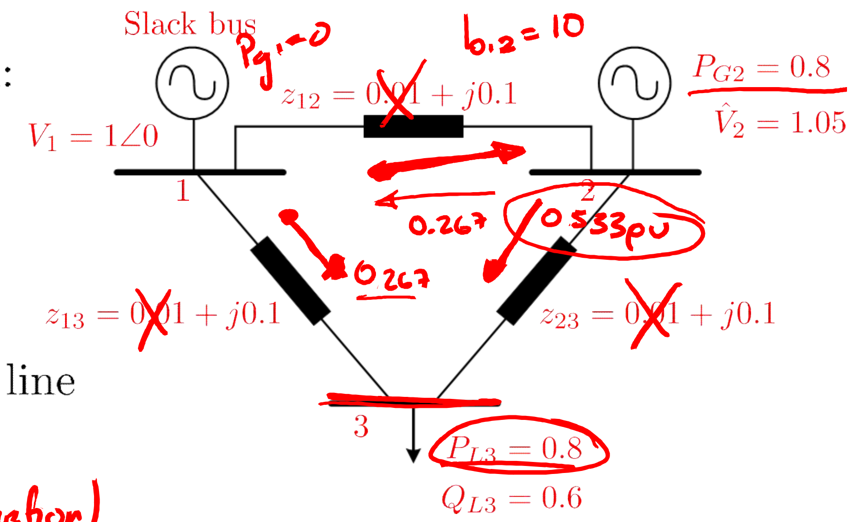
- Compute P_{g1}:

$$P_{g1} = -10\,\delta_2 - 10\,\delta_3 = -10(0.0267) - 10(-0.0267) = 0$$

$$\boxed{P_{g1} = 0}$$

DC Power Flow Example

- Find the dc power flow for the following system:
- The solution is: $\begin{bmatrix} \delta_1 \\ \delta_2 \\ \delta_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 1.53^o \\ -1.53^o \end{bmatrix} = \begin{bmatrix} 0 \\ 0.0267 \\ -0.0267 \end{bmatrix}$
- Can we calculate the power 'flowing' through a line



$P_{ij} = \mathrm{Re}\{V_i I_{ij}^*\} = \mathrm{Re}\{V_i(V_i^* - V_j^*)y_{ij}^*\}$ (Nonlinear equation)

Using DC power flow assumptions (3), we can obtain:

$$\boxed{\mathbf{P_{ij}} = \mathbf{b_{ij}}\,(\delta_\mathbf{i} - \delta_\mathbf{j})\,, \quad \text{for } \mathbf{i} \neq \mathbf{j}, \quad \mathbf{b_{ij}} = \frac{\mathbf{1}}{\mathbf{x_{ij}}}}$$

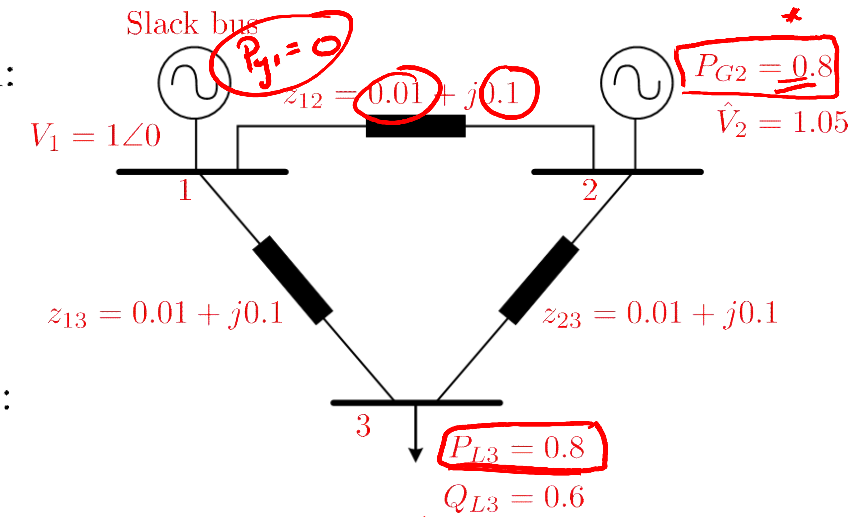
$P_{12} = b_{12}(\overset{0}{\delta_1} - \overset{0.0267}{\delta_2}) = 10\,(0 - 0.0267) = -.267 \text{ pu} \Rightarrow P_{21} = 0.267 \text{ pu}$

$P_{13} = 10\,(\delta_1 - \delta_3) = 0.267$

$P_{23} = 10\,(\delta_2 - \delta_3) = 10\,(0.0267 - (-0.0267)) = 0.533 \text{ pu}$

DC Power Flow Example - Comparison

- Find the dc power flow for the following system:
- The solution is: $\begin{bmatrix}\delta_1\\ \delta_2\\ \delta_3\end{bmatrix} = \begin{bmatrix}0\\ 1.53^o\\ -1.53^o\end{bmatrix} = \begin{bmatrix}0\\ 0.0267\\ -0.0267\end{bmatrix}$



$$P_i = P_{gi} - P_{Li} = \underbrace{\sum_{j=1}^{N} b_{ij}(\delta_i - \delta_j)}_{\text{power flowing through line } ij}$$

- Comparison of solution with regular power flow:

	Approx. DC Power Flow	Actual Regular Power Flow
P_1	0	0.00944*
P_2	0.8	0.8
P_3	-0.8	-0.8
δ_1	0	0
δ_2	0.0267	.02
δ_3	-0.0267	-0.026

Good approximations

DC Power Flow Summary

- For the simplified/DC power flow computation, we make the following assumptions:

  Assumption 1: Voltage magnitudes are close to 1 in per unit

  Assumption 2: Resistances are much smaller than reactances

  Assumption 3: Phase differences across lines are small

- Based on this, we can calculate the active power flows as follows:

$$\mathbf{P_i} \approx \sum_{\mathbf{k=1}}^{\mathbf{N}} \mathbf{B_{ik}}\delta_{\mathbf{k}} \quad \forall \mathbf{i} = \mathbf{1},\ \mathbf{2},\ \ldots,\ \mathbf{N} \quad \text{or} \quad \mathbf{P} = \mathbf{B}\delta$$

Susceptance

- The power flow through lines is given by:

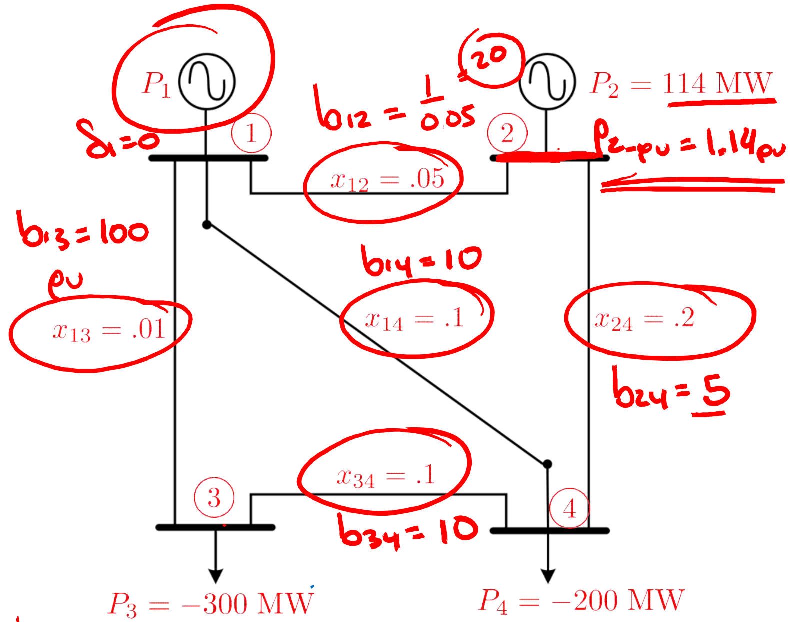
$$\mathbf{P_{ij}} = \mathbf{b_{ij}}\left(\delta_{\mathbf{i}} - \delta_{\mathbf{j}}\right), \quad \text{for } \mathbf{i} \neq \mathbf{j}, \quad \mathbf{b_{ij}} = \frac{\mathbf{1}}{\mathbf{x_{ij}}}$$

