EE 459/611: Smart Grid Economics, Policy, and Engineering

Lecture 4: “Economic Dispatch” and Optimization Theory

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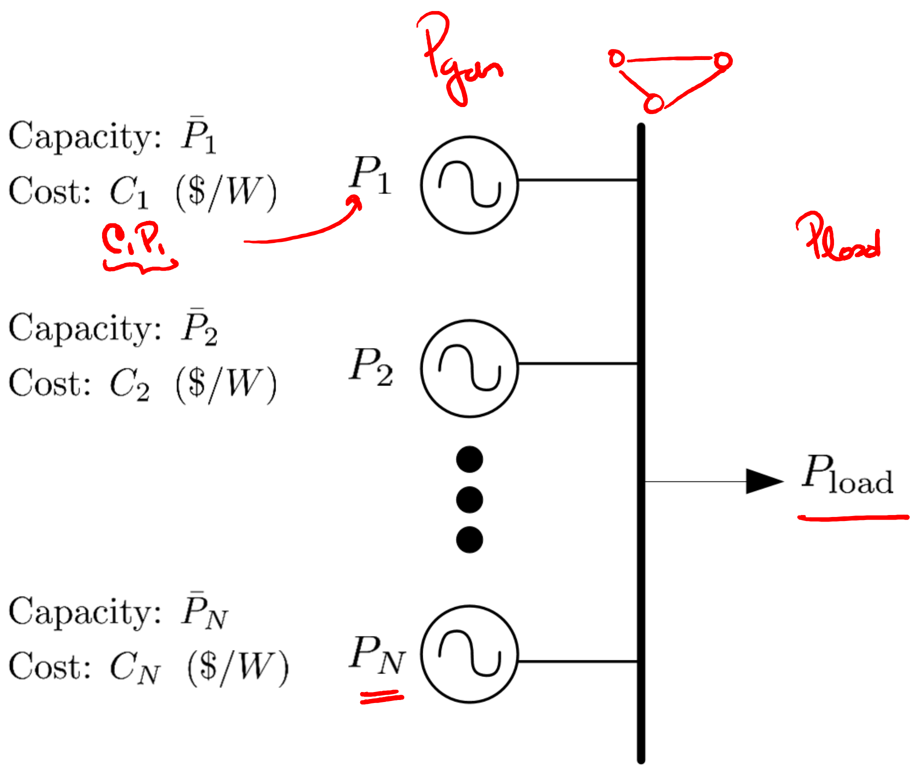
Economic Dispatch Revisited

- The economic dispatch problem attempts to **minimize** cost of generation while **meeting the loads** $P_{gen\text{-TOTAL}} = P_{Load}$
- **Network** influence is not considered – only generation and load



$$\min_{P_1,\, P_2,\, \ldots,\, P_N} \left(C_1 P_1 + C_2 P_2 + \ldots + C_N P_N \right)$$

subject to:

$$\left.\begin{array}{c} 0 \le P_1 \le \bar{P}_1 \\ 0 \le P_2 \le \bar{P}_2 \\ \vdots \\ 0 \le P_N \le \bar{P}_N \end{array}\right\} \text{Ineq. Constraints}$$

$$P_1 + P_2 + \cdots + P_N = P_{\text{load}} \Big\} \text{ Eq. Constraint}$$

Economic Dispatch Revisited

- Closer look at the optimization problem

unknowns.

Cost or Objective function

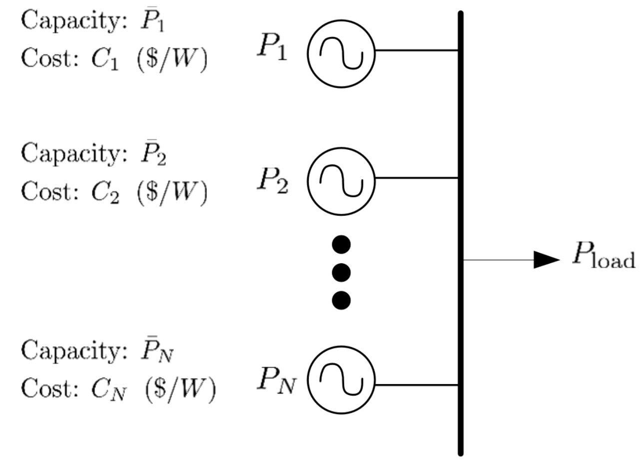
$$\min_{P_1,\, P_2,\, \ldots,\, P_N} \overbrace{\left[C_1 P_1 + C_2 P_2 + \ldots + C_N P_N \right]}^{\text{Cost or Objective function}}$$

subject to:

$$\left.\begin{array}{l} 0 \le P_1 \le \bar{P}_1 \\ 0 \le P_2 \le \bar{P}_2 \\ \quad\vdots \\ 0 \le P_N \le \bar{P}_N \end{array}\right\} \text{Ineq. Constraints.}$$

$$P_1 + P_2 + \cdots + P_N = P_{\text{load}} \;\} \text{ Eq. Constraints.}$$

Constraints



Outline

- Unconstrained optimization $\min_x f(x)$
- Equality constraints – Lagrange Multipliers
- Inequality Constraints – Feasible sets
- Quadratic Programming (Type of optimization problem)
- Economic Dispatch (Application in Power Systems)

General Optimization Problem

- In **general**, an optimization problem has the following form:

cost function

$$\min_x \underline{f(x)}$$

subject to:

$$\begin{aligned} * \; h_i(x) &= 0 \quad i = 1,\ 2,\ \ldots,\ \underline{N_e} \Big\} \text{ Equality constraints} \\ * \; g_j(x) &\le 0 \quad j = 1,\ 2,\ \ldots,\ N_I \Big\} \text{ Inequality "} \end{aligned}$$

- Suppose this occurs at x^*

$\boxed{x^* \ \text{minimizer}}$ or optimal point

$$f(x^*) = \min_x f(x)$$

subject to:

$$\begin{cases} h_i(x) = 0 & i = 1,\ 2,\ \ldots,\ N_e \\ g_j(x) \le 0 & j = 1,\ 2,\ \ldots,\ N_I \end{cases}$$

x^* satisfies all of the constraints

$$\begin{cases} h_i(x^*) = 0 & \forall i \ \checkmark \\ g_j(x^*) \le 0 & \forall j \ \checkmark \end{cases}$$

Unconstrained Optimization Problem

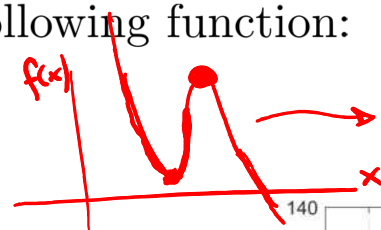
An **unconstrained optimization problem** has the goal of finding the minimum of a function *f(x)*, that is:

$$\min_x f(x)$$

$x \in \mathbb{R}^n \rightarrow x = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$

$\min_{x_1,\dots,x_n} f(x_1, x_2, \cdots, x_n)$

- For example, find the minimum of the following function: $x \in \mathbb{R}$ (one variable)
  - $\min_x \left(3x^2 + 5x + 10\right)$



$f(x) = 5x^3 + 10x^2$

• first order necessary condition for a "minimum"

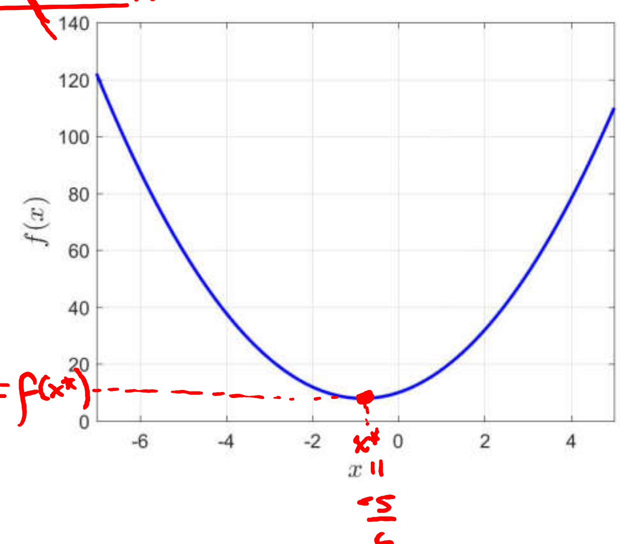
$\rightarrow f'(x) = 0$

$\Rightarrow f'(x) = 6x + 5 = 0 \Rightarrow \boxed{x^* = \frac{-5}{6}}$

• Second order sufficient condition

$\rightarrow f''(x^*) > 0$

$f(x^*) = f\left(\frac{-5}{6}\right)$



- Matlab: fminunc(fun, x0) $\boxed{f''(x) = 6 > 0 \quad \cup}$

$\cap$ < 0

First Order Necessary Conditions for Minimum

- Unconstrained optimization problem: $\min_{x} f(x)$
  - We remember that at x^*, we must have that $f'(x^*) = \underline{\frac{\mathrm{d}f(x^*)}{\mathrm{d}x}} = 0$
- **Why?**

Taylor Expansion @ x^*

$$\underline{f(x)} = \sum_{i=0}^{\infty} \frac{f^{(i)}(x^*)}{i!}(x-x^*)^i = f(x^*) + f'(x^*)(x-x^*) + \frac{f''(x^*)}{2}(x-x^*)^2 + \text{H.O.T.}$$

first degree term

$$f(x) \approx f(x^*) + f'(x^*)(x-x^*)$$

Good approximation for values close to x^*

$\epsilon = x - x^* \Rightarrow e$ is small

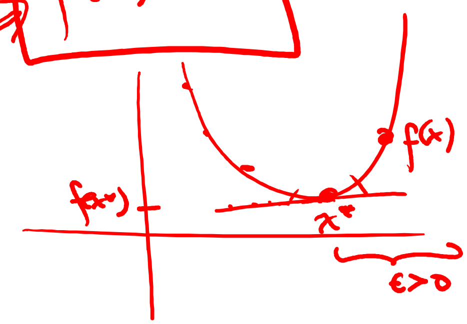
$$* \; f(x) \approx \cancel{f(x^*)} + f'(x^*)e \geq \cancel{f(x^*)}$$

$$\Rightarrow f'(x^*)e \geq 0$$

1 $\begin{cases} \epsilon > 0 \Rightarrow x - x^* > 0 \\ f'(x^*)\epsilon > 0 \Rightarrow f'(x^*) \geq 0 \end{cases}$

2 $\{ \epsilon < 0 \Rightarrow f'(x^*)\epsilon > 0 \Rightarrow f'(x^*) \leq 0$

$$\Rightarrow \boxed{f'(x^*) = 0}$$



- **What is the second order condition?**

$$\boxed{f''(x^*) > 0 \text{ for a minimum}}$$

First Order Necessary Conditions for Minimum

- Unconstrained optimization problem $\min_x f(x)$
  - We remember that at x^*, we must have that $f'(x^*) = \dfrac{\mathrm{d}f(x^*)}{\mathrm{d}x} = 0$

- **More detailed proof:**

$f(x) = f(x^*) + f'(x^*)(x - x^*) + \mathcal{O}(x - x^*)$ we must have that $f(x) \geq f(x^*)$

