EE 459/611: Smart Grid Economics, Policy, and Engineering

Lecture 5: Economic Dispatch* ~~and~~ with linear cost

Linear Programming

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Topics that Will be Covered

- ***Economic dispatch with linear costs***
- ***Linear programming – General Form***
- *Types of Solutions*
- *In-depth example – Matlab code*

Quadratic Programming vs Linear Programming

- Both are types of optimization problems

$$\left.\boxed{\begin{aligned} &\min\ f(x) \\ &\text{s.t.} \\ &\quad h_i(x) = 0 \\ &\quad g_j(x) \leq 0 \end{aligned}}\right\} \text{General Opt. Problem *}$$

- Quadratic Programming: Opt. problem that has quadratic cost function and linear equality and linear inequality constraints

$$\min_x \left(\frac{1}{2}x^T H x + c^T x + \underbrace{e}_{}\right) = f(x)$$

↳ constant term is usually ignored
Doesn't affect optimal solution x^*

s.t.

$$\begin{cases} * \; Ax \leq b \\ * \; Bx = d \end{cases}$$

- Linear Programming: Opt. problem with linear cost and linear eq. and linear ineq constraints

$$\min_x \left(c^T x + \underbrace{e}_{}\right) = f(x)$$

s.t.

* $Ax \leq b$.
* $Bx = d$.

Economic Dispatch With Linear Costs

- Let's consider two generation units with **linear costs** feeding a load of 850 MW

Unit 1: Coal/steam unit

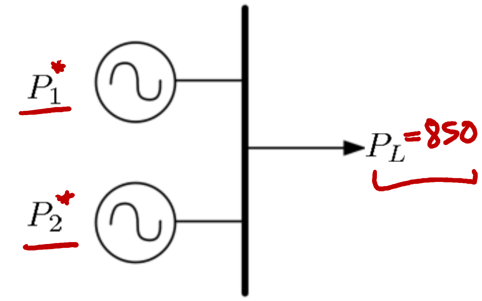
$$F_1(P_1) = 420.42 + 9.09P_1 \ \left(\frac{\$}{\text{h}}\right)$$

Max output: 600 MW
Min Output: 150 MW

Unit 2: Oil/steam unit

$$F_2(P_2) = 232.4 + 8.82P_2 \ \left(\frac{\$}{\text{h}}\right)$$

Max output: 400 MW
Min Output: 100 MW



- Write the optimization problem to minimize cost

$$\min_{x_1,\,x_2} \ \underbrace{9.09x_1 + 420.42}_{F_1(P_1)} + \underbrace{8.82x_2 + 232.4}_{F_2(P_2)}$$

s.t.

$$x_1 + x_2 - 850 = 0$$
$$150 \le x_1 \le 600$$
$$100 \le x_2 \le 400$$

$$P_1 = \boxed{x_1^* = 450, \ x_2^* = 400} = P_2$$

$$\min_{x_1,\,x_2} \ 9.09x_1 + 8.82x_2$$

(ignore constant terms)

s.t.

$$x_1 + x_2 - 850 = 0$$
$$150 \le x_1 \le 600$$
$$100 \le x_2 \le 400$$

$$\boxed{x_1^* = 450, \ x_2^* = 400}$$

($) Total cost = (9.09)(450) + (400)(8.82) + const.

- These two problems give the same solution, **why?** The constant terms do not affect solution $\frac{d(200)}{dt} = 0$
- **Is the cost the same?** No (Be careful)

Economic Dispatch With Linear Costs

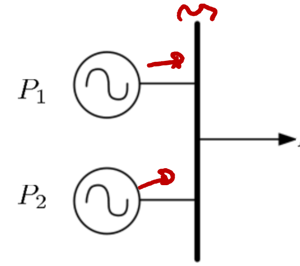
- Let's consider two generation units with **linear costs** feeding a load of 850 MW

Unit 1: Coal/steam unit

Expensive $F_1(P_1) = \underbrace{420.42} + \underline{9.09P_1} \left(\frac{\$}{\text{h}}\right)$ Max output: 600 MW, Min Output: 150 MW

Unit 2: Oil/steam unit

Cheap $F_2(P_2) = \underbrace{232.4} + \underline{8.82P_2} \left(\frac{\$}{\text{h}}\right)$ → Max output: 400 MW, Min Output: 100 MW



Matlab: linprog

$$\min_x \; c^T x \quad \text{s.t.} \quad \left\{ \begin{array}{l} Ax \le b \\ \underbrace{A_{eq}x = b_{eq}} \end{array} \right.$$

- Write the optimization problem as a **Linear Program:** (LP)

$$\min_{x_1,\, x_2} \; \underbrace{9.09x_1 + 8.82x_2}$$

s.t.

$$x_1 + x_2 - 850 = 0$$

$$150 - x_1 \le 0, \; x_1 - 600 \le 0$$

$$100 - x_2 \le 0, \; x_2 - 400 \le 0$$

$$\boxed{x_1^* = 450, \; x_2^* = 400}$$

$$\min_x \; \underbrace{(9.09 \;\; 8.82)}_{c^T} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

s.t.

$$\boxed{\lambda_1}: \; \underbrace{(1 \;\; 1)}_{A_{eq}} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \underbrace{(850)}_{b_{eq}} \leftarrow x_1 + x_2 = 850$$

μ_1: $-x_1 \le -150$

μ_2: $x_1 \le 600$

μ_3: $-x_2 \le -100$

μ_4: $x_2 \le 400$

$$\underbrace{\begin{pmatrix} -1 & 0 \\ 1 & 0 \\ 0 & -1 \\ 0 & 1 \end{pmatrix}}_{A} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \le \underbrace{\begin{pmatrix} -150 \\ 600 \\ -100 \\ 400 \end{pmatrix}}_{b} \begin{array}{l} \leftarrow -x_1 \le -150 \\ \leftarrow x_1 \le 600 \\ \\ \end{array}$$