- **Procedure:**

  1. Calculate $b_{ij} = \dfrac{1}{x_{ij}}$ for every line
  2. Compute the susceptance matrix B and place equations $P = B\delta$
  3. Set $\delta_1 = 0$ (reference bus) and eliminate from equations (eliminate first column of B)
  4. Solve for the angles $\rightarrow P_i$

DC Power Flow Example 2

- Find the dc power flow for the following system:
- Assume $S_{\text{base}} = 100$ MVA



Write the DC Power Flow Equations

$$P = B\,\delta$$

$$\underbrace{\begin{pmatrix} P_1 \\ P_2 \\ P_3 \\ P_4 \end{pmatrix} = \begin{pmatrix} P_{g1} \\ 1.14 \\ -3 \\ -2 \end{pmatrix}}_{P} = \underbrace{\begin{array}{cccc} b_{12}+b_{14}+b_{13} & & & \\ \begin{pmatrix} 130 & -20 & -100 & -10 \\ -20 & 25 & 0 & -5 \\ -100 & 0 & 110 & -10 \\ -10 & -5 & -10 & 25 \end{pmatrix} \end{array}}_{B} \begin{pmatrix} \cancel{\delta_1 = 0} \\ \delta_2 \\ \delta_3 \\ \delta_4 \end{pmatrix}$$

$$\begin{array}{c} 2 \\ 3 \\ 4 \end{array} \begin{pmatrix} 1.14 \\ -3 \\ -2 \end{pmatrix} = \begin{pmatrix} 25 & 0 & -5 \\ 0 & 110 & -10 \\ -5 & -10 & 25 \end{pmatrix} \begin{pmatrix} \delta_2 \\ \delta_3 \\ \delta_4 \end{pmatrix}$$

$$\begin{pmatrix} \delta_2 \\ \delta_1 \\ \delta_4 \end{pmatrix} = \begin{pmatrix} .0279 \\ -.0353 \\ -.0886 \end{pmatrix}$$

$$P_{g1} = -20\delta_2 - 100\delta_3 - 10\delta_4$$

$$P_{g1} = 386\text{pu}$$

Topics that Will be Covered

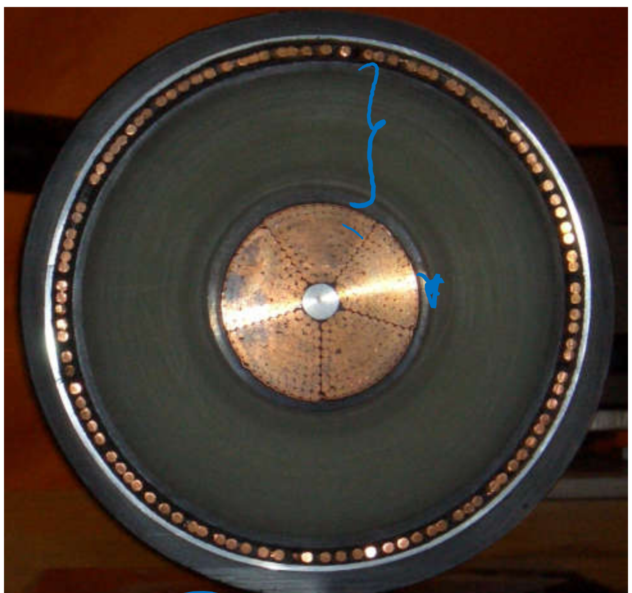
- *Nodal Analysis*
- *Power Flow Equations*
- *DC Power Flow - simplification* (HW4, Problem 2)
- ***DC Optimal Power Flow***
- ***LP Formulation***
- * *Locational Marginal Price* (LMP)

DC Power Flow – Line Power Flows

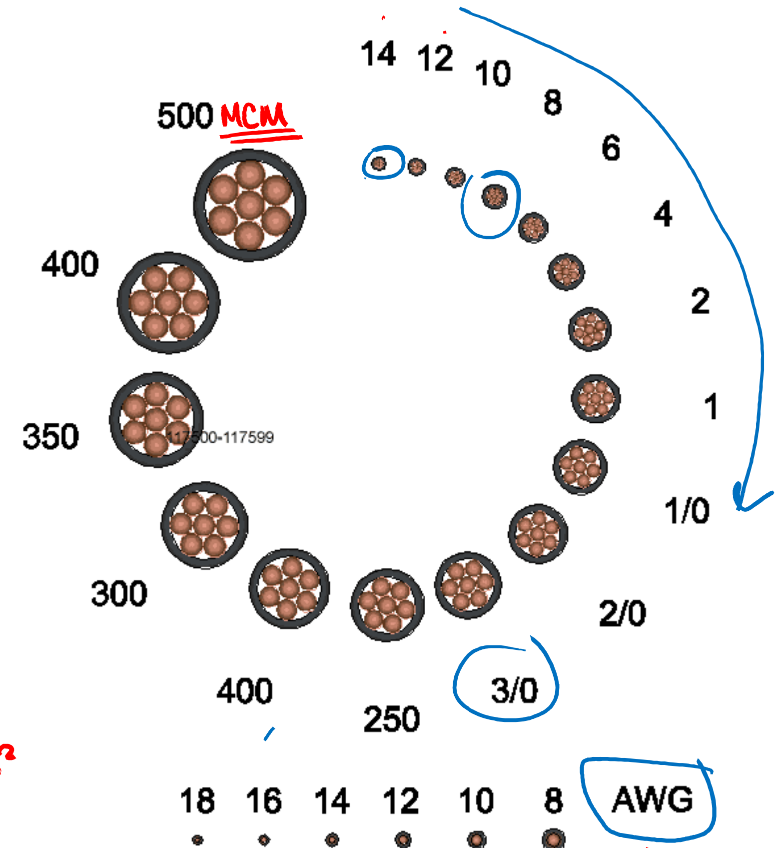
- Transmission lines are physically constrained in the power that they can deliver – proportional to its size and related to material

- Therefore, when dispatching generation sources, it is important to make sure the transmission lines **are not overloaded!**

1 MCM ≈ 0.51 mm^2



400 kV cable cross-section ∼ 500 - 1000 mm^2



DC Power Flow – Line Power Flows

- The general equation to compute the flowing in the lines:

$$P_{ij} = \mathrm{Re}\{V_i I_{ij}^*\} = \mathrm{Re}\{V_i(V_i^* - V_j^*)y_{ij}^*\}$$ (Nonlinear equation)

- The dc power flow simplified equation:

$$\mathbf{P_{ij} = b_{ij}\,(\delta_i - \delta_j)\,,\quad \text{for } i \neq j,\quad b_{ij} = \frac{1}{x_{ij}}}$$

(linear)

$\delta_i \rightarrow$ voltage angle at bus i

- Power flows for the previous example:

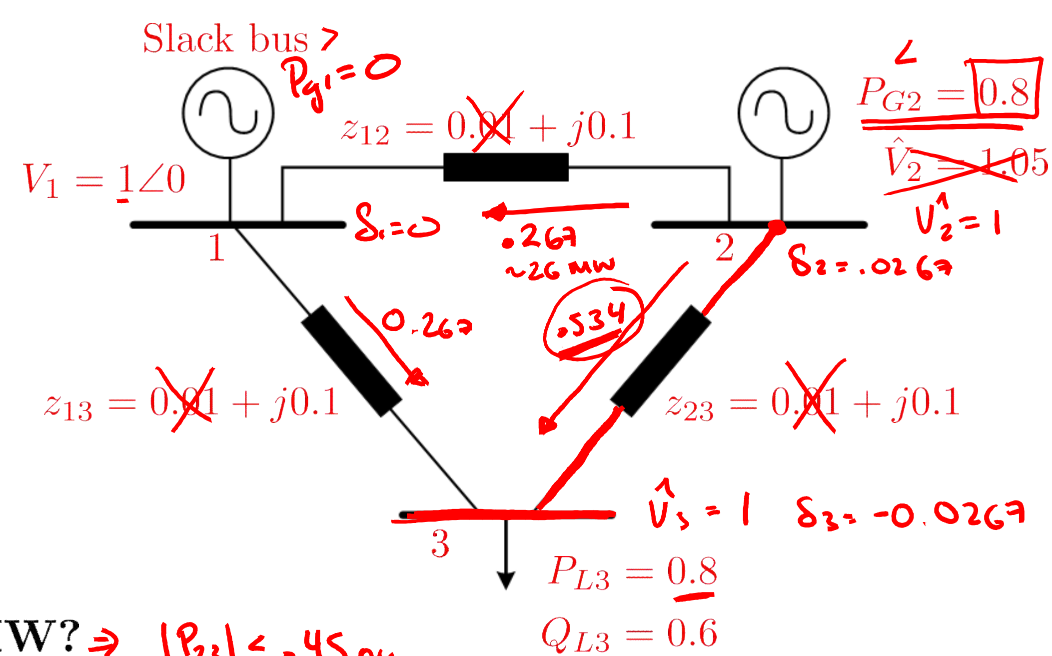
$S_{\text{base}} = 100$ MVA

$$P_{12} = b_{12}\,(\delta_1 - \delta_2) = -0.267$$

$$P_{23} = b_{23}\,(\delta_2 - \delta_3) = 0.534$$

$$P_{13} = b_{13}\,(\delta_1 - \delta_3) = 0.267$$

$$\begin{bmatrix}\delta_1\\ \delta_2\\ \delta_3\end{bmatrix} = \begin{bmatrix}0\\ 1.53^o\\ -1.53^o\end{bmatrix} = \begin{bmatrix}0\\ 0.0267\\ -0.0267\end{bmatrix}$$



- **What if $|P_{23}|$ is limited to 45 MW?** $\Rightarrow$ $|P_{23\,pu}| \leq .45$ pu

Economic Dispatch Revisited

(Before)

Economic Dispatch:

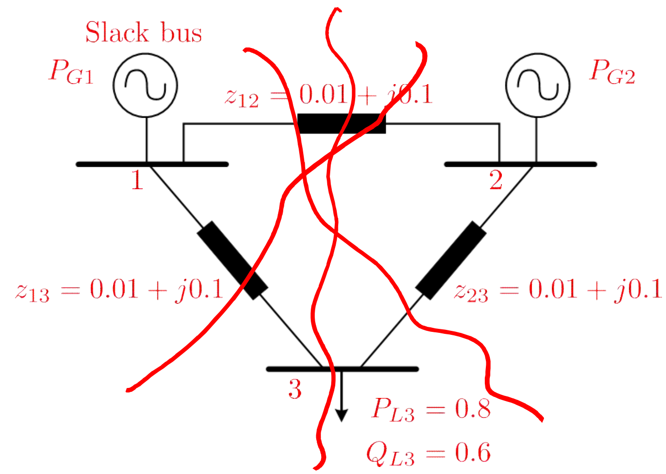
- For a single hour, what is the "best" active power output of generators to meet the load while minimizing the total cost
- In this problem, the network **is ignored**

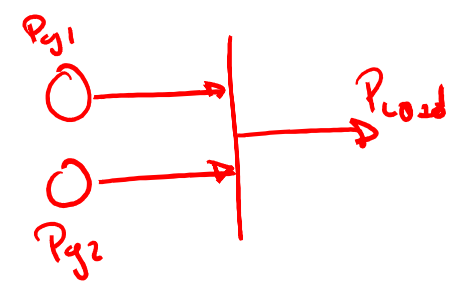
$$\min_{P_{g1},\, P_{g2},\, \ldots,\, P_{gN_g}} f_1(P_{g1}) + \cdots + f_{Ng}(P_{gN_g})$$

s.t.

$$P_{gi-\min} \leq P_{gi} \leq P_{gi-\max} \quad \forall i$$

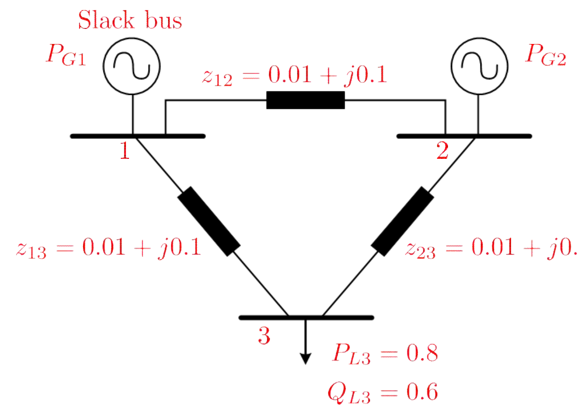
$$* \quad \underbrace{\sum_{i=1}^{Ng} P_{gi}}_{\text{Total generation}} = P_{\text{loads}}$$





$\underbrace{\text{Economic Dispatch}} + \underbrace{\text{Network Constraints}}_{\text{DC power flow}} = \underline{\text{DC OPF}}$

- If we take into account the simplified DC power flow equations, we can impose limits on the transmission lines!



- Economic Dispatch + DC Power flow =
  "**DC Optimal Power Flow**"

$$\min_{\underbrace{P_{g1},\, P_{g2},\, \ldots,\, P_{gN_g}}_{\text{unknown}},\ \&\ \underbrace{\delta_2,\, \ldots,\, \delta_N}_{\text{unknown}}} \underbrace{f_1(P_{g1}) + \cdots + f_{Ng}(P_{gN_g})}_{\text{same}}$$

s.t.

- $P_{gi-\min} \leq P_{gi} \leq P_{gi-\max}$ $\forall i$

e.g. line 23
$|P_{23}| \leq .45$

DC power flow equations

$$\boxed{P_i = P_{gi} - P_{li} = \sum_{k=1}^{N} B_{ik}\delta_{k} \qquad \forall i = 1,\ \ldots,\ N}$$

$$\underbrace{-.45 \leq b_{23}(\delta_2 - \delta_3) \leq .45}$$

$$\underline{P}_{ij} \leq \underbrace{b_{ij}(\delta_i - \delta_j)}_{P_{ij}} \leq \bar{P}_{ij}$$ line constraints

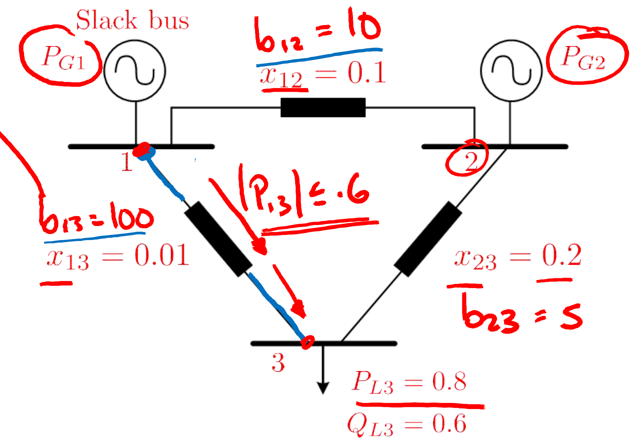
$$\underbrace{\delta_1 = 0}_{\text{reference bus}}$$

Example DC Optimal Power Flow

- For the network as shown, assume the generator parameters are as follows

Generator	$P_{\min}$	$P_{\max}$	Cost
1	.2	.5	5
2	.1	.8	7

$$B = \begin{pmatrix} 110 & -10 & -100 \\ -10 & 15 & -5 \\ -100 & -5 & 105 \end{pmatrix}$$



- *Assume line 13 is limited to **0.6 pu** power flow *
- Compute the **DC OPF**

* $\min_{P_{g1}, P_{g2}, \delta_1, \delta_2, \delta_3} \left(5P_{g1} + 7P_{g2}\right)$

s.t. ~~$P_{g1} + P_{g2} = 0.8$~~

$0.2 \le P_{g1} \le 0.5$
$0.1 \le P_{g2} \le 0.8$ } Generator constraints

$$\left\{ \begin{pmatrix} P_{g1} \\ P_{g2} \\ -0.8 \end{pmatrix} = \begin{pmatrix} 110 & -10 & -100 \\ -10 & 15 & -5 \\ -100 & -5 & 105 \end{pmatrix} \begin{pmatrix} \delta_1 \\ \delta_2 \\ \delta_3 \end{pmatrix} \right\} \text{DC PF}$$

* $-0.6 \le 100(\delta_1 - \delta_3) \le 0.6$ } line constraint

$\delta_1 = 0$ (Reference bus)

$|P_{13}| \le 0.6$

$-0.6 \le b_{13}(\delta_1 - \delta_3) \le 0.6$

LP or QP ?