Let $x = x^* - \epsilon f'(x^*)$ for some $\epsilon > 0$

$$\Rightarrow \quad f(x) = f(x^*) - \epsilon f'(x^*)^2 + \mathcal{O}(\epsilon) \quad \Rightarrow \quad f(x) - f(x^*) = -\epsilon f'(x^*)^2 + \mathcal{O}(\epsilon)$$

$$\Rightarrow \quad \frac{f(x^* - \epsilon f'(x^*)) - f(x^*)}{\epsilon} = -f'(x^*)^2 + \frac{\mathcal{O}(\epsilon)}{\epsilon}$$

$$\Rightarrow \quad \lim_{\epsilon \to 0} \left(\frac{f(x^* - \epsilon f'(x^*)) - f(x^*)}{\epsilon} = -f'(x^*)^2 + \frac{\mathcal{O}(\epsilon)}{\epsilon} \right)$$

$$0 \leq -f'(x^*)^2 \leq 0 \quad \Rightarrow \boxed{f'(x^*) = 0}$$

- **What is the second order condition?**

$f''(x^*) > 0$

First Order Conditions for Minimum with *N* variables

- Unconstrained optimization problem $\min_{x_1,\, x_2,\, \ldots,\, x_N} f(x_1, x_2, \ldots, x_N)$
- **What is the necessary condition for *N* variables?**

$$\nabla_x f = \begin{pmatrix} \frac{\partial f}{\partial x_1} \\ \vdots \\ \frac{\partial f}{\partial x_n} \end{pmatrix} = \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix}$$

$$\boxed{\nabla_x f = 0}$$

Taylor expansion for multiple variables

$$x = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$$

1st Order: $f(x) \approx f(x^*) + \frac{\partial f(x_1^*)}{\partial x_1}(x_1 - x_1^*) + \ldots + \frac{\partial f(x_n^*)}{\partial x_n}(x_n - x_n^*)$

$$\Rightarrow f(x) \approx f(x^*) + \underbrace{\nabla_x f(x^*)}^{T}(x - x^*)$$

- **What is the second order condition?**

positive definite

$$\underbrace{\nabla_x^2 f(x)}_{\text{Hessian Matrix}} = \begin{pmatrix} \frac{\partial^2 f}{\partial x_1^2} & \cdots & \frac{\partial^2 f}{\partial x_1 \partial x_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial^2 f}{\partial x_n \partial x_1} & \cdots & \frac{\partial^2 f}{\partial x_n^2} \end{pmatrix} \succ 0 \rightarrow \boxed{\lambda_i > 0}$$

eigenvalues of Hessian > 0

Unconstrainted Optimization with Multiple Variables

- Suppose the cost of operating generator 1 is: $C_1(x_1) = 3x_1^2 - 30x_1 + 200$

  and the cost of operating generator 2 is: $C_2(x_2) = 3x_2^2 - 24x_2$

- If the two generators are operating in parallel, what is the power of the two generators for which the total cost of operation is a minimum? What is this cost?

Total cost $= f(x) = \underbrace{C_1(x_1) + C_2(x_2)}$ $= 3x_1^2 - 30x_1 + 200 + 3x_2^2 - 24x_2$

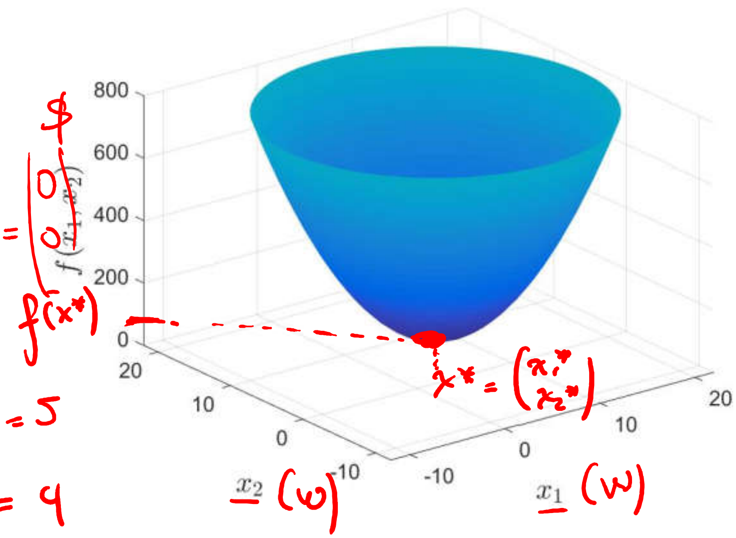
$$\boxed{\min_{x_1, x_2} f(x_1, x_2)}$$

- 1st order test: $\nabla_x f = \underline{0} \Rightarrow \begin{pmatrix} \frac{\partial f}{\partial x_1} \\ \frac{\partial f}{\partial x_2} \end{pmatrix} = \begin{pmatrix} 6x_1 - 30 \\ 6x_2 - 24 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

$x_1^* = 5$

$x_2^* = 4$

$\Rightarrow \begin{cases} 6x_1 - 30 = 0 \Rightarrow x_1^* = 5 \\ 6x_2 - 24 = 0 \Rightarrow x_2^* = 4 \end{cases}$



Unconstrainted Optimization with Multiple Variables

- Suppose the cost of operating generator 1 is: $C_1(x_1) = 3x_1^2 - 30x_1 + 200$

  and the cost of operating generator 2 is: $C_2(x_2) = 3x_2^2 - 24x_2$

- How do we know it is a minimum?

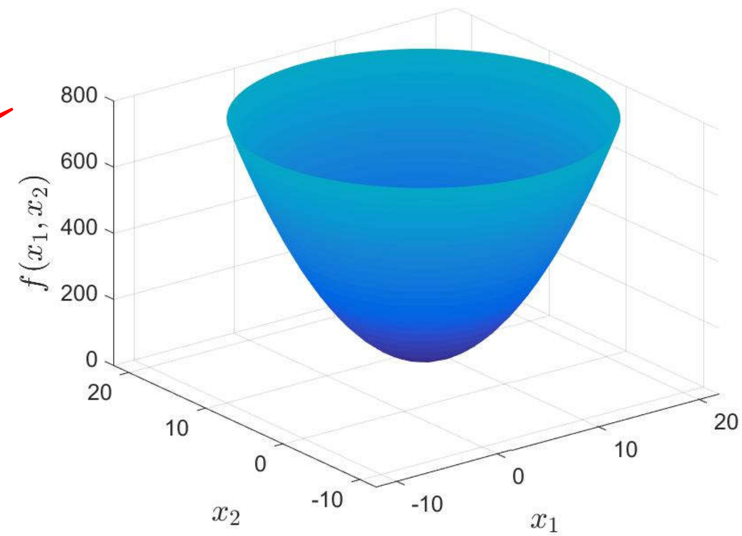
Second order Test

Hessian

$\nabla_x^2 f(x^*) \succ 0$

$$\nabla_x f(x) = \begin{pmatrix} 6x_1 - 30 \\ 6x_2 - 24 \end{pmatrix} = \begin{pmatrix} \frac{\partial f}{\partial x_1} \\ \frac{\partial f}{\partial x_2} \end{pmatrix}$$

$$\begin{pmatrix} \frac{\partial^2 f}{\partial x_1^2} & \frac{\partial^2 f}{\partial x_1 \partial x_2} \\ \frac{\partial^2 f}{\partial x_2 \partial x_1} & \frac{\partial^2 f}{\partial x_2^2} \end{pmatrix} = \underbrace{\begin{pmatrix} 6 & 0 \\ 0 & 6 \end{pmatrix}}$$



eigenvalues $\Rightarrow$ $\lambda_1 = 6 > 0$

$\lambda_2 = 6 > 0$

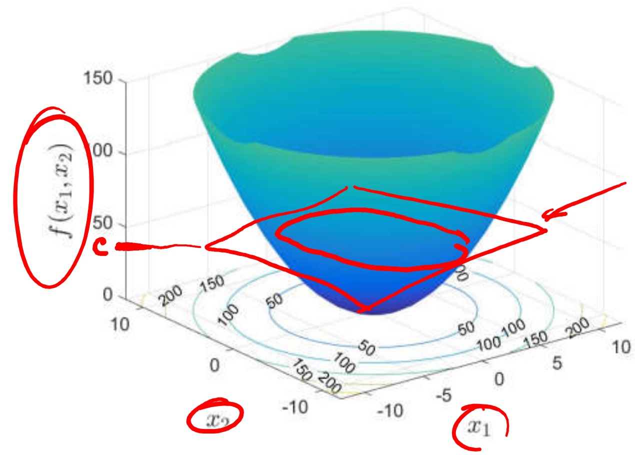
$\rightarrow$ $\boxed{\text{pos. definite}}$ $\Rightarrow$ minimum

Contour Lines and the Gradient

- What are contour lines? or level sets

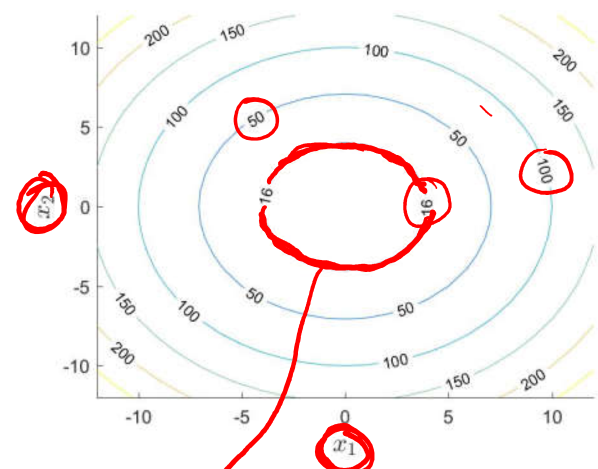
  Curve/points at which the function has the same **value**

$$f(x_1,\ x_2) = x_1^2 + x_2^2$$



- A level set or contour line are all points $x \in \mathbb{R}^n$ ~~that satisfy~~ $f(x) = c$, where c is a constant.