$\mu_4 > 0$

Economic Dispatch With Linear Costs

- Let's consider two generation units with **linear costs** feeding a load of 850 MW

Unit 1: Coal/steam unit

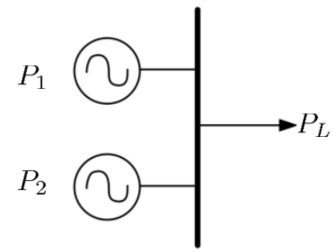
(Expensive) $F_1(P_1) = 420.42 + 9.09 P_1 \quad \left(\frac{\$}{\text{h}}\right)$

Max output: 600 MW
Min Output: 150 MW

Unit 2: Oil/steam unit

(cheaper) $F_2(P_2) = 232.4 + 8.82 P_2 \quad \left(\frac{\$}{\text{h}}\right)$

Max output: $\boxed{\text{400 MW}}$
Min Output: 100 MW



- Write the optimization problem as a **Linear Program:**

$$\min_x \underbrace{\begin{bmatrix} 9.09 & 8.82 \end{bmatrix}}_{c^T} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

s.t.

$$\underbrace{\begin{bmatrix} 1 & 1 \end{bmatrix}}_{A_{eq}} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \underbrace{850}_{b_{eq}}$$

$$\underbrace{\begin{bmatrix} 150 \\ 100 \end{bmatrix}}_{lb} \leq \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \leq \underbrace{\begin{bmatrix} 600 \\ 400 \end{bmatrix}}_{ub}$$

rather than $Ax \leq b$

$lb \leq x$

$x \leq ub$

```
%% Now solve using linear programming
c = [9.09 8.82]';

% equality constraints
Aeq = [1 1];
beq = 850;

% lb and ubs
lb = [150; 100];
ub = [600; 400];

xlp = linprog(c, [], [], Aeq, beq, lb, ub)
```

A b

$lb = \begin{pmatrix} 150 \\ 100 \end{pmatrix} \leq \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$

$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \leq \begin{pmatrix} 600 \\ 400 \end{pmatrix} = ub$

$\cancel{Ax \leq b}$

$$\boxed{x_1^* = 450, \; x_2^* = 400}$$

Economic Dispatch With Linear Costs

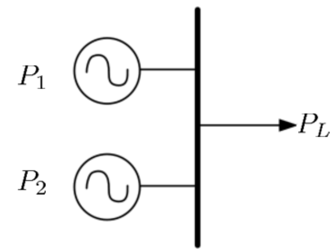
- Let's consider two generation units with **linear costs** feeding a load of 850 MW

Unit 1: Coal/steam unit

$F_1(P_1) = 420.42 + 9.09 P_1 \quad \left(\frac{\$}{\text{h}}\right)$ Max output: 600 MW, Min Output: 150 MW

Unit 2: Oil/steam unit

$F_2(P_2) = 232.4 + 8.82 P_2 \quad \left(\frac{\$}{\text{h}}\right)$ Max output: 400 MW, Min Output: 100 MW



- Write the optimization problem as a **Linear Program:**

$$\min_{x} \quad \begin{bmatrix} 9.09 & 8.82 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

s.t.

$$\begin{bmatrix} 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 850$$

$$\begin{bmatrix} 150 \\ 100 \end{bmatrix} \leq \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \leq \begin{bmatrix} 600 \\ 400 \end{bmatrix}$$

we can also write this as $Ax \leq b$

(see slide 5)

$x_1^* = 450, \; x_2^* = 400$

Try it again in Matlab

Linear Programming Formulation – General Form

- **General form** for a linear optimization program:
  - Linear cost function ✓
  - Linear equality constraints ✓
  - Linear inequality constraints ✓
  - Component wise non-negativity *

$$\min_x \quad c^T x + d$$

s.t.

$$Ax \le b \quad ✓$$

$$A_{eq}x = b_{eq}$$

$$* \; x \ge 0 \; * \quad \text{(ignore)}$$

min $10x_1 + 20x_2$
s.t.
$x_1 + x_2 = 50$
$-15 \le x_1 \le 20$
$0 \le x_2 \le 50$

slack variables

decompose x_1 → $x_{1p} \ge 0$, $x_{1n} \ge 0$

- **Example:** write the following problem in **LP general form**: ✓

$$\min_x \left(10x_1 + 2x_2\right) = f(x) \quad \text{linear cost function}$$

s.t.

$$* \; x_1 + 2x_2 \le 15 * \quad \text{— linear ineq.} ✓$$

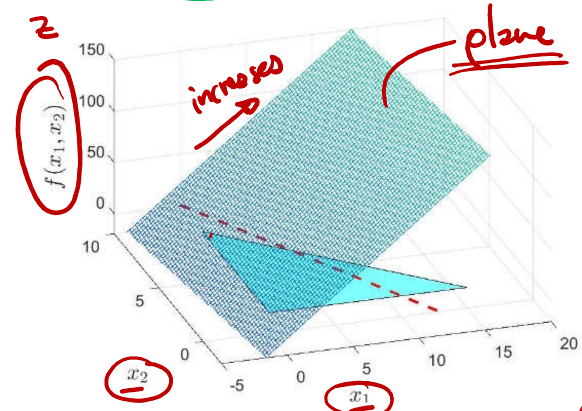
$$* \; x_1 + x_2 = 10 * \quad \text{— linear equality} ✓$$