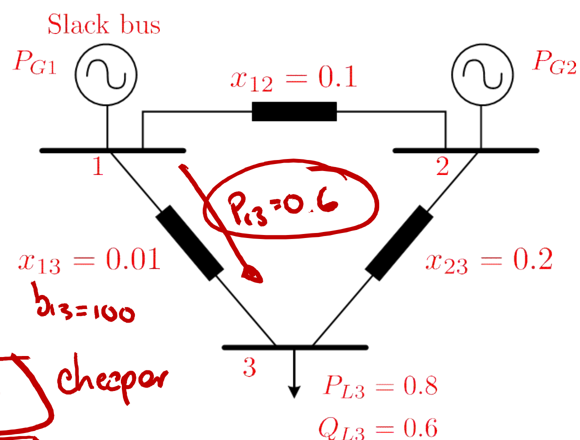
$\min c^T x$
$Ax \le b$
$A_{eq} x = b_{eq}$
$x = [P_{g1}\ P_{g2}\ \delta_1\ \delta_2\ \delta_3]^T$

Example DC Optimal Power Flow – Matlab Code

- For the network as shown, assume the generator parameters are as follows

Generator	$P_{\min}$	$P_{\max}$	Cost
1	.2	.5	5
2	.1	.8	7

- Assume line 13 is limited to **0.6 pu** power flow
- Compute the **DC OPF**



$\boxed{P_{g1} = 0.26}$ cheaper

$\boxed{P_{g2} = 0.54}$

$\delta_1 = 0$

$\delta_2 = .34$

$\delta_3 = -.06$

$P_{13} = 100(\delta_1 - \delta_3)$

$P_{13} = 0.6$ (at limit)

```
%% Write the susceptance matrix
b12 = 1/0.1;
b13 = 1/0.01;
b23 = 1/0.2;

B = [b12+b13 -b12     -b13;
      -b12    b12+b23  -b23;
      -b13     -b23    b13+b23];
```

```
%% Use CVX to obtain a solution
cvx_begin
variables P1 P2
variables d1 d2 d3
```

dual variable μ,

```
minimize  5*P1 + 7*P2
subject to
    b13*(d1-d3) <= 0.6;
    b13*(d3-d1) <= 0.6;
    0.2 <= P1 <= 0.5;
    0.1 <= P2 <= 0.8;
    [P1; P2; -0.8] == B*[d1; d2; d3];
    d1 == 0;
cvx_end
```

* binding

μ :

$P = B\delta$

Compare to prev slide

Example 2 DC Optimal Power Flow

$P_{\text{base}} = 100\text{ MW}$ *

- Consider a three bus system with local loads
- The generator costs are as follows

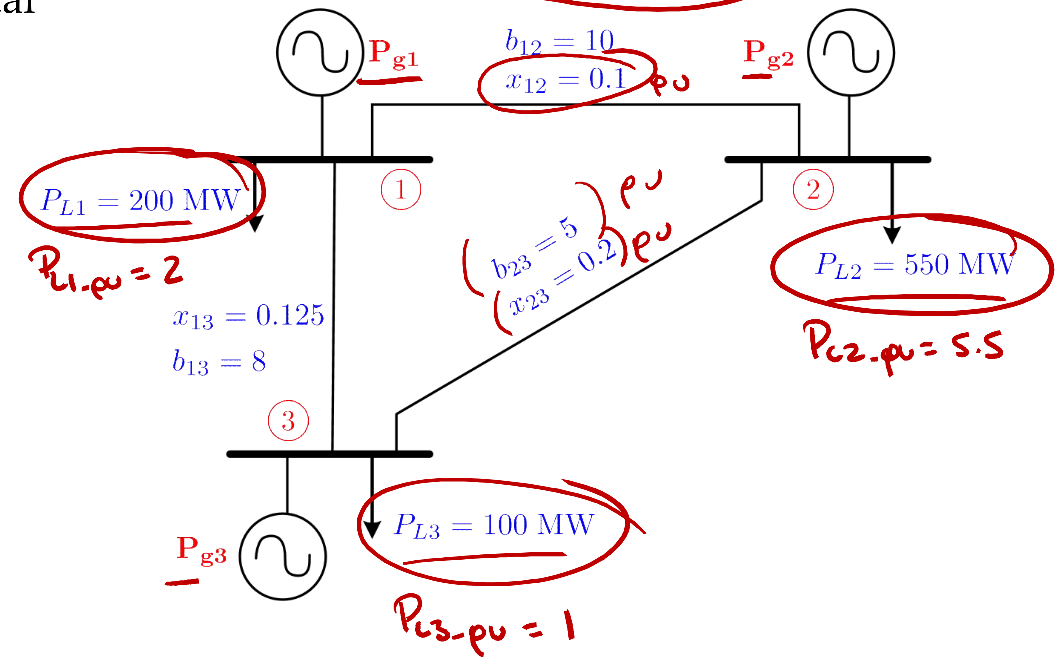
$$* \; F_1 = 792P_{g1} + 15.62P_{g1}^2 \quad (P_{g1} \text{ in pu})$$

$$* \; F_2 = 785P_{g2} + 19.4P_{g2}^2$$

$$* \; F_3 = 797P_{g3} + 48.2P_{g3}^2$$

- With limits as (pu):

Generator	$P_{\min}$	$P_{\max}$
1	1.50	6.0
2	1.00	4.0
3	0.5	2.0

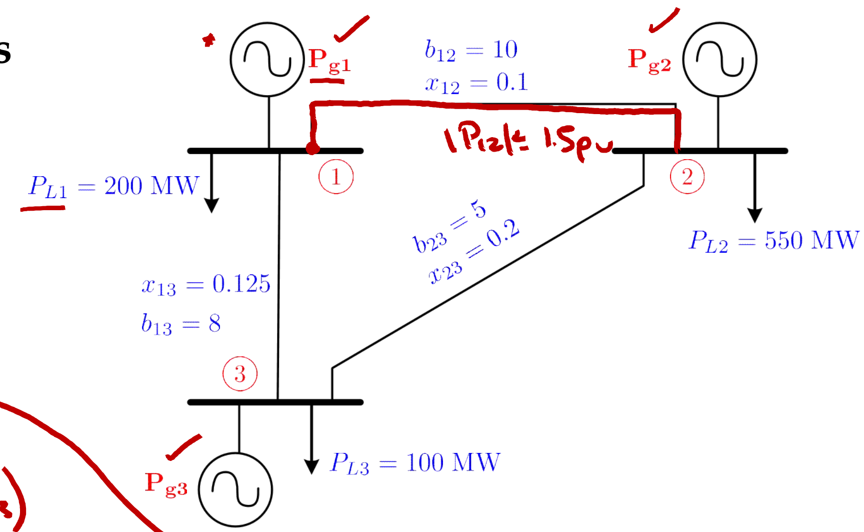


$$P_{\text{pu}} = \frac{P_{\text{actual}}}{P_{\text{base}}}$$

Example 2 DC Optimal Power Flow

- **Find the optimal operation of the generators to minimize the cost of operation**
- Take into account dc power flow
- * Assume line 1-2 is limited to 150 MW *

$|P_{12}| \le 1.5\,pu$



DC OPF

$$\min_{P_{g1},P_{g2},P_{g3},\delta_1,\delta_2,\delta_3} F_1(P_{g1}) + F_2(P_{g2}) + F_3(P_{g3})$$

s.t.

$$\boxed{\begin{aligned} 1.5 &\le P_{g1} \le 6 \\ 1 &\le P_{g2} \le 4 \\ 0.5 &\le P_{g3} \le 2 \end{aligned}}$$

$$\begin{array}{c} 1 \\ 2 \\ 3 \end{array} \begin{pmatrix} P_{g1} - 2 \\ P_{g2} - 5.5 \\ P_{g3} - 1 \end{pmatrix} = B \begin{pmatrix} \delta_1 \\ \delta_2 \\ \delta_3 \end{pmatrix}$$

$$P = B\delta$$

$$-1.5 \le b_{12}(\delta_1 - \delta_2) \le 1.5$$

$$\delta_1 = 0$$

$$F_1 = 792P_{g1} + 15.62P_{g1}^2$$
$$F_2 = 785P_{g2} + 19.4P_{g2}^2$$
$$F_3 = 797P_{g3} + 48.2P_{g3}^2$$