- $\{x \in \mathbb{R}^n \mid f(x) = c\}$ level set



$f(x) = 16$

Examples

$c = 16 \Rightarrow f(x_1, x_2) = 16$

$\Rightarrow x_1^2 + x_2^2 = 16 = 4^2$

circle with radius of 4

$c = 25 \Rightarrow$ circle of radius 5

$c = -16 = x_1^2 + x_2^2$

level set $\overset{?}{=} \phi$ empty null set

Contour Lines and the Gradient

- Relationship to the gradient

$$f(x_1,\ x_2) = x_1^2 + x_2^2$$

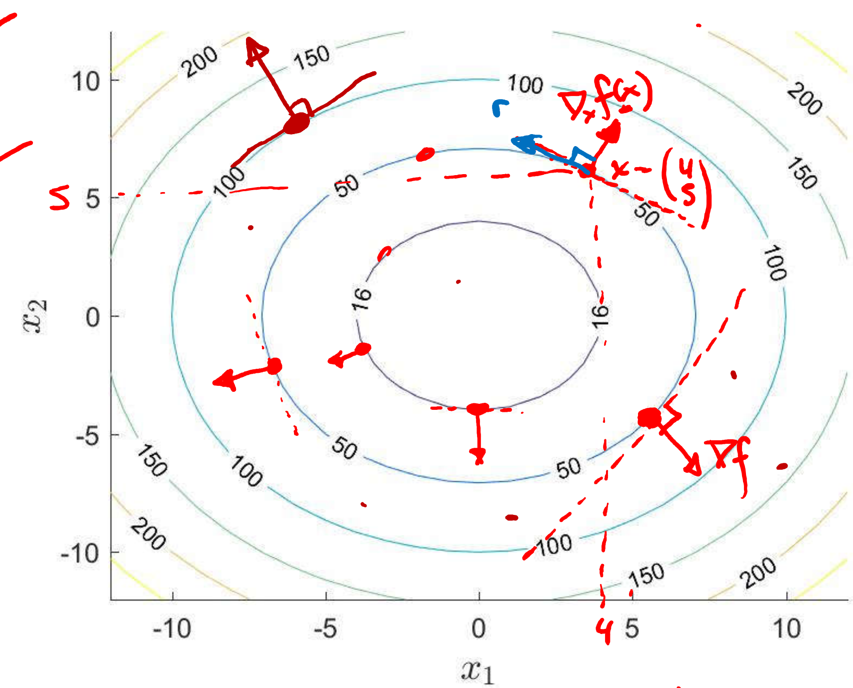
Properties of Gradient (Vector)

1) $\nabla_x f(x)$ shows direction of max increase ✓

2) $\nabla_x f(x)$ is perpendicular to any tangent vector ✓

$$\nabla_x f(x)^T \cdot \vec{r} = 0$$

$\vec{r}$ tangent vector at x



$x = \begin{pmatrix} 4 \\ 5 \end{pmatrix}$

Gradient Descent Algorithm Basics

- What if analytically finding the minimum based on the gradient is complicated?

$$\min_{x_1, x_2, \ldots, x_N} f(x_1, x_2, \ldots, x_N), \quad \underbrace{\nabla_x f(x) = 0}_{\text{solve for } x}$$

$f(x_1, x_2) = \sqrt{\cos(x_1 \log(x_2)) + e^{x_1 x_2}}$

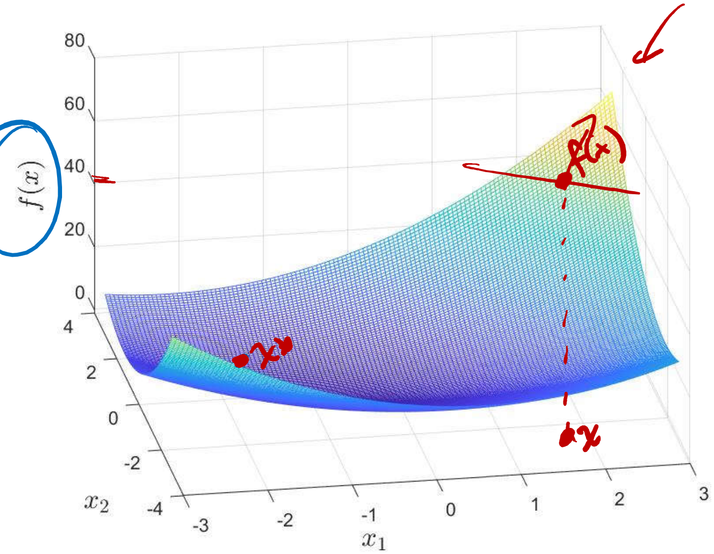
$\nabla_x f = \begin{pmatrix} \frac{\partial f}{\partial x_1} \\ \frac{\partial f}{\partial x_2} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ not always possible to solve

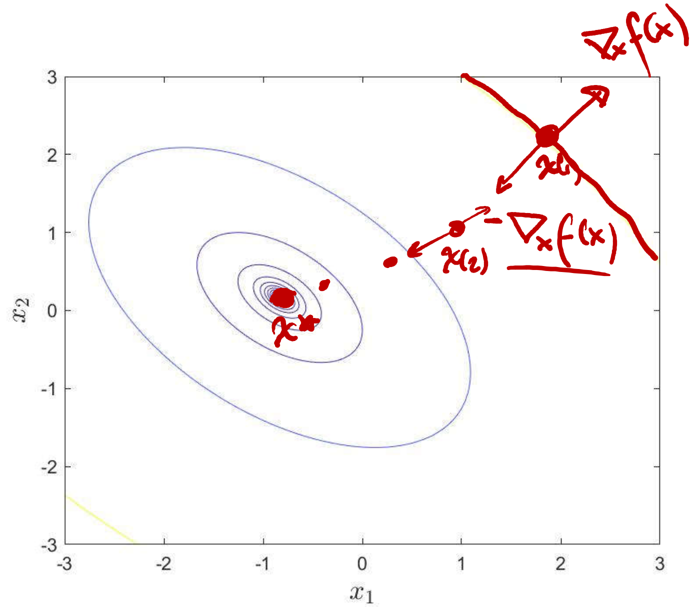
- **Gradient or steepest descent** is a common numerical algorithm to find optimal point
  - Which direction does the gradient point?

    where $f(x)$ increases the most

  - What about the negative of the gradient?

    $-\nabla_x f(x)$ where $f(x)$ decreases the most





Gradient Descent Algorithm

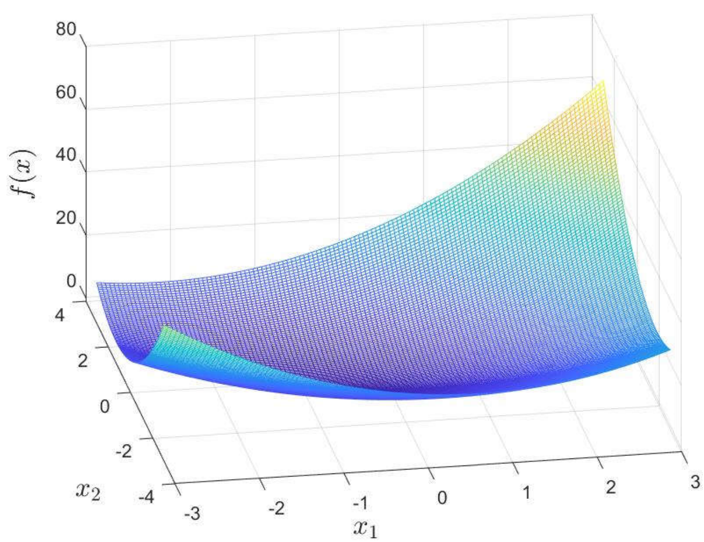
- The gradient points in the direction where the function increases the most
- Therefore, we can take a guess (initial point) and compute the following:

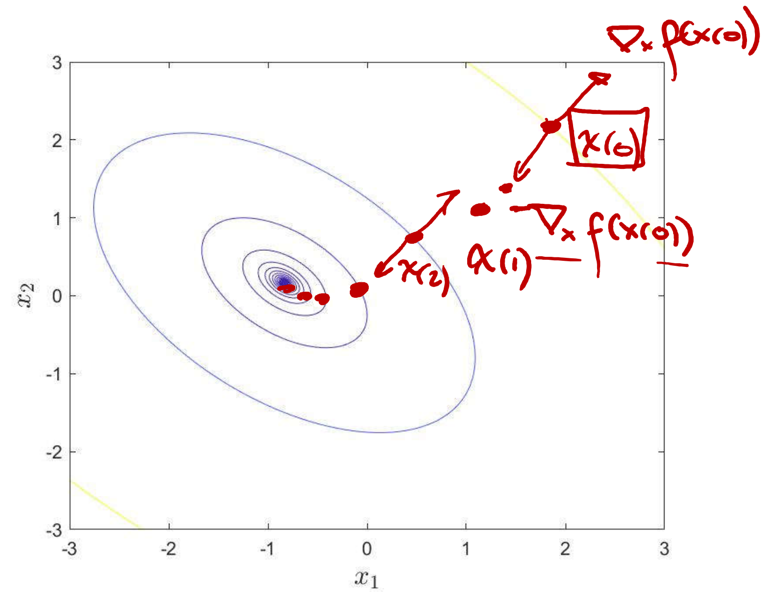
$$\min_{x_1,\, x_2,\, \ldots,\, x_N} \quad f(x_1, x_2, \ldots, x_N)$$

$$\nabla_x f(x) = 0$$

step size α (small)

$$\begin{aligned} x(0) &\quad \text{initial guess} \\ x(1) &= x(0) - \alpha \nabla_x f(x(0)) \\ x(2) &= x(1) - \alpha \nabla_x f(x(1)) \\ &\ \vdots \qquad \vdots \\ x(k+1) &= x(k) - \alpha \nabla_x f(x(k)) \end{aligned}$$



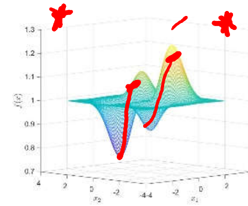


Gradient Descent Algorithm Results

- The gradient points in the direction where the function increases the most
- Therefore, we can take a guess (initial point) and compute the following:

$$\min_{x_1,\, x_2,\, \ldots,\, x_N} f(x_1, x_2, \ldots, x_N)$$

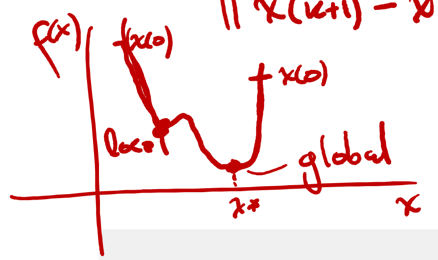
$$\nabla_x f(x) = 0$$

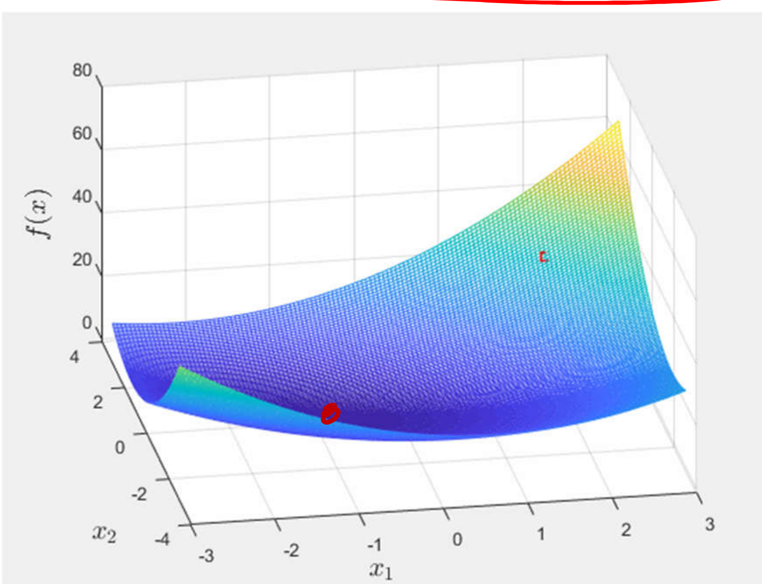


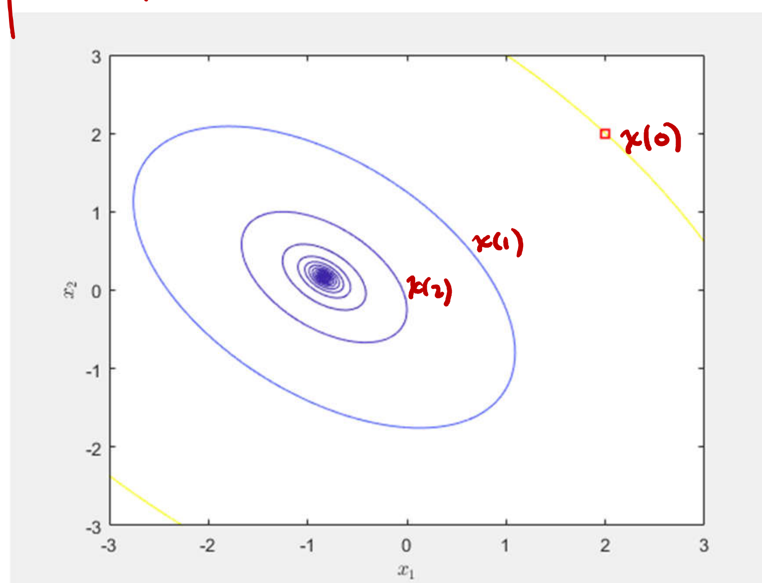
when to stop?