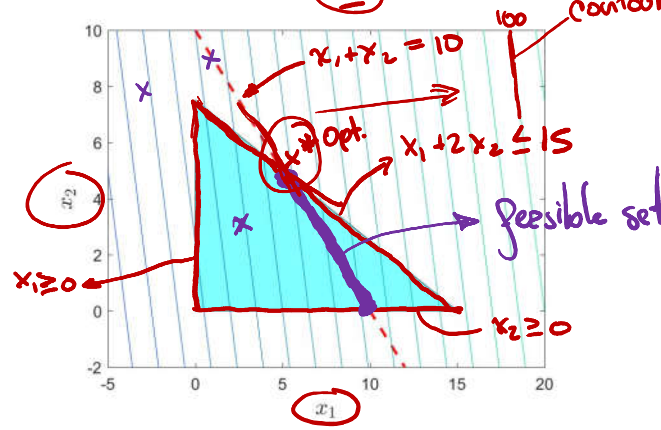
$$\left.\begin{array}{l} x_1 \ge 0 \\ x_2 \ge 0 \end{array}\right\} \text{non-negative} \quad x \ge 0 \quad \text{(linear)}$$

cost

contour lines or level sets
$10x_1 + 2x_2 = C$

$x_1^* =$
$x_2^* =$





General Form for Linear Programming

General form

$$\boxed{\begin{aligned} &\min_x \quad c^T x + d \\ &\text{s.t.} \\ &\quad Ax \le b \\ &\quad \boxed{A_{eq} x = b_{eq}} \end{aligned}}$$

$\cancel{x \ge 0}$ (ignore)

$x \in \mathbb{R}^2$

$$\begin{cases} 0.3x_1 + 0.8x_2 = 10 \\ 0.7x_1 + 0.1x_2 = 5 \\ \vdots \end{cases}$$

Any system of linear equations

$\boxed{A_{eq} x = b_{eq}}$ *

Topics that Will be Covered

- *Economic dispatch with linear costs*
- *Linear programming – General Form*
- ***Types of Solutions***
- *In-depth example – Matlab code*

Types of Solutions for Linear Programs - Example

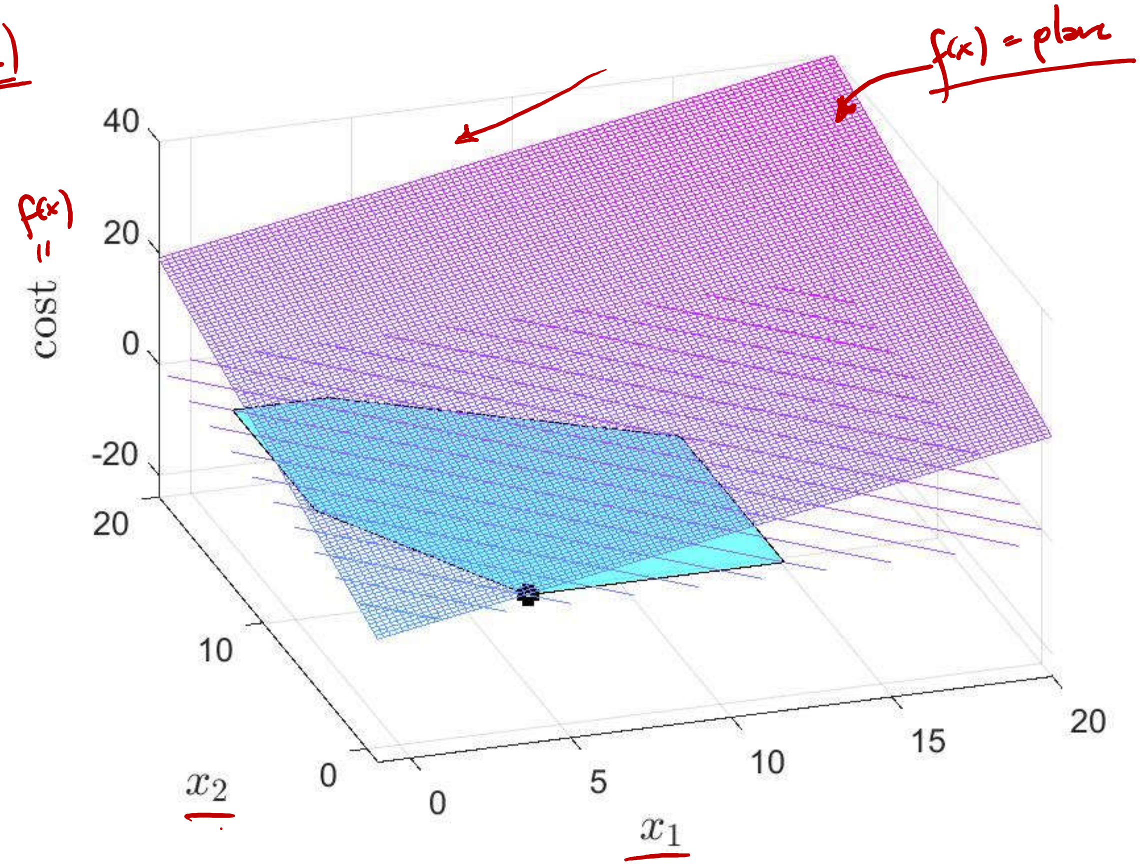
- Let's consider the following Linear Program:

$$\min_x \quad \underbrace{x_1 + x_2}_{} = f(x) \quad (\text{plane})$$

s.t.

$$\left\{\begin{aligned} &\frac{2}{3}x_1 + x_2 \leq 18 \\ &2x_1 + x_2 \geq 8 \\ &x_1 \leq 12 \\ &x_2 \leq 16 \\ &x_1 \geq 0 \\ &x_2 \geq 0 \end{aligned}\right.$$