Generator	$P_{\min}$	$P_{\max}$
1	1.50	6.0
2	1.00	4.0
3	0.5	2.0

$$B = \begin{pmatrix} 18 & -10 & -8 \\ -10 & 15 & -5 \\ -8 & -5 & 13 \end{pmatrix}$$

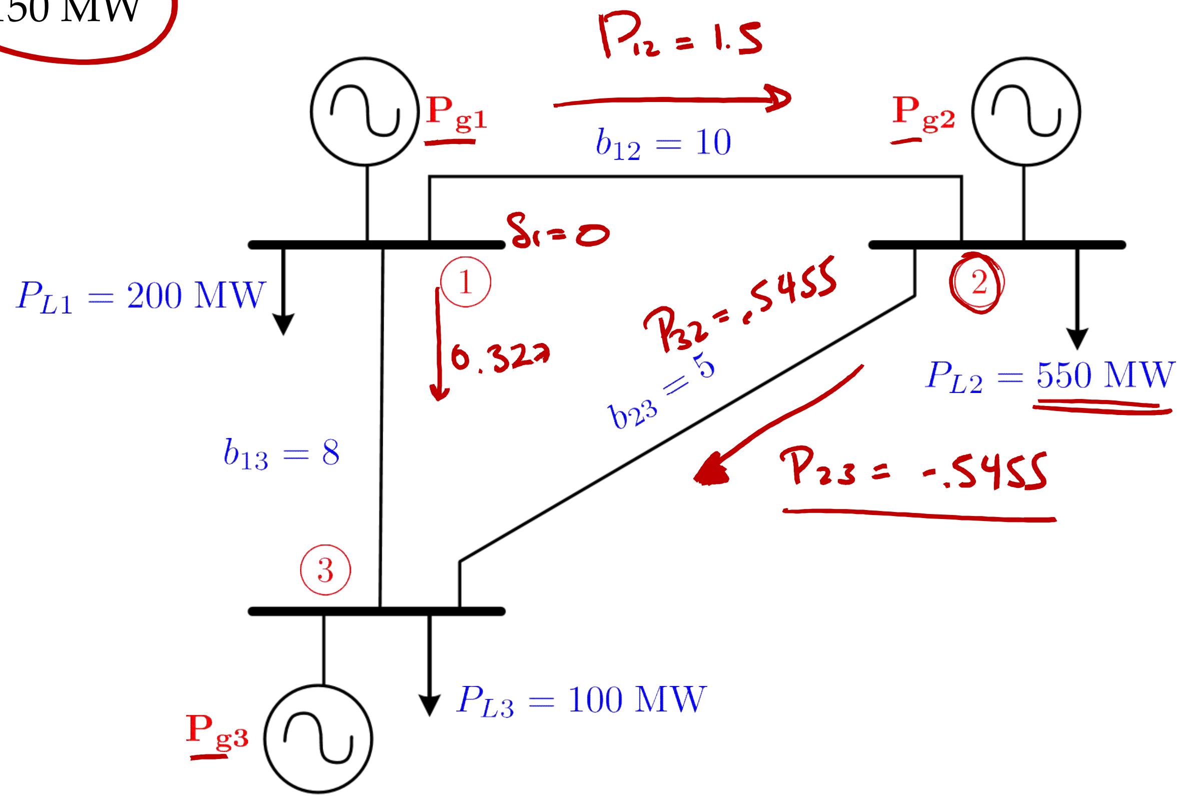
- **What kind of optimization problem is this? (LP or QP)**

$$\min \tfrac{1}{2}x^T H x + c^T x$$
s.t. $Ax \le b,\ A_{eq}x = b_{eq}$

Example 2 DC Optimal Power Flow

- **Find the optimal operation of the generators to minimize the cost of operation**
- Take into account dc power flow
- Assume line 1-2 is limited to 150 MW

$$x^* = \begin{pmatrix} P_{g1} \\ P_{g2} \\ P_{g3} \\ \delta_1 \\ \delta_2 \\ \delta_3 \end{pmatrix} = \begin{pmatrix} 3.8272 \\ 3.4545 \\ 1.2182 \\ 0 \\ -0.15 \\ -0.0409 \end{pmatrix}$$



$$P_{12} = b_{12}(\delta_1 - \delta_2) = 1.5$$
$$P_{13} = b_{13}(\delta_1 - \delta_3) = 0.3272$$
$$P_{23} = b_{23}(\delta_2 - \delta_3) = -.5455$$

Economic Dispatch with Network Constraints = DC OPF

- If we take into account the simplified DC power flow equations, we can impose limits on the transmission lines!

- Economic Dispatch + DC Power flow = **DC Optimal Power Flow**

$$\min_{P_{g1},\, P_{g2},\, \ldots,\, P_{gN_g},\, \delta_1,\, \delta_2,\, \ldots,\, \delta_N} \underbrace{f_1(P_{g1}) + \cdots + f_{Ng}(P_{gN_g})}_{\text{Total cost of generation}}$$

s.t.

$$P_{gi-\min} \le P_{gi} \le P_{gi-\max}$$

$$P_i = P_{gi} - P_{li} = \sum_{k=1}^{N} B_{ik}\delta_{ik} \quad \forall i = 1,\, \ldots,\, N \quad * \text{ DC PF}$$

$$\underline{P}_{ij} \le b_{ij}(\delta_i - \delta_j) \le \bar{P}_{ij} \quad \text{line constraints}$$

$$\delta_1 = 0 \quad \text{ref. bus}$$

Topics that Will be Covered

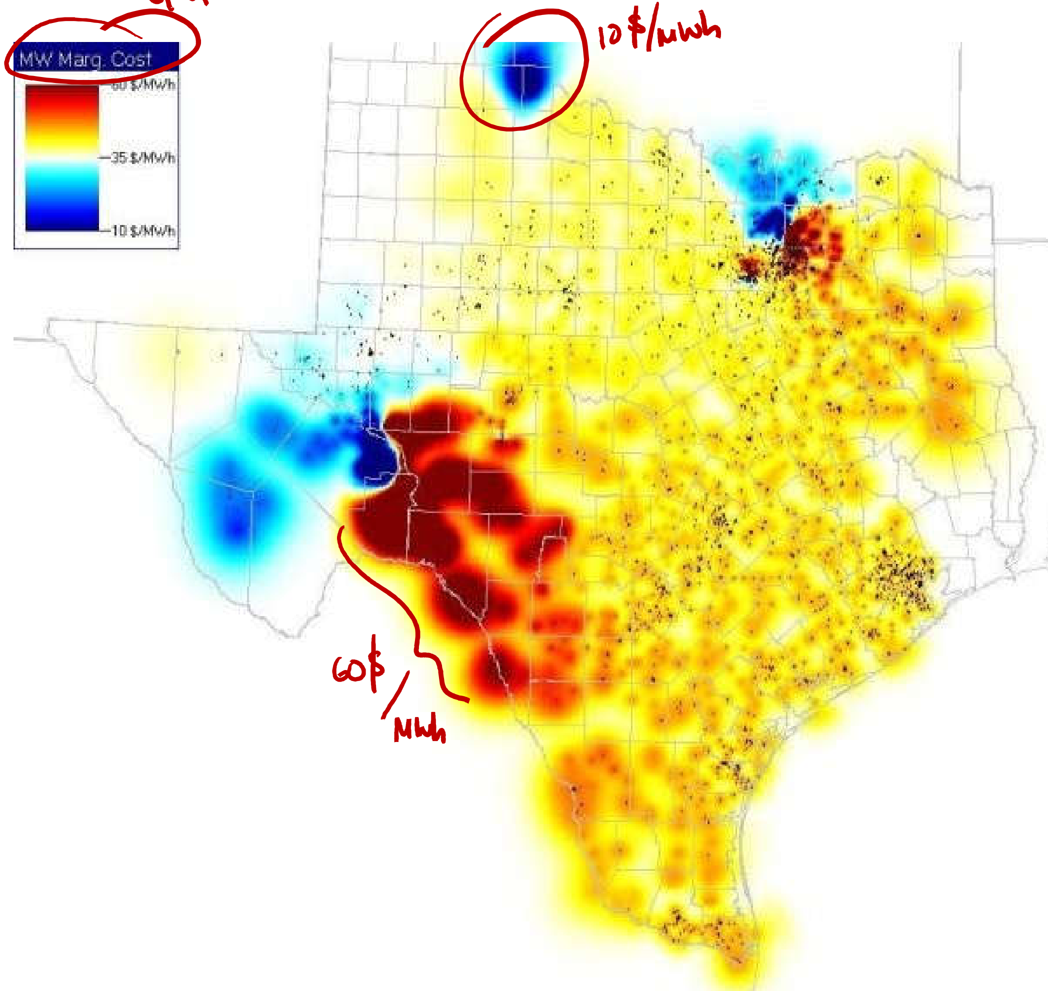
- *Nodal Analysis*
- *Power Flow Equations*
- *DC Power Flow - simplification*
- *Economic Dispatch with Network Constraints – DC OPF*
- *LP Formulation*
- ***Locational Marginal Price***