$$\|x(k+1) - x(k)\|_2 \le \epsilon \approx 0.00001$$

$$
\begin{aligned}
&x(0) && \text{initial guess} \\
&x(1) && = x(0) - \alpha \nabla_x f(x(0)) \\
&x(2) && = x(1) - \alpha \nabla_x f(x(1)) \\
&\vdots && \vdots \\
&\boxed{x(k+1) = x(k) - \alpha \nabla_x f(x(k))}
\end{aligned}
$$







Summary for Unconstrained Optimization

- **For an unconstrained optimization problem**

$$\min_{x_1,\, x_2,\, \ldots,\, x_N} f(x_1, x_2, \ldots, x_N)$$

- **First order necessary condition for a local min:**

$$\nabla_x f(x) = 0' \quad \Leftrightarrow \quad \begin{pmatrix} \frac{\partial f}{\partial x_1} \\ \frac{\partial f}{\partial x_2} \\ \vdots \\ \frac{\partial f}{\partial x_N} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \Rightarrow \text{Solve for } x^*$$

- * **Second order condition:** (for a min)

$$\nabla_x^2 f(x) \succ 0 \qquad \begin{pmatrix} \frac{\partial^2 f}{\partial x_1 \partial x_1} & \cdots & \frac{\partial^2 f}{\partial x_1 \partial x_N} \\ \vdots & \ddots & \vdots \\ \frac{\partial^2 f}{\partial x_N \partial x_1} & \cdots & \frac{\partial^2 f}{\partial x_N \partial x_N} \end{pmatrix} \succ 0 \quad \text{Positive definite}$$

Matrix has only positive eigenvalues

Outline

- Unconstrained optimization
- **Equality constraints – Lagrange Multipliers**
- Inequality Constraints – Feasible sets
- Quadratic Programming
- Economic Dispatch

Equality Constrained Optimization Problem

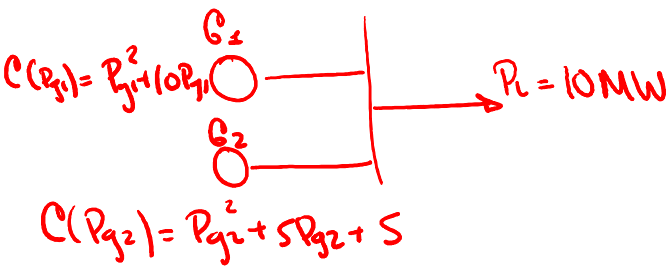
- An optimization problem with equality constrains is defined as follows:

$$\min_{x} f(x)$$

subject to:

$$h_i(x) = 0 \quad i = 1,\ 2,\ \ldots,\ N_e \Big\}\ \text{Equality Constraints}$$

- **Example:**



~~Minimize~~ cost of supplying the load

$$\min_{P_{g1},P_{g2}} \underbrace{\left(P_{g1}^2 + 10P_{g1} + P_{g2}^2 + 5P_{g2} + 5\right)}_{f(x)}$$

s.t.

$$\left(P_{g1} + P_{g2} = 10\,MW\right) \Rightarrow h(x) = 0$$

now we have eq. constraint.

To solve:

Lag. Multiplier

- $\nabla_x f = -\lambda \nabla_x h$
- $h(x) = 0$

solve for P_{g1}, P_{g2}, λ

Lagrange Multipliers

$$\min_x f(x) \qquad \nabla_x f = 0$$

- Suppose that we have the following problem

$$\min_x f(x)$$

subject to:

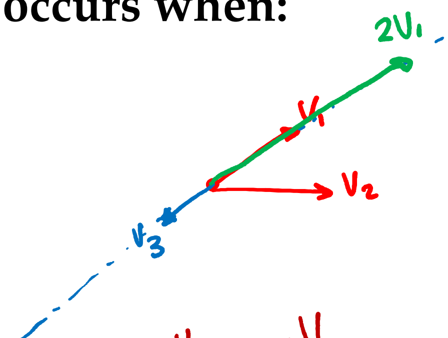
$$h_1(x) = 0 \longrightarrow 1 \text{ eq. constraint.} \qquad \nabla_x f = -\lambda \nabla_x h$$

- **The first order necessary condition for a minimum occurs when:**

$$\boxed{\nabla_x f \parallel \nabla_x h}$$

parallel

$$\boxed{\nabla_x f(x) = -\lambda \nabla_x h(x)}$$



$$v_1 = \alpha v_3$$

$\lambda \in \mathbb{R}$

λ: Lagrange Multiplier

Equality Constrained Opt. Problem Example

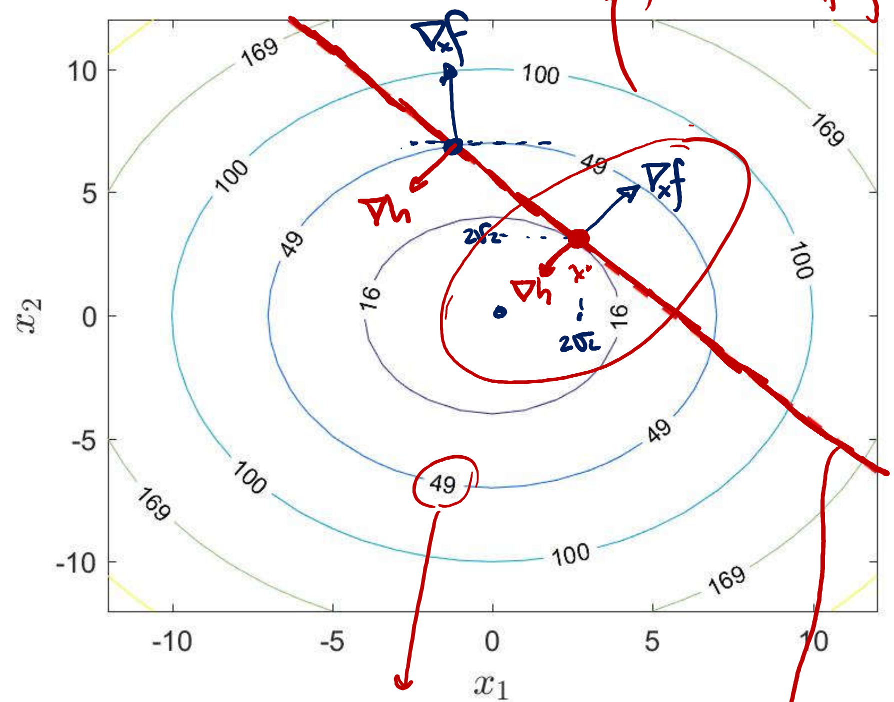
- **Example:** $\min_{x_1,\, x_2} \left(x_1^2 + x_2^2\right)$

  subject to

  • $\left(x_1 + x_2 = 4\sqrt{2}\right) * \cdots \lambda$

  $\underbrace{x_1 + x_2 - 4\sqrt{2}}_{* h(x)} = 0$

level sets of $f(x)$



$f(x) = 49$

$h(x) = 0$

$x_1 + x_2 - 4\sqrt{2} = 0$

First order condition

$\nabla_x f = -\lambda \nabla_x h$

• $\begin{pmatrix} 2x_1 \\ 2x_2 \end{pmatrix} = -\lambda \begin{pmatrix} 1 \\ 1 \end{pmatrix} \Rightarrow$ $2x_1 = -\lambda$, $2x_2 = -\lambda$ $\Big\}$ 2 eqs, 3 unknowns x_1, x_2, λ

+   +

• $x_1 + x_2 - 4\sqrt{2} = 0$ $\}$ 1 equation

$\Longrightarrow$ $\boxed{\begin{aligned} x_1^* &= 2\sqrt{2} \\ x_2^* &= 2\sqrt{2} \\ \lambda^* &= 4\sqrt{2} \end{aligned}}$

Eq. Constrained Optimization Problem

- Suppose that we have the following problem
- The first order necessary condition for a minimum occurs when: $\nabla \mathbf{f} + \lambda \nabla \mathbf{h} = \mathbf{0}$

$$\min_{x_1,\ x_2} f(x_1, x_2)$$

subject to

$$h(x_1,\ x_2) = 0$$

- **Why?**

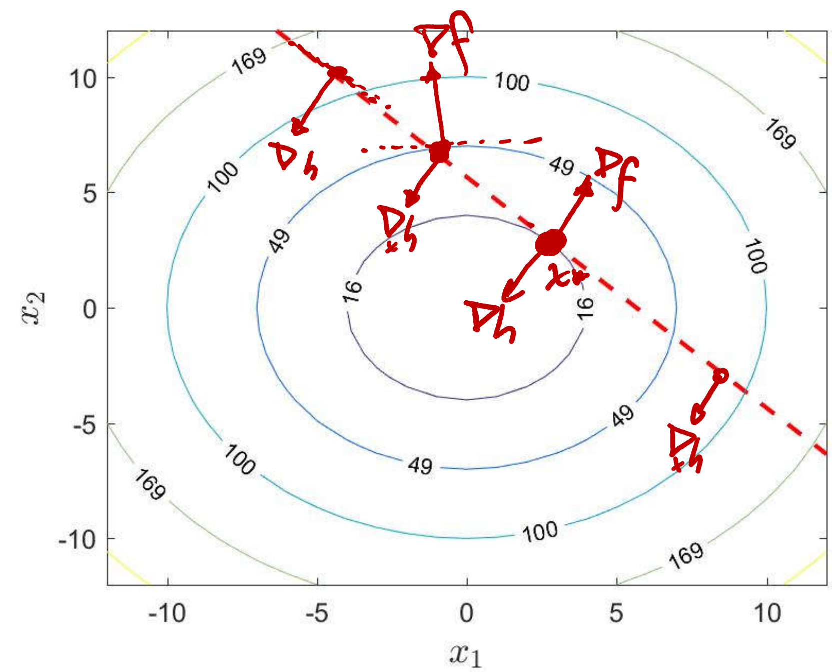
The rate of change of $f(x_1,\ x_2)$ along the constraint curve $h(x_1,\ x_2) = 0$ must be 0

- $\nabla_x f \cdot \underbrace{\vec{r}_h}_{\text{tangent vector to a point in } h(x)=0} = 0$

- But we know from gradient properties

$\nabla_x h \cdot \vec{r}_h = 0$

$\Rightarrow \quad \nabla_x f \parallel \nabla_x h \Rightarrow \boxed{\nabla_x f = -\lambda \nabla_x h}$