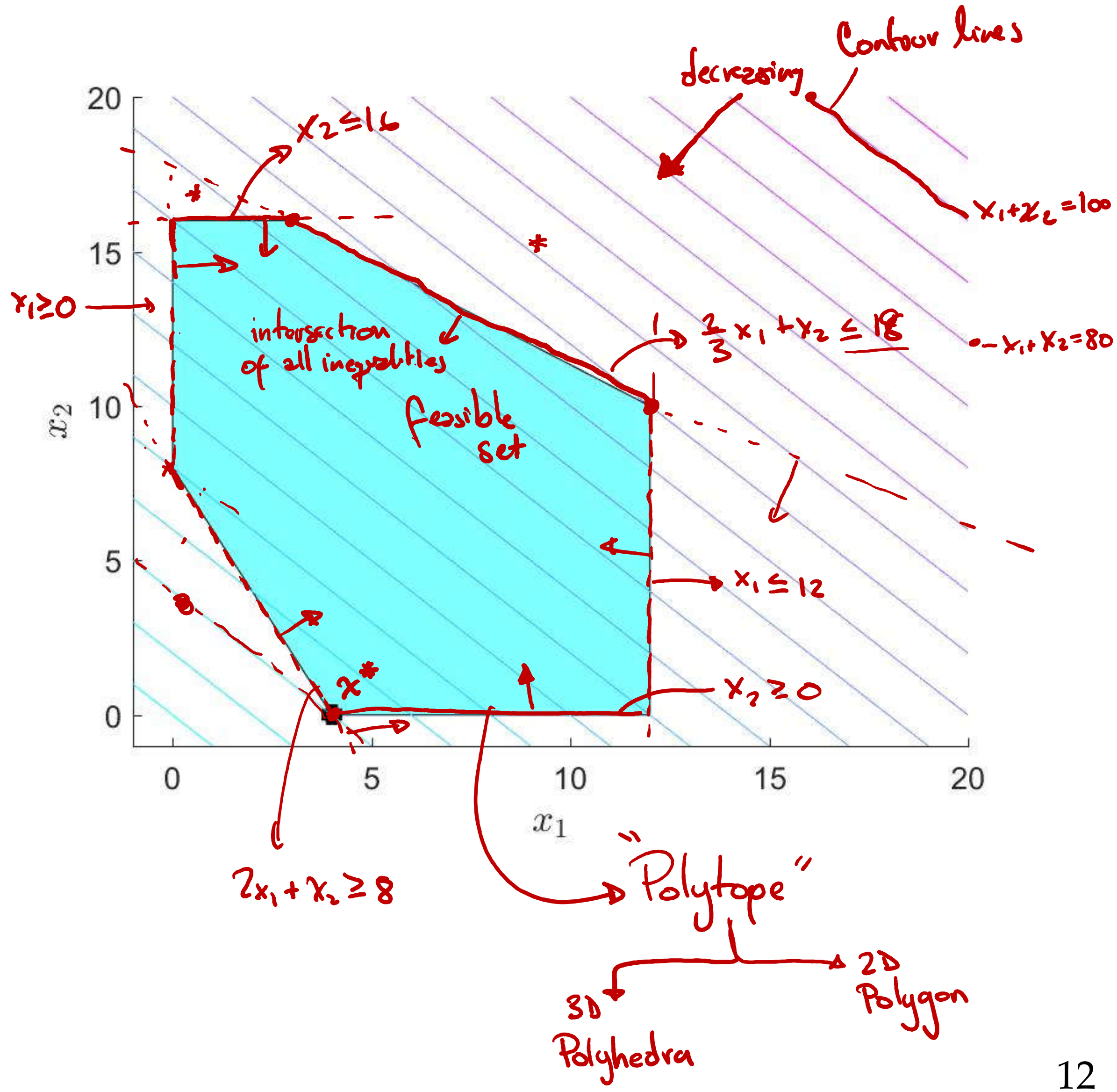
inequalities

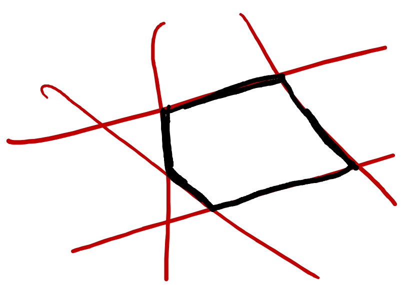


Types of Solutions for Linear Programs - Example

- Let's consider the following Linear Program:

$$
\begin{aligned}
\min_{x} \quad & x_1 + x_2 \\
\text{s.t.} \quad & \left\{
\begin{aligned}
&\tfrac{2}{3}x_1 + x_2 \leq 18 \\
&2x_1 + x_2 \geq 8 \\
&x_1 \leq 12 \\
&x_2 \leq 16 \\
&x_1 \geq 0 \\
&x_2 \geq 0
\end{aligned}
\right.
\end{aligned}
$$