Concept of Locational Marginal Price

- **Locational Marginal Pricing:**
  A method to reflect the value of electric energy at different locations (buses) accounting for:
  - Generation (Cost, Constraints)
  - Losses
  - Transmission line limits

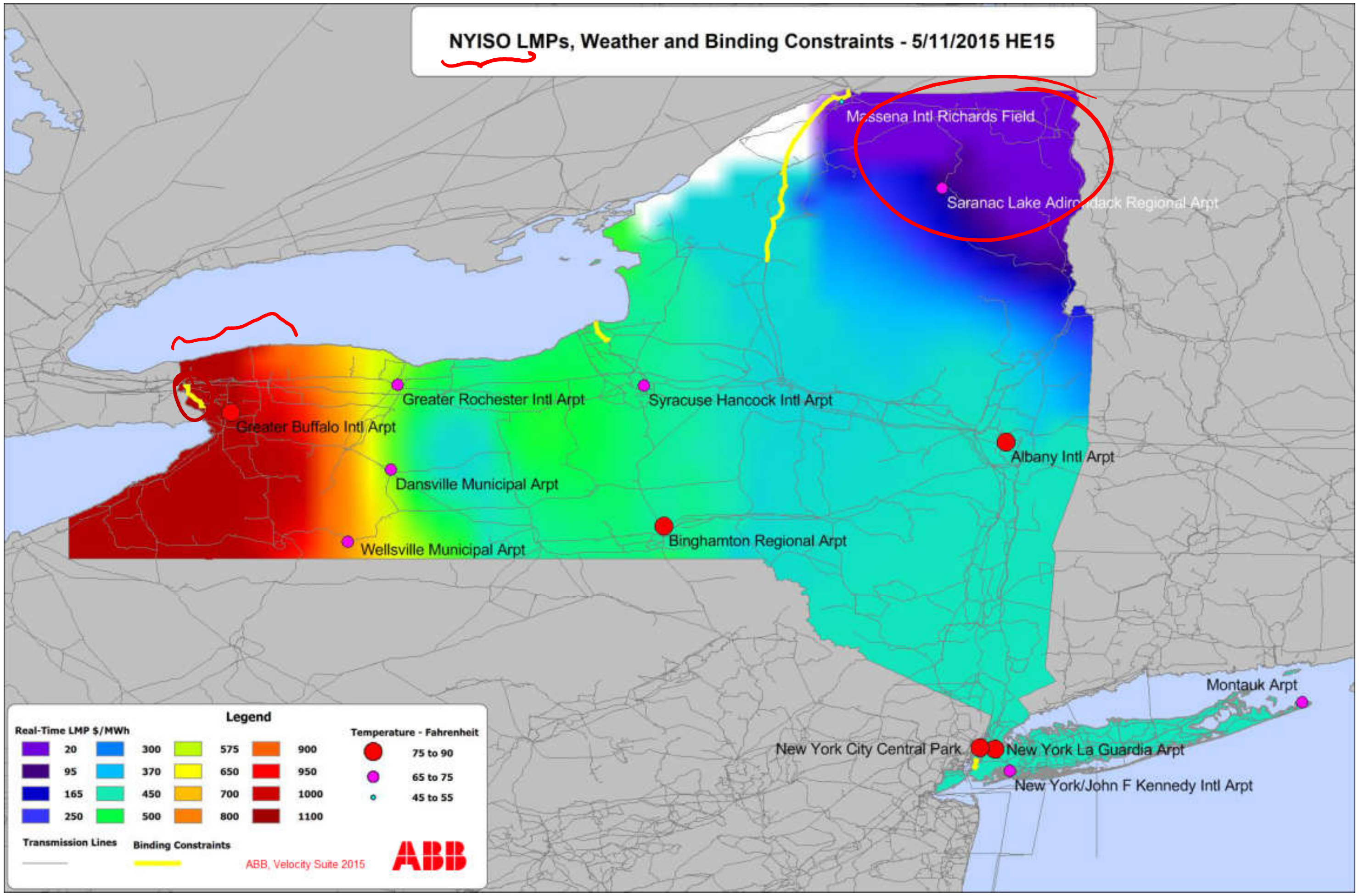
    $|P_{ij}| \le P_{ij,max}$

LMP



Concept of Locational Marginal Price

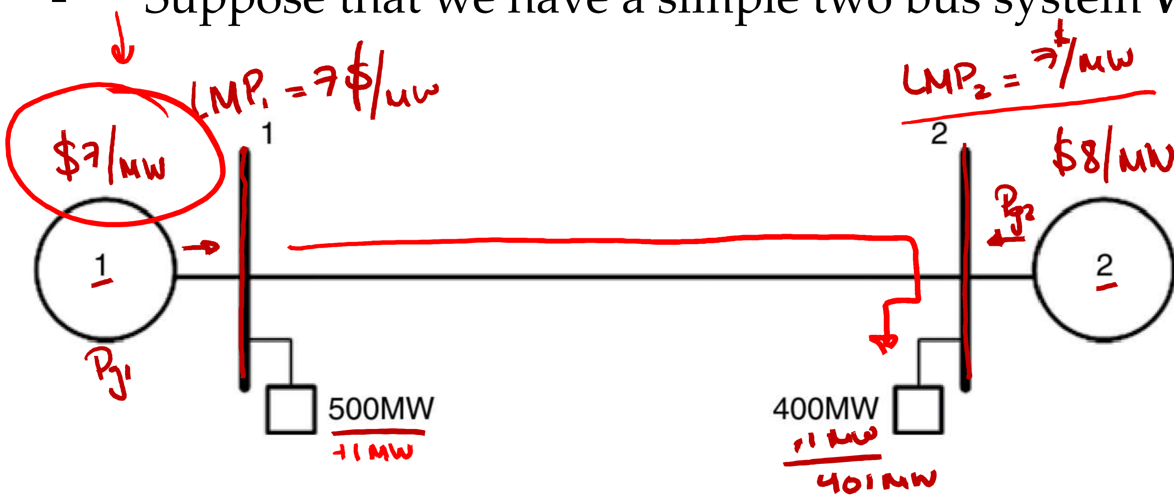
- **Locational Marginal Pricing:**
  A method to reflect the value of electric energy at different locations (buses) accounting for generation, losses, and transmission line limits



Locational Marginal Pricing – Two Bus System

3 factors (LMP)
1) Gen.
2) Losses ×
3) Trans. limits

- Suppose that we have a simple two bus system **without** line constraints



Generator	$P_{\min}$	$P_{\max}$	Cost
1	20	1000	7
2	40	500	8

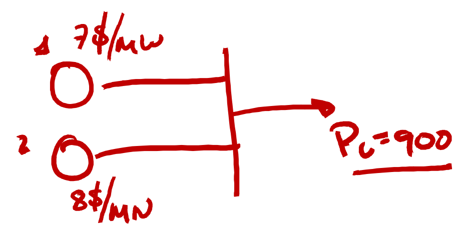
- The total cost $= 7P_{g1} + 8P_{g2}$
- No line constraints. network

$P_L = 900MW$

$P_{g1} + P_{g2} = 900$

$$\boxed{\begin{aligned} P_{g1}^* &= 860 \\ P_{g2} &= 40 \end{aligned}}$$

Cheaper
Exp.