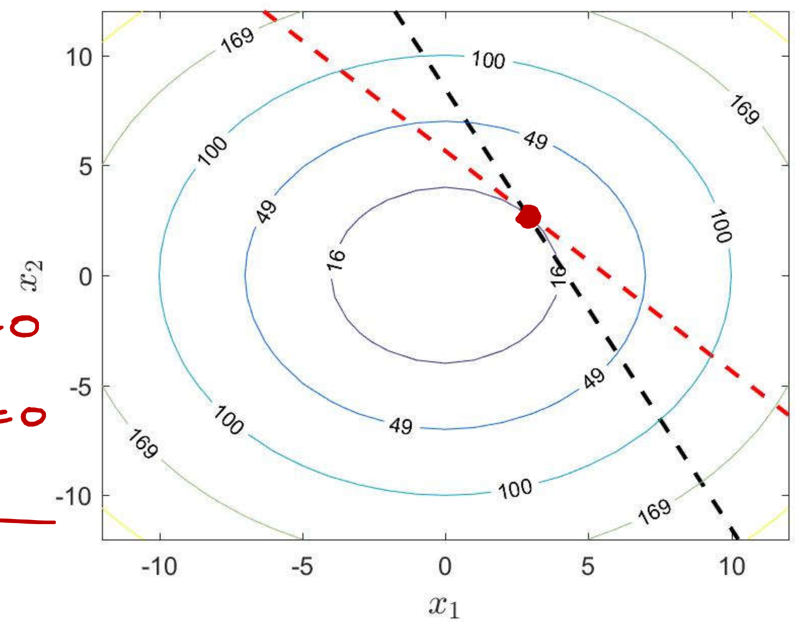
Equality Constrained Opt. Problem Example

- **Example:**

$$\min_{x_1,\, x_2} \boxed{x_1^2 + x_2^2}$$

subject to

$$* \begin{cases} x_1 + x_2 = 4\sqrt{2} & \cdots\ h_1(x) = 0 \Rightarrow x_1 + x_2 - 4\sqrt{2} = 0 \\ 2x_1 + x_2 = 6\sqrt{2} & h_2(x) = 0 \Rightarrow 2x_1 + x_2 - 6\sqrt{2} = 0 \end{cases}$$



First order Necessary condition

$$\rightarrow \nabla_x f = -\lambda_1 \nabla_x h_1 - \lambda_2 \nabla_x h_2 \iff \boxed{\nabla_x f + \lambda_1 \nabla_x h_1 + \lambda_2 \nabla_x h_2 = 0}$$

4 eqs, 4 unknowns

$$\underbrace{\begin{pmatrix} 2x_1 \\ 2x_2 \end{pmatrix}}_{\nabla_x f} + \lambda_1 \underbrace{\begin{pmatrix} 1 \\ 1 \end{pmatrix}}_{\nabla_x h_1} + \lambda_2 \underbrace{\begin{pmatrix} 2 \\ 1 \end{pmatrix}}_{\nabla_x h_2} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow \begin{cases} 2x_1 + \lambda_1 + 2\lambda_2 = 0 \\ 2x_2 + \lambda_1 + \lambda_2 = 0 \\ \bullet\ x_1 + x_2 - 4\sqrt{2} = 0 \quad (h_1(x) = 0) \\ \bullet\ 2x_1 + x_2 - 6\sqrt{2} = 0 \quad (h_2(x) = 0) \end{cases}$$

Optimization Problem with Equality Constraints

- For a general optimization problem with N_e equality constraints:

$$\min_{x_1,\, x_2,\, \ldots,\, x_N} f(x_1,\, x_2,\, \ldots,\, x_N)$$

s.t.

$$\begin{cases} h_1(x_1,\, \ldots,\, x_N) = 0 \quad \cdots \lambda_1 \\ \quad \vdots \\ h_{N_e}(x_1,\, \ldots,\, x_N) = 0 \quad \cdots \lambda_{N_e} \end{cases}$$

Define the Lagrangian function as

$$* \; \mathcal{L}(x,\, \lambda_1,\, \ldots,\, \lambda_{N_e}) = f + \lambda_1 h_1 + \cdots + \lambda_{N_e} h_{N_e}$$

$$\tilde{x} = \begin{pmatrix} x \\ \lambda_1 \\ \vdots \\ \lambda_{N_e} \end{pmatrix} \Rightarrow \mathcal{L}(\tilde{x}) \Rightarrow \nabla_{\tilde{x}} \mathcal{L} \Rightarrow \begin{pmatrix} \nabla_x \mathcal{L} \\ \nabla_{\lambda_1} \mathcal{L} \\ \vdots \\ \nabla_{\lambda_{N_e}} \mathcal{L} \end{pmatrix}$$

- **The first order necessary conditions for a minimum are:**

$$\nabla_x f + \lambda_1 \nabla_x h_1 + \lambda_2 \nabla_x h_2 + \cdots + \lambda_{N_e} \nabla h_{N_e} = 0 \qquad \nabla_x \mathcal{L} = 0$$

$$h_1(x_1,\, \ldots,\, x_N) = 0 \qquad \nabla_{\lambda_1} \mathcal{L} = 0$$

$$\vdots \qquad\qquad \vdots$$

$$h_{N_e}(x_1,\, \ldots,\, x_N) = 0 \qquad \nabla_{\lambda_{N_e}} \mathcal{L} = 0$$

Outline

- Unconstrained optimization

$\min_x f(x)$ $\boxed{\nabla_x f = 0}$

- Equality constraints – Lagrange Multipliers

$\min_x f(x)$ s.t. $h(x) = 0$

$\nabla_x f = -\lambda \nabla_x h$, $h(x) = 0$

- **Inequality Constraints – Feasible sets**

$\min f(x)$ s.t. $h(x) = 0$, $g(x) \le 0$

- Quadratic Programming
- Economic Dispatch

Inequality Constrained Optimization Problem

- Consider an optimization problem with N_I inequality constrains

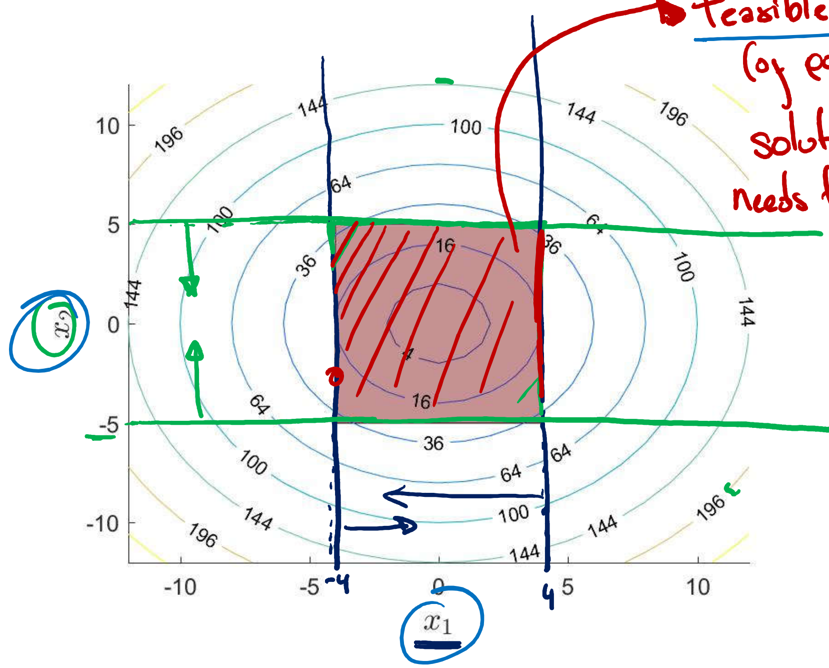
$$\min_x f(x)$$
s.t.
$$g_j(x) \leq 0 \quad \forall j = 1, \ldots, N_I$$

- Consider an optimization problem with N_I inequality constrains

$$\min_{x_1,\, x_2} x_1^2 + x_2^2$$
subject to
$$x_1 \leq 4$$
$$x_1 \geq -4$$
$$x_2 \leq 5$$
$$x_2 \geq -5$$

Feasible set (of points)

solution x^* needs to be inside this set



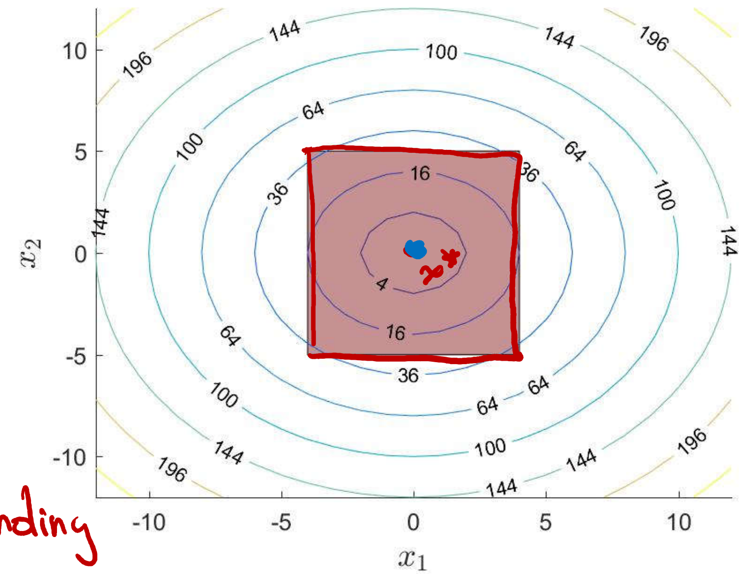
Inequality Constrained Optimization Problem

- Consider an optimization problem with N_I inequality constrains

$$\min_{x_1,\, x_2} \; x_1^2 + x_2^2$$

subject to

$$\begin{cases} x_1 \leq 4 \\ x_1 \geq -4 \\ x_2 \leq 5 \\ x_2 \geq -5 \end{cases}$$



- Attempt # 1 assume inequalities are not active/binding

$\Downarrow$ $\Rightarrow$ ignore the inequalities

$$\min_{x} \underbrace{(x_1^2 + x_2^2)}_{f(x)} \Rightarrow \nabla_x f = 0 \Rightarrow \begin{pmatrix} 2x_1 \\ 2x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow \boxed{\begin{matrix} x_1^* = 0 \\ x_2^* = 0 \end{matrix}}$$

Does it satisfy inequalities? (inside feasible set?) Yes $\Rightarrow$ also the solution to the original problem

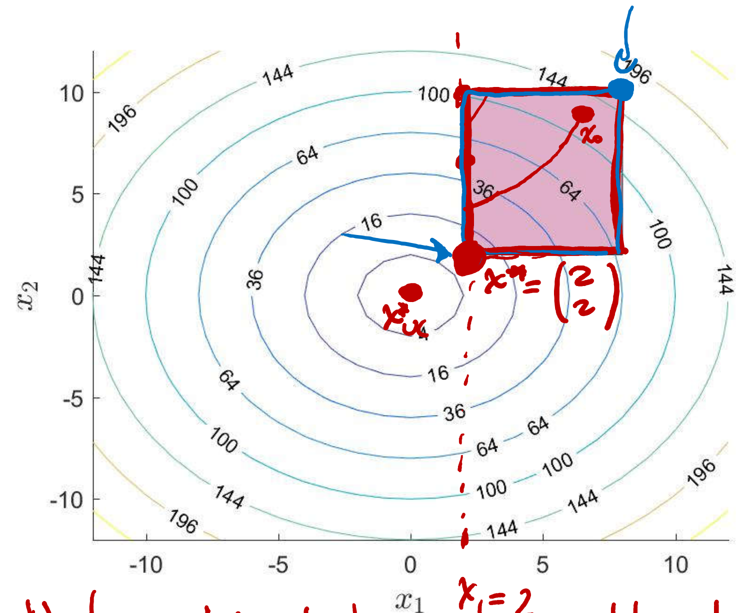
Inequality Constrained Optimization Problem

- Consider an optimization problem with N_I inequality constrains

$$\min_{x_1,\, x_2} \; x_1^2 + x_2^2$$

subject to

$$x_1 \le 8 \quad \text{Not binding}$$
$$\boxed{x_1 \ge 2} \quad \text{Binding}$$
$$x_2 \le 10 \quad \text{Not binding}$$
$$\boxed{x_2 \ge 2} \quad \text{Binding}$$



- **Attempt 1:** ignore constraints, $x_{uc}^* = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

  Does it satisfy inequalities? No! $x_1^* = 0 \not\geq 2$, $x_2^* = 0 \not\geq 2$. Not a solution to original problem!