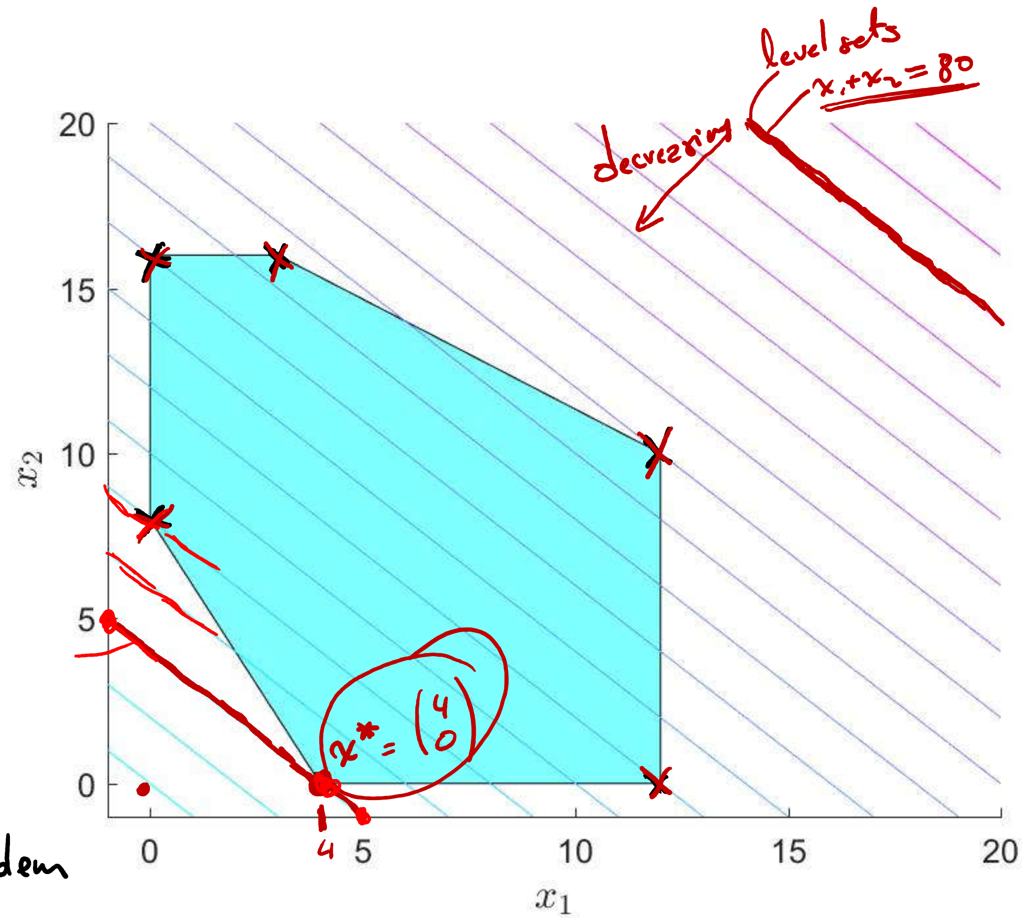
Types of Solutions for Linear Programs - Example

- **Case 1:** The minimum is in one of the extreme points (ideal case)

$$\min_{x} \quad x_1 + x_2$$

s.t.

$$\begin{cases} \frac{2}{3}x_1 + x_2 \leq 18 \\ 2x_1 + x_2 \geq 8 \\ x_1 \leq 12 \\ x_2 \leq 16 \\ x_1 \geq 0 \\ x_2 \geq 0 \end{cases}$$



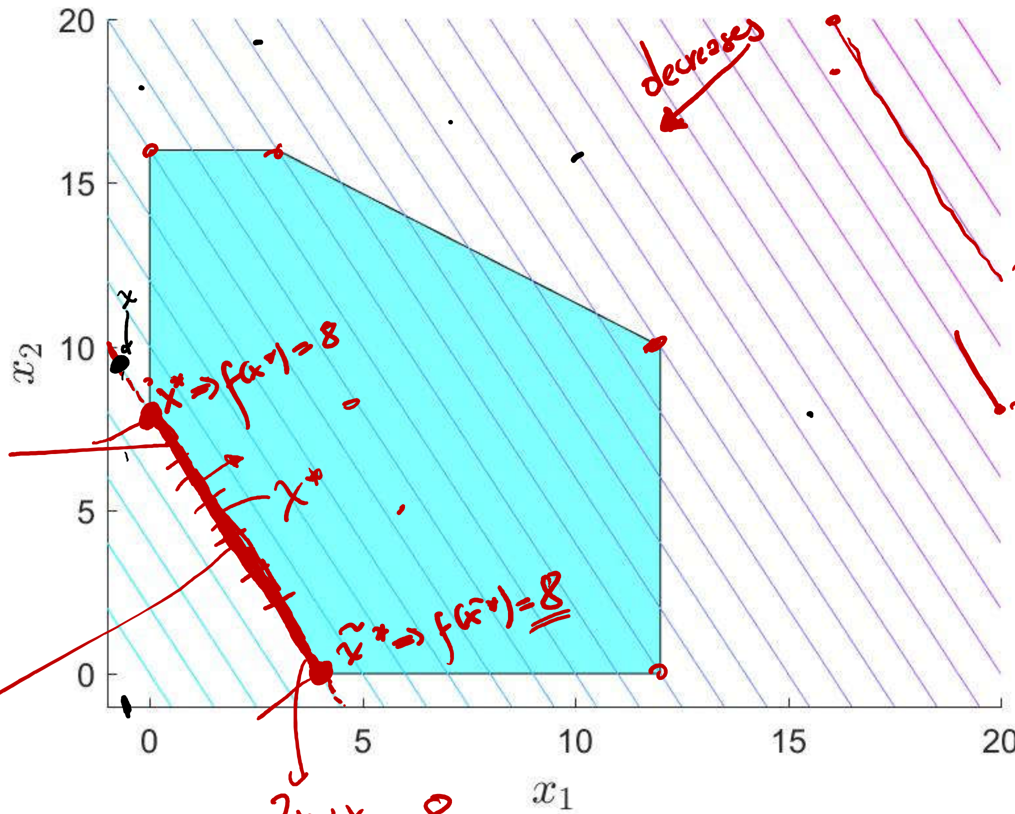
- A well formulated LP problem will have optimal solution, x^*, on a vertex/extreme

Types of Solutions for Linear Programs - Example

- **Case 2:** There are an infinite amount of optimal solutions (a side of polygon)
  - This happens when a side of the polygon is parallel to a contour line of the cost function

$$
\begin{aligned}
&\min_{x} \ \boxed{2x_1 + x_2} \\
&\text{s.t.} \\
&\quad \frac{2}{3}x_1 + x_2 \leq 18 \\
&\quad 2x_1 + x_2 \geq 8 \\
&\quad x_1 \leq 12 \\
&\quad x_2 \leq 16 \\
&\quad x_1 \geq 0 \\
&\quad x_2 \geq 0
\end{aligned}
$$