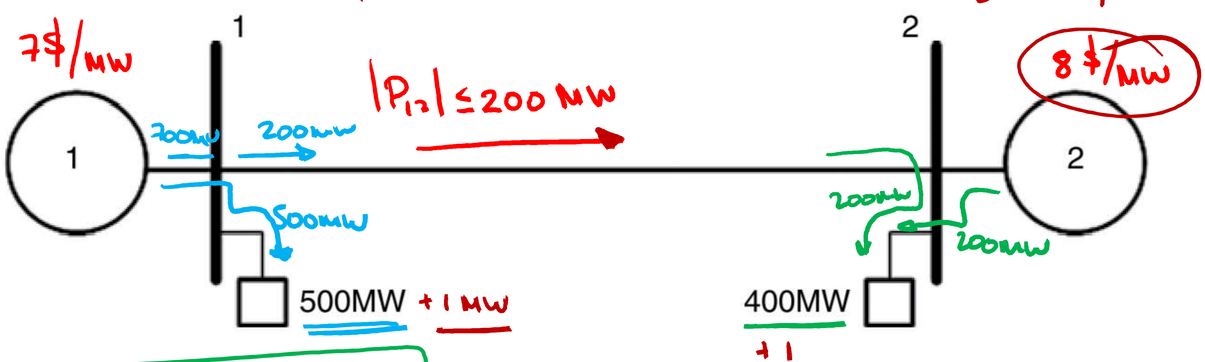


- If we add 1MW of load at bus 1, which generator will supply? Gen 1 ⇒ $LMP_1 = 7 \$/MW$
- " " " " " " " bus 2 " " " " ? Gen 1 ⇒ $LMP_2 = 7 \$/MW$

Locational Marginal Pricing – Two Bus System

- Suppose that we have a simple two bus system
- **Assume line 1-2 is limited to 200 MW**

$LMP_1 = 7 \$/MW$ $LMP_2 = 8 \$/MW$



Generator	$P_{\min}$	$P_{\max}$	Cost
1	20	1000	7
2	40	500	8

$P_{g_1}^* = 700$ MW
$P_{g_2}^* = 200$ MW

- If we add 1MW of load at bus 1, which generator will supply? Gen 1 ⇒ $LMP_1 = 7 \frac{\$}{MW}$
- " " " 1MW " " " " 2, " " " " ? Gen 2 ⇒ $LMP_2 = 8 \frac{\$}{MW}$

- **What is the LMP at bus 2? At bus 1?**

Locational Marginal Pricing – Derivation DC OPF

- The main idea with Locational Marginal Pricing (LMP) is to answer:
  - **What is the cost of increasing the power demand at any bus?**

(load)

i: bus number

change in cost

$$LMP_i = \frac{\Delta C}{\Delta P_{Li}} \frac{\$}{MW} \xrightarrow{\lim_{\Delta \to 0}} \frac{\partial C}{\partial P_{Li}}$$

change in load

- What is this total cost and how can we find it?
- **We need to consider:** generation, losses, and transmission line limits

DC OPF

$$\min \left(F_1(P_{g1}) + \dots + F_N(P_{gN}) \right)$$

s.t.

$$\left\{ \begin{array}{l} P_{gi\text{-}min} \le P_{gi} \le P_{gi\text{-}max} \\ P = P_g - P_L = B\delta \\ \vdots \end{array} \right.$$

Total cost of generation doesn't include
- losses
- Trans. line limits

$\boxed{\text{What should be the the cost } C?}$

Locational Marginal Pricing – Derivation DC OPF

- The main idea with Locational Marginal Pricing (LMP) is to answer:
  What is the cost of increasing the power demand at any bus? $\frac{\partial C}{\partial P_{Li}}$
- Remember the DC OPF:

$$\min_{P_{g1},\, P_{g2},\, \ldots,\, P_{gN_g},\, \delta_2,\, \ldots,\, \delta_N} \left(f_1(P_{g1}) + \cdots + f_{Ng}(P_{gN_g}) \right)$$

s.t.

$$P_{gi-\min} \le P_{gi} \le P_{gi-\max}$$

$$P_i = P_{gi} - P_{Li} = \sum_{k=1}^{N} B_{ik}\delta_{ik} \qquad \forall i = 1, \ldots, N$$

$$\underline{P}_{ij} \le b_{ij}(\delta_i - \delta_j) \le \bar{P}_{ij}$$

$$\delta_1 = 0$$

Write the problem in general form (LN 4)

$$\left\{ \begin{array}{lll} \min & f(x) & \\ \text{s.t.} & & \\ & \bullet\ h_i(x) = 0 & i = 1, \ldots, N_e \} \rightarrow \lambda_1, \ldots, \lambda_{N_e} \\ & \bullet\ g_j(x) \le 0 & j = 1, \ldots, N_I \rightarrow \mu_1, \ldots, \mu_{N_I} \end{array} \right.$$

Write DC OPF in this way

Locational Marginal Pricing – Derivation DC OPF

LMP
1 Generation
2 losses
3 limits

- The main idea with Locational Marginal Pricing (LMP) is to answer:
  What is the cost of increasing the power demand at any bus? $\frac{\partial C}{\partial P_{Li}}$
- Write the Lagrangian for the problem:

$$\min_{P_{gi},\,\delta_i} \left(f_1(P_{g1}) + \cdots + f_{Ng}(P_{gN_g}) \right)$$

s.t.

Ineq. lag. multipliers

$P_{gi} - P_{gi-\max} \le 0 \quad : \quad \eta_i^+$

$-P_{gi} + P_{gi-\min} \le 0 \quad : \quad \eta_i^-$

$\underbrace{\sum_{k=1}^{N} B_{ik}\delta_{ik}}_{P_{flow,i}} - P_{gi} + P_{Li} = 0 \quad \forall i = 1, \ldots, N \ : \ \lambda_i$ — Equality const. lag. mult.

$b_{ij}(\delta_i - \delta_j)$

$P_{ij} - \bar{P}_{ij} \le 0 \quad : \quad \mu_{ij}^+$

$\underline{P}_{ij} - P_{ij} \le 0 \quad : \quad \mu_{ij}^-$

$\boxed{\delta_1 = 0}$ λ_0

Ineq. lag. mult.

Recall from LN4 Lagrangian function

$\sum_{k=1}^{N} B_{ik}\delta_{ik}$

$$\mathcal{L}(P_{gi}, \delta_i, \eta_i^+, \eta_i^-, \lambda_i, \mu_{ij}^+, \mu_{ij}^-, \lambda_0) = \underbrace{\left(f_1(P_{g1}) + \cdots + f_{N_g}(P_{gN_g})\right)}_{\$} + \sum_{i=1}^{N} \lambda_i \left(P_{flow,i} - P_{gi} + P_{Li}\right)$$
$$+ \sum_{i=1}^{N_g} \eta_i^+ (P_{gi} - P_{gi-\max}) + \sum_{i=1}^{N} \eta_i^- (-P_{gi} + P_{gi-\min})$$
$$+ \sum_{ij} \mu_{ij}^+ (P_{ij} - \bar{P}_{ij}) + \underbrace{\sum_{ij} \mu_{ij}^- (\underline{P}_{ij} - P_{ij})}_{\$}$$
$$+ \lambda_0(\delta_0)$$

- **The Lagrangian can be seen as the total cost that we are looking for!**

Locational Marginal Pricing – Derivation DC OPF

- The main idea with Locational Marginal Pricing (LMP) is to answer:
  What is the cost of increasing the power <u>demand</u> at any bus? $\frac{\partial C}{\partial P_{Li}} \rightarrow \frac{\partial \mathcal{L}}{\partial P_{Li}}$
- Write the Lagrangian for the problem: DC Power flow equations

$$(\$)\ \mathcal{L} = (f_1^{(P_g)} + \cdots + f_{Ng}^{(P_{g,N_g})}) + \boxed{\sum_{i=1}^{N} \lambda_i \left(-P_{gi} + P_{Li} + \underbrace{\text{pflow}_i}_{\sum_{k=1}^{N} B_{ik}\delta_k}\right)} + \sum_{j=1}^{N_g} \eta_j^+ \left(P_{gi} - P_{gi-\max}\right)$$

$$+ \sum_{j=1}^{N_g} \eta_j^- \left(-P_{gi} + P_{gi-\min}\right) + \mu_{ij}^+ \left(\underbrace{P_{ij}}_{b_{ij}(\delta_i - \delta_j)} - \bar{P}_{ij}\right) + \mu_{ij}^- \left(-P_{ij} + \underline{P}_{ij}\right)$$