- **Attempt 2:** Assume some inequalities are "active"/"binding".

  By an algorithm $x^* = \begin{pmatrix} 2 \\ 2 \end{pmatrix} \Rightarrow$ ineqs. $\begin{matrix} x_1 \ge 2 \\ x_2 \ge 2 \end{matrix}$ are "binding" $\Rightarrow$ become equalities

  $x_1 = 2$
  $x_2 = 2$

$$\min \; x_1^2 + x_2^2$$
s.t.
$$\left.\begin{matrix} x_1 = 2 \\ x_2 = 2 \end{matrix}\right\}$$

Karush Kuhn Tucker (KKT) Conditions for Optimality

- For a general optimization problem with N_e equality constraints:

$$\min_x f(x)$$

s.t.

$$h_i(x) = 0 \quad i = 1, \ldots, N_e \quad \cdots \lambda = \begin{pmatrix} \lambda_1 \\ \vdots \\ \lambda_{N_e} \end{pmatrix}$$

$$g_j(x) \le 0 \quad j = 1, \ldots, N_I \quad \cdots \underbrace{\mu = \begin{pmatrix} \mu_1 \\ \vdots \\ \mu_{N_I} \end{pmatrix}}$$

Lagrangian function

Lag. Mult.   Lag. Mult.

$$\mathcal{L}(x, \lambda, \mu) = f(x) + \underbrace{\sum_{i=1}^{N_e} \lambda_i h_i(x)}_{\text{equality const.}} + \underbrace{\sum_{j=1}^{N_I} \mu_j g_j(x)}_{\text{ineq. constraints.}}$$

- **The first order necessary conditions for a minimum are:**

$$\nabla_x \mathcal{L} = 0 \Rightarrow \nabla_x f + \sum_{i=1}^{N_e} \lambda_i \nabla_x h_i + \sum_{j=1}^{N_I} \mu_j \nabla_x g_j = 0$$

$$h_i(x) = 0 \ \forall\, i = 1, \ldots, N_e$$

$$g_j(x) \le 0 \ \forall\, j = 1, \ldots, N_I$$

"Primal feasibility"

$$\mu_j \ge 0 \ \forall\, j = 1, \ldots, N_I \rightarrow \text{"Dual feasibility"}$$

$$\mu_j g_j(x) = 0 \ \forall\, j = 1, \ldots, N_I \rightarrow \text{"Complementary Slackness"}$$

associated w/ ineq.

$\mu_j = 0 \Rightarrow \mu_j g_j(x) = 0$

$g_j(x) < 0$ Not b.

$\mu_j > 0 \Rightarrow \mu_j g_j = 0$

$\rightarrow g_j(x) = 0$ Binding

Inequality Constrained Optimization Problem

- Consider an optimization problem with 4 inequality constrains

$$\min_{x_1,\, x_2} x_1^2 + x_2^2$$

subject to

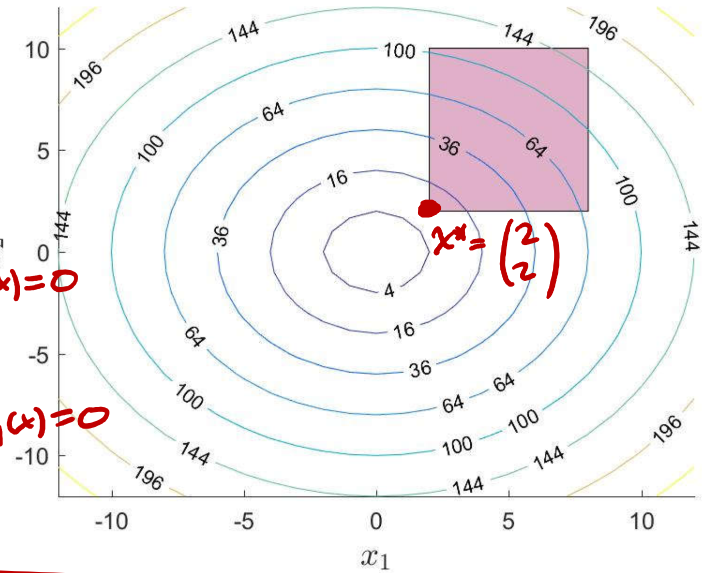
$x_1 \leq 8 \longrightarrow x_1 - 8 \leq 0 \qquad g_1(x) \leq 0 \;,\; \mu_1 = 0$

$* \; x_1 \geq 2 \longrightarrow 2 - x_1 \leq 0 \qquad g_2(x) \leq 0 \;,\; \mu_2 > 0 \Rightarrow g_2(x) = 0$

$x_2 \leq 10 \longrightarrow x_2 - 10 \leq 0 \qquad g_3(x) \leq 0 \;,\; \mu_3 = 0$

$* \; x_2 \geq 2 \longrightarrow 2 - x_2 \leq 0 \qquad g_4(x) \leq 0 \;,\; \mu_4 > 0 \Rightarrow g_4(x) = 0$

$g_j(x) \leq 0$ for $j = 1, 2, 3, 4 \qquad \mu_j \; 1, \cdots 4$



KKT 1st Order Conds.

$$\left\{ \begin{array}{l} \mu_j \geq 0 \;\; \forall \, j = 1, \, ..., \, N_I \\ \mu_j g_j(x) = 0 \;\; \forall \, j = 1, \, ..., \, N_I \end{array} \right.$$

$\mu_j g_j = 0$	$\mu_j g_j = 0$	$\mu_j = 0 \;\; g_j = 0$
$\mu_j = 0$ **or**	$g_j(x) = 0$	**or both**
$\Rightarrow g_j < 0$	$\mu_j > 0$	
Not binding	Binding	

Applies independently to each ineq. constraint

General Optimization Problem Example

- Consider an optimization problem with N_I inequality constrains

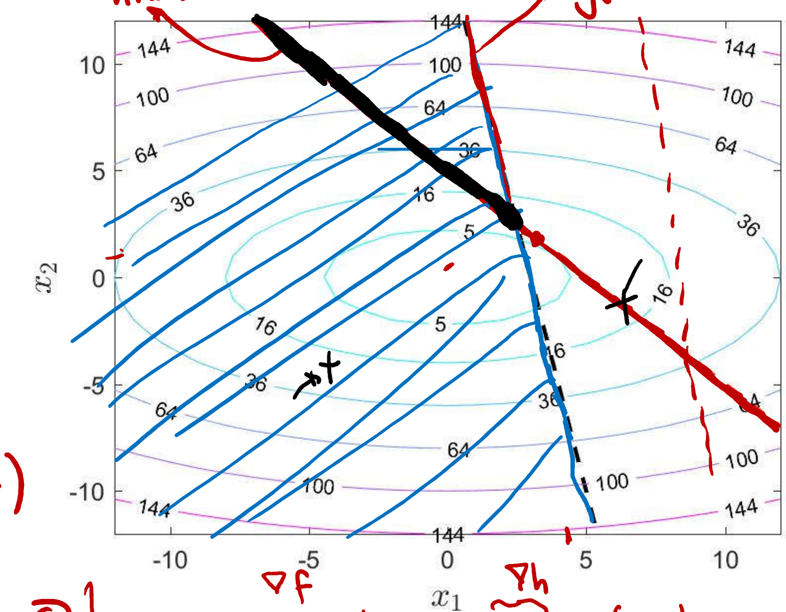
$$\min_{x_1,\, x_2} \left(0.25x_1^2 + x_2^2\right)$$

subject to

$\rightarrow 5 - x_1 - x_2 = 0$ ... $\lambda_1 \Longleftrightarrow h_1(x) = 0$

$\rightarrow \underbrace{x_1 + 0.2x_2 - 3}_{g_1(x)} \le 0$ ... $\mu_1 \Longleftrightarrow g_1(x) \le 0$

$h_1(x)=0$ $\quad$ $g_1(x) \le 0$



Attempt 1: Assume ineq. is not binding (ignore it)

$$\left.\begin{aligned} &\min \; 0.25x_1^2 + x_2^2 \\ &\text{s.t.} \\ &\quad 5 - x_1 - x_2 = 0 \end{aligned}\right\} \Rightarrow \left.\begin{aligned} &\nabla_x f + \lambda_1 \nabla_x h = 0 \\ &h(x) = 0 \end{aligned}\right\} \Rightarrow \underbrace{\begin{pmatrix} 0.5x_1 \\ 2x_2 \end{pmatrix}}_{\nabla f} + \lambda \underbrace{\begin{pmatrix} -1 \\ -1 \end{pmatrix}}_{\nabla h} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$5 - x_1 - x_2 = 0$$

$x_1^* = 4$

$x_2^* = 1$

$\lambda^* = 2$

$\boxed{\text{Does it satisfy ineq? No}}$

$g_1(x^*) = 4 + 0.2(1) - 3 = 1.2 \not\le 0$

General Optimization Problem Example

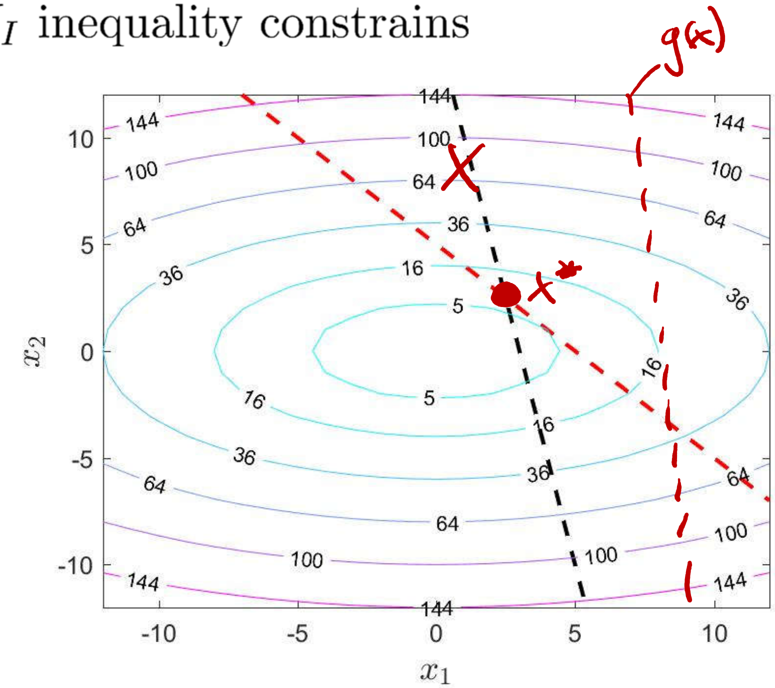
- Consider an optimization problem with N_I inequality constrains

$$\min_{x_1,\, x_2} 0.25x_1^2 + x_2^2$$

subject to

$5 - x_1 - x_2 = 0$ *

$x_1 + 0.2x_2 - 3 \boxed{\leq} 0$



Attempt 2 Assume ineq. is "binding"

$\Rightarrow \min 0.25x_1^2 + x_2^2$

s.t.