$2x_1+x_2 \geq 8$



$2x_1+x_2 = 8$

x^* is an infinite # of points defined on this line ∩ polytope (not desired)

Types of Solutions for Linear Programs - Example

- **Case 3:** The LP problem <u>is unbounded</u> (e.g. the cost is –infinity)
  - Occurs if the feasible region is *open*, one side is missing

and unbounded

$$\min_{x} \; -x_1 - x_2$$

s.t.

$$\cancel{\left(\frac{2}{3}x_1 + x_2 \leq 18\right) \text{ ignore}}$$

$$2x_1 + x_2 \geq 8$$

$$\cancel{(x_1 \leq 12) \text{ ignore}}$$

$$x_2 \leq 16$$

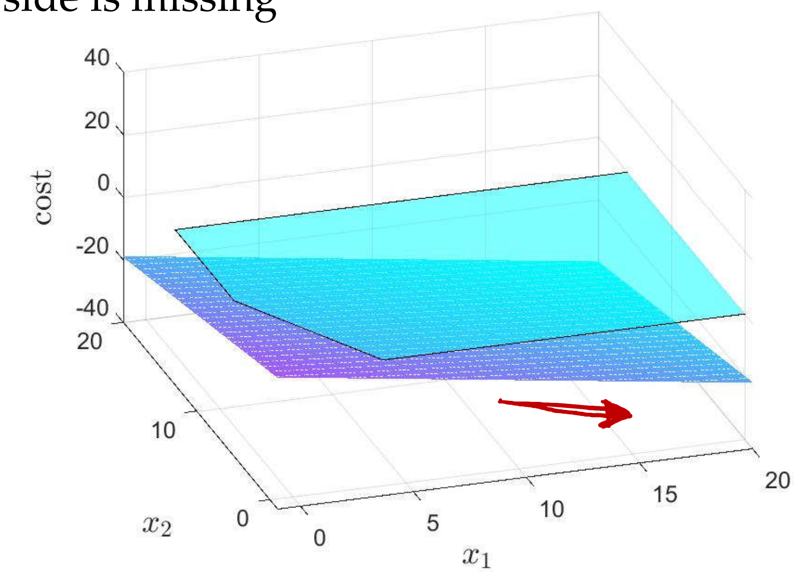
$$x_1 \geq 0$$

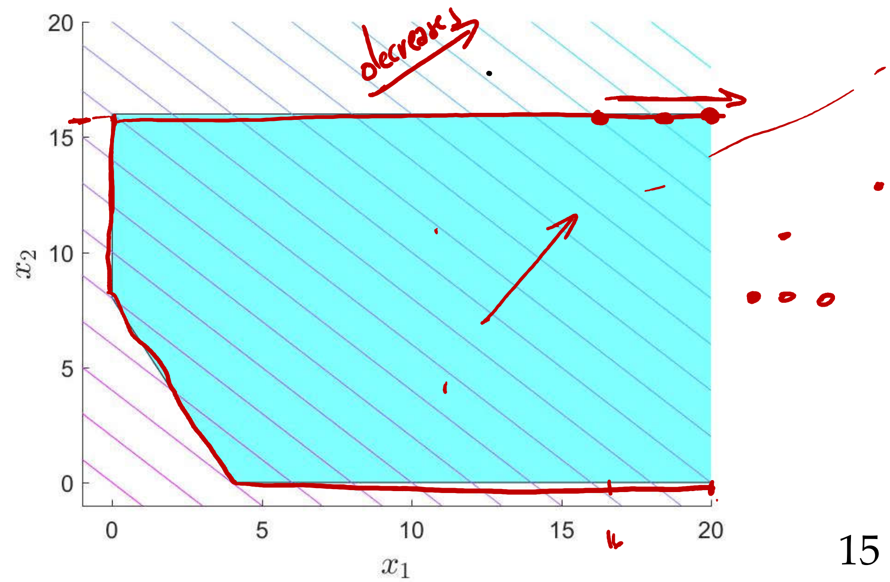
$$x_2 \geq 0$$

not desired

$$x_1^* = \infty, \quad x_2^* = 16$$

$$\underbrace{\text{The problem is unbounded below}}$$

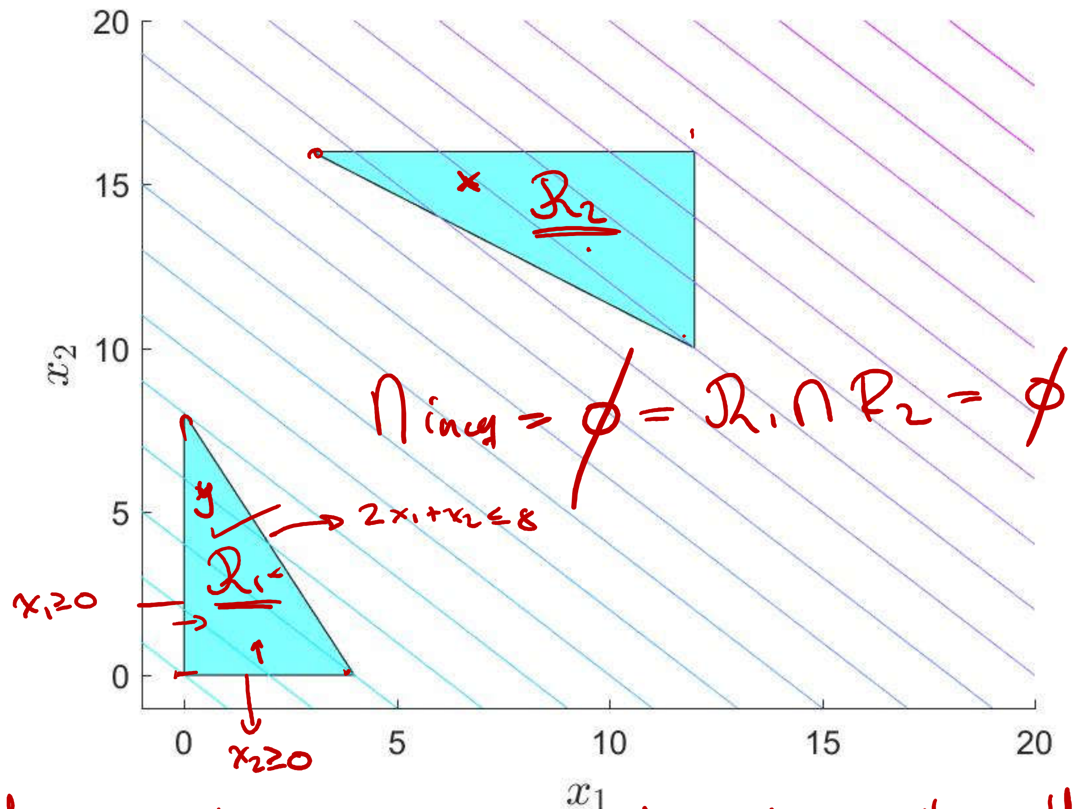




Types of Solutions for Linear Programs - Example

- **Case 4:** Feasible set is empty! $\Rightarrow$ no solutions
  - Occurs when the inequality constraints are not well formulated!

$$
\begin{aligned}
&\min_{x} \quad x_1 + x_2 \\
&\text{s.t.} \\
&\left\{
\begin{aligned}
&\tfrac{2}{3}x_1 + x_2 \geq 18 \\