- **What is the LMP at bus *i*?** $\frac{\partial C}{\partial P_{Li}} \rightarrow \frac{\partial \mathcal{L}}{\partial P_{Li}} = \text{LMP}_i$ $\quad i$: bus number

$$\boxed{\text{LMP}_i = \frac{\partial \mathcal{L}}{\partial P_{Li}} = \lambda_i}$$

(E_4) Lag. multiplier associated with DC power flow equation at bus i

Locational Marginal Pricing – Derivation DC OPF

- The main idea with Locational Marginal Pricing (LMP) is to answer:
  What is the cost of increasing the power demand at any bus? $\frac{\partial C}{\partial P_{Li}} \rightarrow \frac{\partial \mathcal{L}}{\partial P_{Li}}$
- Write the Lagrangian for the problem:

$$(*)\ \mathcal{L} = (f_1 + \cdots + f_{Ng}) + \sum_{i=1}^{N} \lambda_i \left(-P_{gi} + P_{Li} + \text{pflow}_i\right) + \sum_{j=1}^{N_g} \eta_j^+ \left(P_{gi} - P_{gi-\max}\right)$$

$$+ \sum_{j=1}^{N_g} \eta_j^- \left(-P_{gi} + P_{gi-\min}\right) + \mu_{ij}^+ \left(P_{ij} - \bar{P}_{ij}\right) + \mu_{ij}^- \left(-P_{ij} + \underline{P}_{ij}\right)$$

- **Therefore, the Locational Marginal Price at bus i is:**

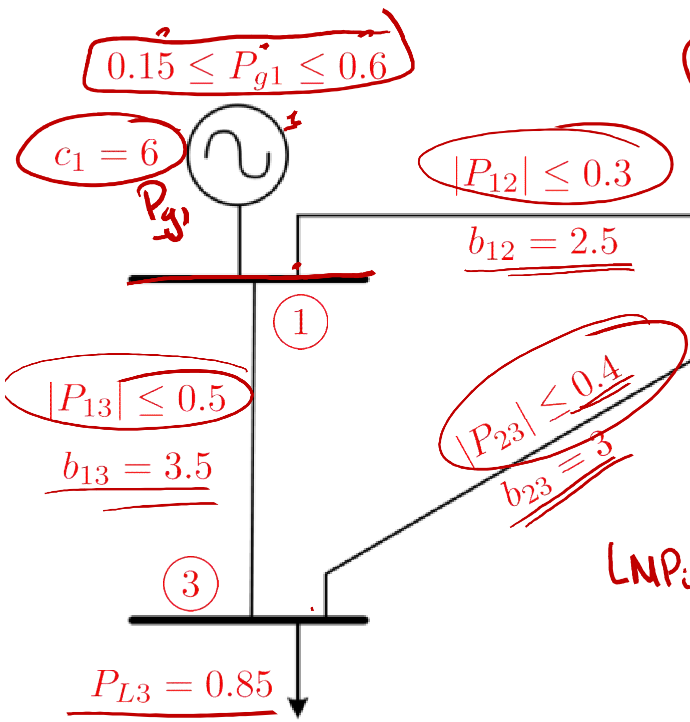
$$\text{LMP}_i = \frac{\partial \mathcal{L}}{\partial \mathbf{P_{Li}}} = \lambda_i$$

Lagrange Multiplier associated with active DC power flow at bus i

LMP Example – DC OPF

$P_{gi} - P_{ci} = \sum_{k=1}^{N} B_{ik}\delta_k$

- **Example:** Solve using DC OPF and find **LMP at every bus**



Write DC OPF problem

$$\min_{P_{g1}, P_{g2}, \delta_1, \delta_2, \delta_3} \quad 6P_{g1} + 7P_{g2}$$

s.t.

$0.15 \leq P_{g1} \leq 0.6$

$0.1 \leq P_{g2} \leq 0.4$

$\lambda_1: \; -P_{g1} + \sum_{k=1}^{3} B_{1k}\delta_k = 0$

$\lambda_2: \; -P_{g2} + \sum_{k=1}^{3} B_{2k}\delta_k = 0$

$\lambda_3: \; 0.85 + \sum_{k=1}^{3} B_{3k}\delta_k = 0$

LMP_i = Lag. multipliers

$-0.3 \leq 2.5(\delta_1 - \delta_2) \leq 0.3$

$-0.5 \leq 3.5(\delta_1 - \delta_3) \leq 0.5$

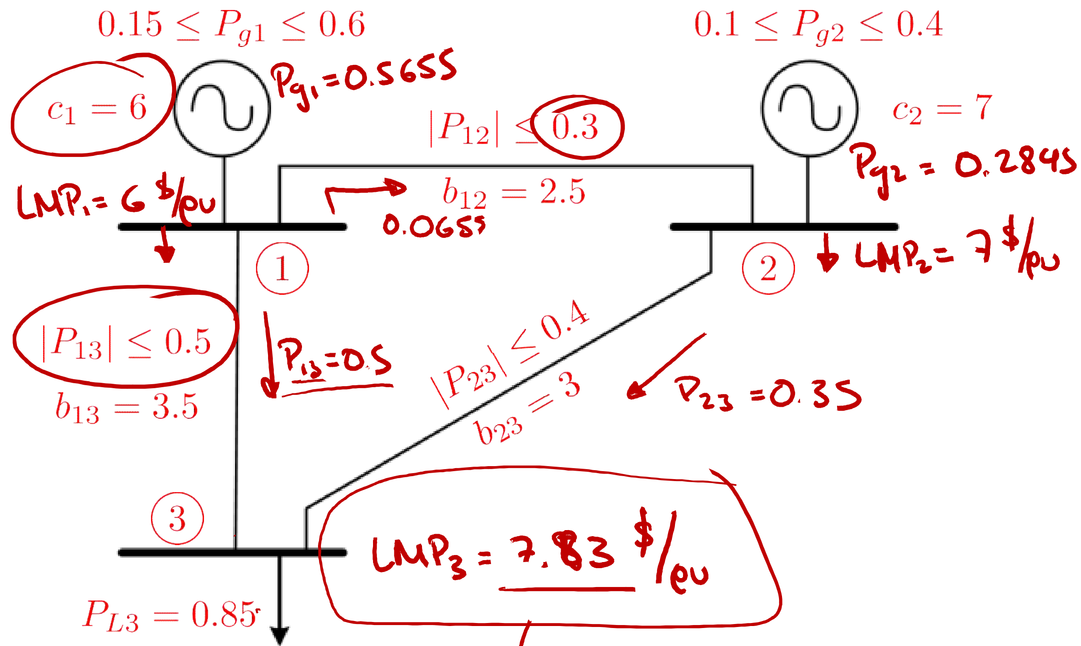
$-0.4 \leq 3(\delta_2 - \delta_3) \leq 0.4$

$\delta_1 = 0$

LMP Example – DC OPF

- **Example:** Solve using DC OPF and find **LMP at every bus**

$$x^* = \begin{pmatrix} P_{g1} \\ P_{g2} \\ \delta_1 \\ \delta_2 \\ \delta_3 \end{pmatrix} = \begin{pmatrix} 0.5655 \\ 0.2845 \\ 0 \\ -0.0262 \\ -0.1429 \end{pmatrix}$$



$$P_{12} = b_{12}(\delta_1 - \delta_2) = 0.0655$$
$$P_{13} = b_{13}(\delta_1 - \delta_3) = 0.5$$
$$P_{23} = b_{23}(\delta_2 - \delta_3) = 0.35$$

$$\lambda_1 = 6$$
$$\lambda_2 = 7$$
$$\lambda_3 = 7.83$$

Why is LMP_3 more expensive?

1 Gen ✓
2 losses
3 line limits — $P_{13} = 0.5$

Summary of Topics Covered

- *Nodal Analysis*
- *Power Flow Equations* (nonlinear) → Extra credit.
- *DC Power Flow - simplification*
- *Economic Dispatch with Network Constraints – DC OPF*
- *Formulation (LP or QP)*
- *Locational Marginal Price*