$h(x) = 5 - x_1 - x_2 = 0 \cdots \lambda_1$

$g(x) = x_1 + 0.2x_2 - 3 = 0 \cdots \mu_1$

$$\begin{cases} \nabla_x f + \lambda_1 \nabla_x h + \mu_1 \nabla_x g = 0 \\ h(x) = 0 \\ g(x) = 0 \end{cases}$$

4 eqs, 4 unk.

$$\begin{cases} \underbrace{\begin{pmatrix} 0.5x_1 \\ 2x_2 \end{pmatrix}}_{\nabla f} + \lambda \underbrace{\begin{pmatrix} -1 \\ -1 \end{pmatrix}}_{\nabla h} + \mu \underbrace{\begin{pmatrix} 1 \\ 0.2 \end{pmatrix}}_{\nabla g} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \\ h(x) = 0 \\ g(x) = 0 \end{cases}$$

*Solution

$$\boxed{\begin{aligned} x_1^* &= 2.5 \\ x_2^* &= 2.5 \\ \lambda^* &= 5.94 \\ \mu^* &= 4.7 > 0 \end{aligned}}$$

Feasible Set Definition

- For a general optimization problem with N_e equality constraints/ N_I inequality constraints

$$\min_x f(x)$$

s.t.

$$\begin{cases} h_i(x) = 0 \quad i = 1, \; \ldots, \; N_e \\ g_j(x) \leq 0 \quad j = 1, \; \ldots, \; N_I \end{cases}$$

Define a "feasible set": The set of all x that satisfy the constraints

- The **feasible set** is defined as follows:

$$\mathcal{X} \triangleq \{\underbrace{x \in \mathbb{R}^n}_{\text{The set of all } x\in\mathbb{R}^n} \;\Big|\; \underbrace{h_i(x) = 0 \text{ for } i = 1, \; \ldots, \; N_e}_{\text{equality functions are satisfied}}, \; \underbrace{g_j(x) \leq 0 \text{ for } j = 1, \; \ldots, \; N_I}_{\text{ineq. functions are satisfied}}\}$$

such that

Feasible Set Example

- Plot the feasible set for the following problem:

$x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$

$$\min_{x_1,\, x_2} \; x_1^2 + x_2^2$$

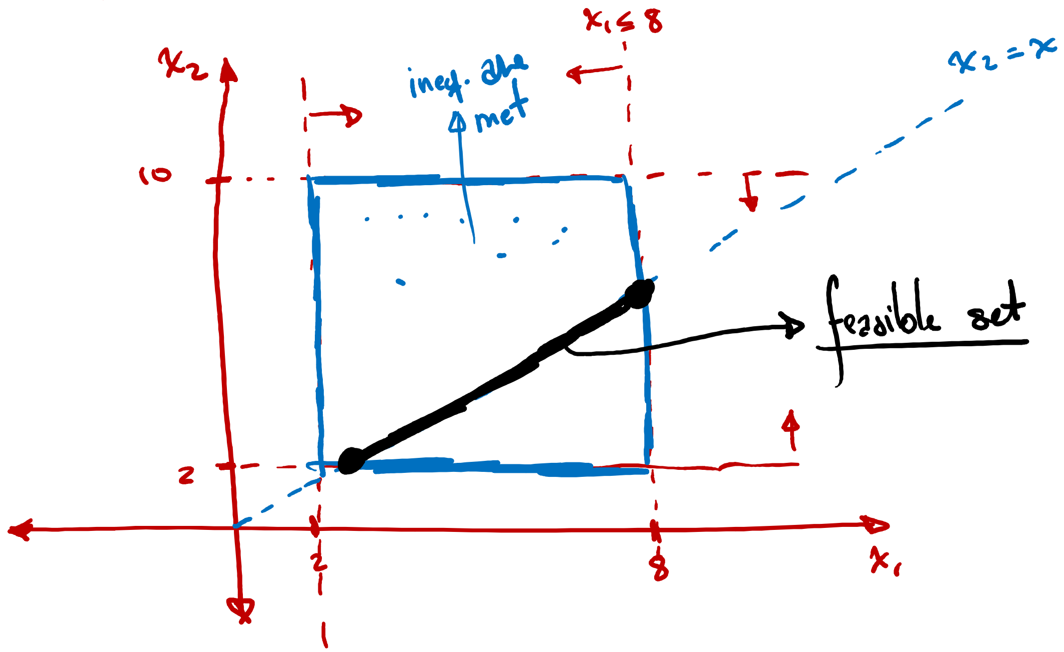
subject to

$$\left\{\begin{aligned} x_1 &\le 8 \\ x_1 &\ge 2 \\ x_2 &\le 10 \\ x_2 &\ge 2 \\ -x_1 + x_2 &= 0 \end{aligned}\right.$$

$x_2 = x_1$

"$y = x$"

$X = \{ x \in \mathbb{R}^2 \mid x_1 \le 8,\ x_1 \ge 2,\ x_2 \le 10,\ x_2 \ge 2,\ -x_1 + x_2 = 0 \}$



Outline

- Unconstrained optimization
- Equality constraints – Lagrange Multipliers
- Inequality Constraints – Feasible sets
- **Quadratic Programming**
- **Economic Dispatch**

Quadratic Programming Definition

- A **quadratic programming** (QP) **problem** is a type of **optimization problem** which has a quadratic cost function with **linear inequality** and **linear equality constraints**

Type of opt. Problem

General Optimization Problem

$$\min_x f(x)$$

s.t.

$$h_i(x) = 0 \quad i = 1, \ldots, N_e$$

$$g_j(x) \le 0 \quad j = 1, \ldots, N_I$$

$\supset$

Quadratic Programming Problem

$$\min_x \left(\frac{1}{2} x^T H x + f^T x \right) \rightarrow \text{cost is quadratic}$$

s.t.

$$Ax \le b \; \} \text{ ineq. are linear}$$

$$A_{eq} x = b_{eq} \; \} \text{ eq. are linear}$$

Examples: - Support Vector Machine is a quadratic prog. prob.

- Least squares QP

Quadratic Programming Example

- Write the following problem in QP form:

$$\min_{x} \left(\underbrace{x_1^2 + x_2^2 + 4x_1x_2} + \underbrace{10x_1 + 12x_2}\right) \quad \text{cost function is quadratic}$$

s.t.

$$\begin{aligned} * \quad & x_1 \leq 10 \\ & x_1 \geq 2 \quad \rightarrow \quad -x_1 \leq -2 \\ & x_2 \leq 20 \\ & x_2 \geq 2 \quad \rightarrow \quad -x_2 \leq -2 \end{aligned} \Bigg\} \text{ linear ineq.}$$

$$x_1 + x_2 = 5 \} \text{ linear eq.}$$

QP form:

$$\boxed{\begin{aligned} & \min_{x} \ \tfrac{1}{2} x^T \underline{H} x + \underline{c}^T x \\ & \text{s.t.} \\ & \quad *Ax \leq b* \\ & \quad A_{eq} x = b_{eq} \end{aligned}}$$

$$f(x) = \tfrac{1}{2} x^T H x$$

$$\nabla_x f = \tfrac{1}{2}(\cancel{2} H x)$$

$$\min_{x \in \mathbb{R}^2} \ \boxed{\tfrac{1}{2}} (x_1 \ x_2)\, 2 \cdot \underbrace{\begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}}_{H} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \underbrace{(10 \quad 12)}_{c^T} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

s.t.

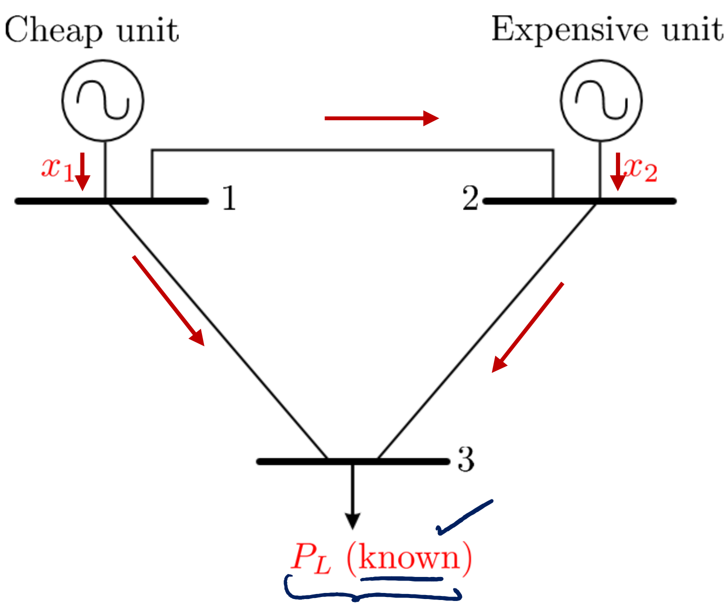
$$\begin{array}{cc} & 2 \\ 4 & \underbrace{\begin{pmatrix} 1 & 0 \\ -1 & 0 \\ 0 & 1 \\ 0 & -1 \end{pmatrix}}_{\underline{A}} \end{array} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \leq \underbrace{\begin{pmatrix} 10 \\ -2 \\ 20 \\ -2 \end{pmatrix}}_{\underline{b}}, \quad \underbrace{(1 \quad 1)}_{A_{eq}} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \underbrace{(5)}_{b_{eq}}$$

Economic Dispatch Example

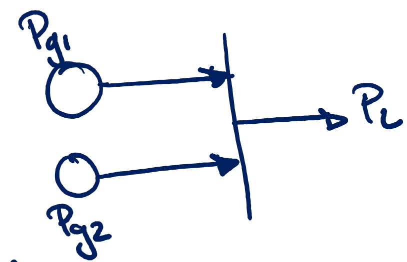
- **Economic Dispatch** is to find out, for a **single period of time**, the output power of every generation unit so that demands are satisfied at a **minimum costs**

load

- Example formulation #1:



ignore network



$$\min_{P_{g1}, P_{g2}} f_1(P_{g1}) + f_2(P_{g2})$$

s.t.

$$P_{g1\text{-}min} \le P_{g1}, \quad P_{g1} \le P_{g1\text{-}max}$$

$$P_{g2\text{-}min} \le P_{g2}, \quad P_{g2} \le P_{g2\text{-}max}$$

$$P_{g1} + P_{g2} = P_L$$

Example Economic Dispatch

- Suppose we have $\underbrace{\text{three generator units}}_{}$ (Thermal) with the following characteristics