&2x_1 + x_2 \leq 8 \\
&x_1 \leq 12 \\
&x_2 \leq 16 \\
&x_1 \geq 0 \\
&x_2 \geq 0
\end{aligned}
\right.
\end{aligned}
$$

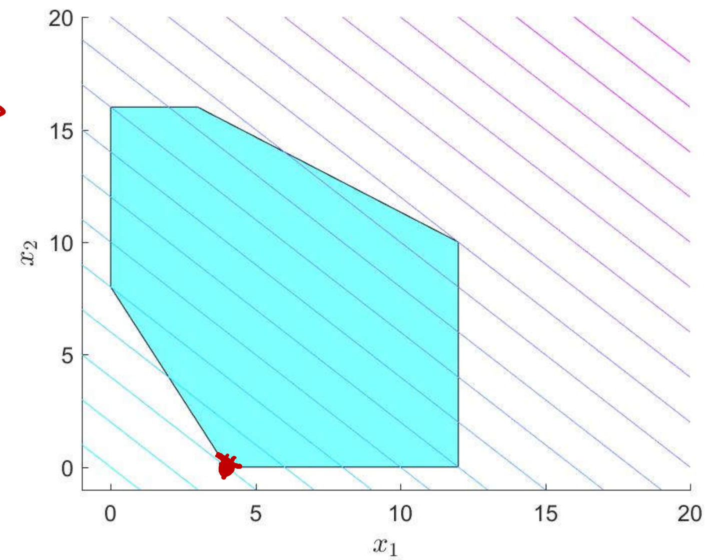


Feasible set is empty $\Rightarrow$ There doesn't exist any $x \in \mathbb{R}^n$ that satisfies all of the constraints

$\Rightarrow$ $\boxed{\text{No solution to this problem}}$

Types of Solution for LP - Summary

- **Case 1:** The minimum is in one of the extreme points (well defined)
  Vertex



- **Case 2:** There are an infinite amount of optimal solutions (a side of polygon)

- **Case 3:** The LP problem is unbounded (e.g. the cost is –infinity)

$\Rightarrow$ problem is not well formulated

- **Case 4:** Feasible set is empty!
  - Occurs when the inequality constraints are not well formulated!

R_1 R_2 $\Rightarrow$ no $x \in \mathbb{R}^n$ that satisfy all constraints

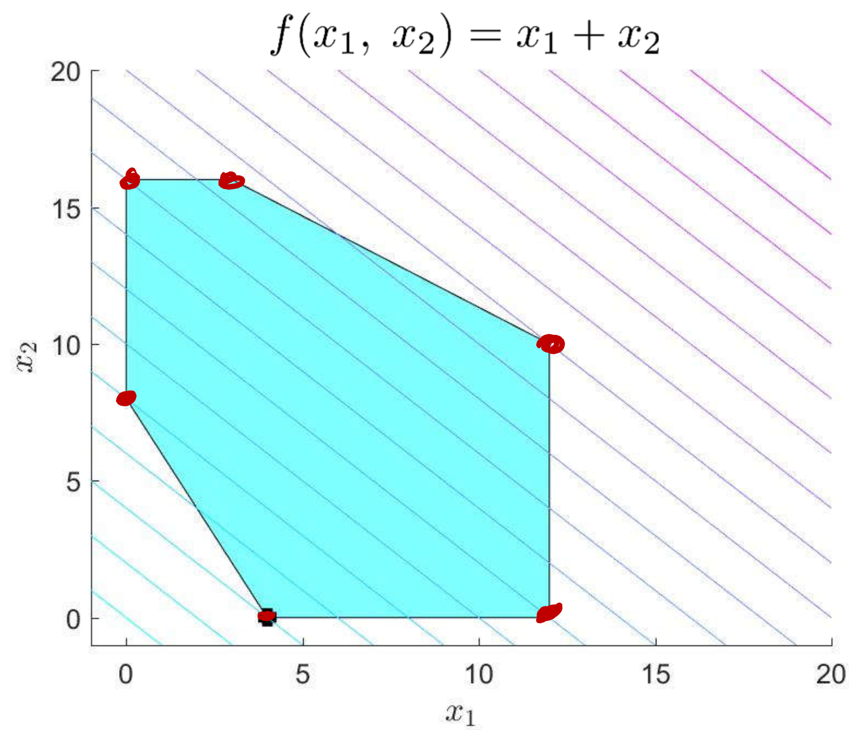
$R_1 \cap R_2 = \phi$

Solution Methods

- Focusing on well formulated problems (**Case 1**), we conclude:
- The optimal solution needs to be an extreme point of the feasible set