Heat as a function of output power

Unit 1: Coal/steam unit

$$H_1\left(\frac{\text{MBtu}}{\text{h}}\right) = 510 + 7.2P_1 + 0.00142P_1^2$$

Fuel cost $= 1.1$ \$/MBtu

Cost as function of power output

$$F_1(P_1) = \underbrace{1.1}_{(\$/\text{MBtu})} \times \underbrace{H_1(P_1)}_{(\text{MBtus}/\text{MW})} = 561 + 7.92P_1 + 0.001562P_1^2 \quad \left(\frac{\$}{\text{h}}\right)$$

Unit 2: Oil/steam unit

$$H_2\left(\frac{\text{MBtu}}{\text{h}}\right) = 310 + 7.85P_2 + 0.00194P_2^2$$

Fuel cost $= 1.0$ \$/MBtu

Cost as a function of P_2

$$F_2(P_2) = 1 \times H_2(P_2) = 310 + 7.85P_2 + 0.00194P_2^2 \quad \left(\frac{\$}{\text{h}}\right)$$

Unit 3: Oil/steam unit

$$H_3\left(\frac{\text{MBtu}}{\text{h}}\right) = 78 + 7.97P_3 + 0.00482P_3^2$$

Fuel cost $= 1.0$ \$/MBtu

cost as a function of P_3

$$F_3(P_3) = 1 \times H_3(P_3) = 78 + 7.97P_3 + 0.00482P_3^2 \quad \left(\frac{\$}{\text{h}}\right)$$

Example Economic Dispatch

- Suppose we have three generator units with the following characteristics

Unit 1: Coal/steam unit 1.1

$$F_1(P_1) = 561 + 7.92P_1 + 0.001562P_1^2 \quad \left(\frac{\$}{\text{h}}\right)$$

Max output: 600 MW
Min Output: 150 MW

Unit 2: Oil/steam unit

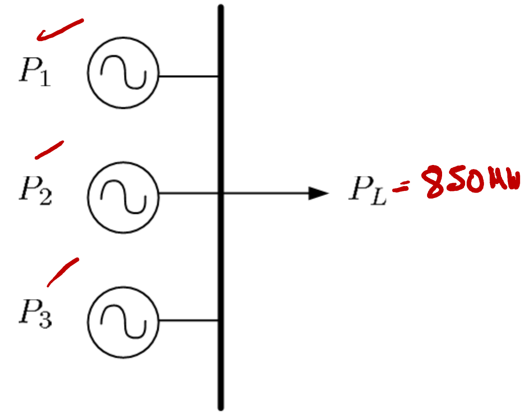
$$F_2(P_2) = 310 + 7.85P_2 + 0.00194P_2^2 \quad \left(\frac{\$}{\text{h}}\right)$$

Max output: 400 MW
Min Output: 100 MW

Unit 3: Oil/steam unit

$$F_3(P_3) = 78 + 7.97P_3 + 0.00482P_3^2 \quad \left(\frac{\$}{\text{h}}\right)$$

Max output: 200 MW
Min Output: 50 MW



- The three generators feed a load of $P_L = 850$ MW. What is the power level of each generator to minimize the cost of operation?

1) Write the opt. problem

$$\min_{P_1, P_2, P_3} \left[\underbrace{F_1(P_1) + F_2(P_2) + F_3(P_3)} \right]$$

s.t.

$P_1 + P_2 + P_3 = 850$ } 1 eq. const. λ_1

$P_1 \leq 600$
$P_1 \geq 150$
$P_2 \leq 400$
$P_2 \geq 100$
$P_3 \leq 200$, $P_3 \geq 50$

6 ineq. constraint $\mu_1, \mu_2, \ldots, \mu_6$

Attempt 1: assume ineqs. are not binding (ignore)

$$\min \; F_1(P_1) + F_2(P_2) + F_3(P_3) = \tilde{F}(P_1, P_2, P_3)$$

s.t. $(P_1 + P_2 + P_3 = 850) \rightarrow (h(x) = 0)$

$\nabla_x \tilde{F} + \lambda \nabla_x h = 0$, $h(x) = 0$

$P_1^* = 393$
$P_2^* = 334$
$P_3^* = 122$
$\lambda^* = ?$

Does it satisfy ineqs? yes!

Example Economic Dispatch v2 with Capacity Constraints

- Suppose we have three generator units with the following characteristics

Unit 1: Coal/steam unit Coal: 0.9 \$/MBtu

$F_1(P_1) = 459 + 6.48P_1 + 0.00128P_1^2 \left(\frac{\$}{\mathrm{h}}\right)$

Max output: 600 MW*
Min Output: 150 MW

Unit 2: Oil/steam unit

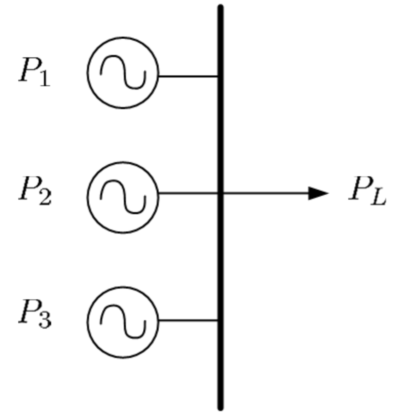
$F_2(P_2) = 310 + 7.85P_2 + 0.00194P_2^2 \left(\frac{\$}{\mathrm{h}}\right)$

Max output: 400 MW
Min Output: 100 MW

Unit 3: Oil/steam unit

$F_3(P_3) = 78 + 7.97P_3 + 0.00482P_3^2 \left(\frac{\$}{\mathrm{h}}\right)$

Max output: 200 MW
Min Output: 50 MW*



- The three generators feed a load of $P_L = 850$ MW. What is the power level of each generator to minimize the cost of operation?

We have a different cost function than prev. slide

Attempt 1: ignore inequalities

$$\min_{P_1,P_2,P_3} (F_1 + F_2 + F_3)$$
s.t. $P_1 + P_2 + P_3 = 850$

$$\nabla_x (F_1 + F_2 + F_3) + \lambda \nabla_x h = 0$$
$$h(x) = 0$$

$\boxed{P_1^* = 704*} \not\leq 600$

$P_2^* = 111$

$P_3^* = 32 \not\geq 50$

* Did not meet ineq. const.
* Cannot be the solution.

Example Economic Dispatch v2 with Capacity Constraints

- Suppose we have three generator units with the following characteristics

Unit 1: Coal/steam unit Coal: 0.9 \$/MBtu

$F_1(P_1) = 459 + 6.48P_1 + 0.00128P_1^2 \quad \left(\frac{\$}{\text{h}}\right)$ Max output: 600 MW; Min Output: 150 MW

$P_1 \geq 150$

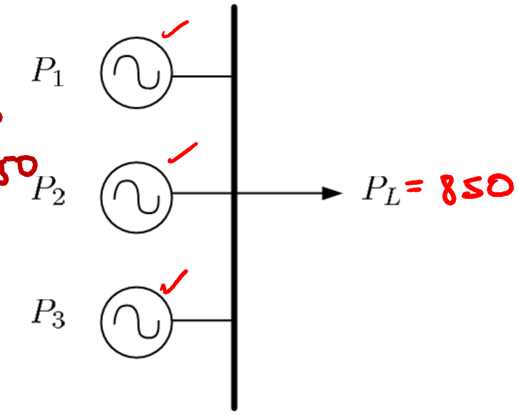
$-P_1 \leq -150$

Unit 2: Oil/steam unit

$F_2(P_2) = 310 + 7.85P_2 + 0.00194P_2^2 \quad \left(\frac{\$}{\text{h}}\right)$ Max output: 400 MW; Min Output: 100 MW

Unit 3: Oil/steam unit

$F_3(P_3) = 78 + 7.97P_3 + 0.00482P_3^2 \quad \left(\frac{\$}{\text{h}}\right)$ Max output: 200 MW; Min Output: 50 MW



- The three generators feed a load of $P_L = 850$ MW. What is the power level of each generator to minimize the cost of operation?

Use Matlab to solve this problem (QP)

$$\begin{aligned} \min\ & \tfrac{1}{2}x^T H x + c^T x \\ \text{s.t.}\ & Ax \leq b \\ & A_{eq}\, x = b_{eq} \end{aligned}$$

$$\min_{x_1, x_2, x_3} \frac{1}{2} \begin{pmatrix} x_1 & x_2 & x_3 \end{pmatrix} \underbrace{2 \cdot \begin{pmatrix} 0.00128 & 0 & 0 \\ 0 & 0.00194 & 0 \\ 0 & 0 & 0.00482 \end{pmatrix}}_{H} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} + \underbrace{\begin{pmatrix} 6.48 & 7.85 & 7.97 \end{pmatrix}}_{c^T} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

s.t.

$$\underbrace{\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}}_{A} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \leq \underbrace{\begin{pmatrix} 600 \\ 400 \\ 200 \\ -150 \\ -100 \\ -50 \end{pmatrix}}_{b} \begin{matrix} \mu_1 \geq 0 \\ \mu_2 \\ \vdots \\ \\ \\ \mu_6 \end{matrix}$$

$$\left[\underbrace{\begin{pmatrix} 1 & 1 & 1 \end{pmatrix}}_{A_{eq}} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \underbrace{850}_{b_{eq}} \right] \quad \lambda_1$$

quadprog

$\mu_i > 0 \Rightarrow i^{th}$ ineq is binding (equality)

Example Economic Dispatch v2 with Capacity Constraints

- Suppose we have three generator units with the following characteristics

$F_1(P_1) = 459 + 6.48P_1 + 0.00128P_1^2 \ \left(\frac{\$}{\text{h}}\right)$	Max output: 600 MW Min Output: 150 MW
$F_2(P_2) = 310 + 7.85P_2 + 0.00194P_2^2 \ \left(\frac{\$}{\text{h}}\right)$	Max output: 400 MW Min Output: 100 MW
$F_3(P_3) = 78 + 7.97P_3 + 0.00482P_3^2 \ \left(\frac{\$}{\text{h}}\right)$	Max output: 200 MW Min Output: 50 MW

- The three generators feed a load of $P_L = 850$ MW. What is the power level of each generator to minimize the cost of operation?

- Matlab code:

```
% Economic dispatch with quadratic costs
% Quadprog needs min 0.5*x'*H*x + f'x
H = diag([2*0.00128, 2*0.00194, 2*0.00482]);
f = [6.48; 7.85; 7.97];

Aeq = [1 1 1];
beq = 850;

lb = [150 100 50]';
ub = [600 400 200]';

[x,fvalx, exitflag , output, lambda] = ...
    quadprog(H, f, [], [], Aeq, beq, lb, ub)
```

$\boxed{lb \le x \ , \ x \le ub}$ or rather than $\boxed{Ax \le b}$

```
x =
  600.0000
  187.1302
   62.8698
```

$= P_1^*$, $= P_2^*$, $= P_3^*$

satisfy constraints

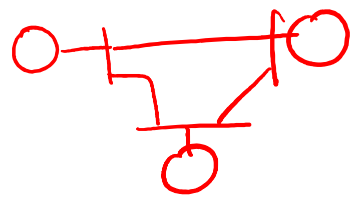
```
>> lambda.upper

ans =
    0.5601
    0.0000
         0
```

Lagrange multiplier assoc. with upper bound

$\mu_1 > 0$ $\quad P_{g1} \le 600$ binding $\Rightarrow P_{g1} = 600$

Next Topics

- **Economic Dispatch – Linear Programming** *
- Optimal Power Flow



- Unit Commitment