Algorithms

- This is where the standard form makes it simpler to find these vertex (extreme) points
- If we know these points, we can evaluate cost function at each point and compare
- * **Simplex method** * works in this way (not covered)



Standard form

$$\min_x \quad c^T x + d$$

s.t.

$$A_{eq} x = b_{eq}$$

$$x \geq 0$$

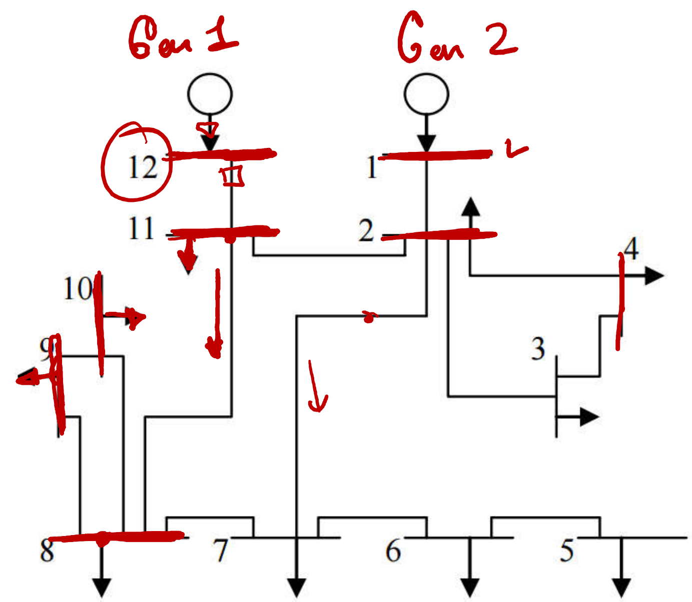
what about $Ax \leq b$?

Topics that Will be Covered

- *Economic dispatch with linear costs*
- *Linear programming – General Form*
- *Types of Solutions*
- ***In-depth example – Matlab code***

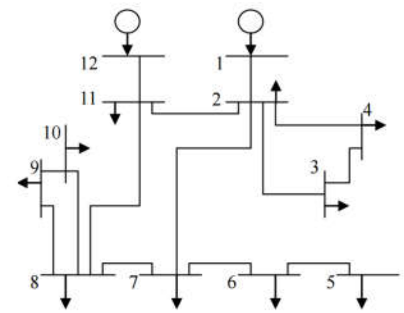
Economic Dispatch Example

- Consider two generator system as shown in the figure
- **Generator 1:**
  - **Cost:** $F_1(P_1) = 2000 + 160P_1$
  - **Capacity limits:** $30 \le P_1 \le 510$
- **Generator 2:**
  - **Cost:** $F_2(P_2) = 3000 + 145P_2$
  - **Capacity limits:** $10 \le P_2 \le 70$
- **Load power:** $P_L = 505$ (Total load)
- ***Losses:** $\underbrace{P_{\text{Losses}} = 0.0189P_1 + 0.0924P_2}_{\text{transmission losses in lines } (I^2R)}$ *



Economic Dispatch Example

- Consider two generator system as shown in the figure



Generator 1:
$F_1(P_1) = 2000 + \underline{160P_1}$
$\underline{30 \le P_1 \le 510}$ ✓

Generator 2:
\$ $F_2(P_2) = 3000 + \underline{145P_2}$
$\underline{10 \le P_2 \le 70}$

- **Load power:** $\boxed{P_L = 505}$ *
- **Losses:** $\boxed{P_{\text{Losses}} = 0.0189P_1 + 0.0924P_2}$ *

$$\min_{P_1, P_2} \left(2000 + 160P_1 + 3000 + 145P_2 \right)$$

s.t

$30 \le P_1$
$P_1 \le 510$

$10 \le P_2$
$P_2 \le 70$

$$\underbrace{P_1 + P_2}_{\text{Generation}} = \underbrace{505 + 0.0189P_1 + 0.0924P_2}_{\text{load + losses}}$$

$$(1 - 0.0189)P_1 + (1 - 0.0924)P_2 = 505$$

minimize cost of supplying load + losses

$$\min c^T x + d$$
s.t.
$Ax \le b$
$A_{eq} x = b_{eq}$

*
$$\boxed{\begin{aligned} P_1^* &= 449.9725 \\ P_2 &= 70 \end{aligned}}$$
*

$$\min_{P_1, P_2} \begin{pmatrix} 160 & 145 \end{pmatrix} \begin{pmatrix} P_1 \\ P_2 \end{pmatrix} + \underbrace{5000}_{\text{ignore}}$$

s.t.

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \\ -1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} P_1 \\ P_2 \end{pmatrix} \le \begin{pmatrix} 510 \\ 70 \\ -30 \\ -10 \end{pmatrix}$$

$$\begin{pmatrix} (1 - 0.0189) & (1 - 0.0924) \end{pmatrix} \begin{pmatrix} P_1 \\ P_2 \end{pmatrix} = 505$$

Next Topics

- Next topic we will look at:

    - *Economic Dispatch with Network Constraints*
    - *LP Formulation*
    - *Locational Marginal Price*