Motivation: Linear State Space Systems

- We will analyze linear time invariant (LTI) models in state space form:

$$\dot{x} = Ax + Bu, \quad x(0) = x_0 \qquad x \in \mathbb{R}^n,\ u \in \mathbb{R}^m,\ y \in \mathbb{R}^p$$
$$y = Cx$$

- Let's consider for example $n = 2,\ m = 1,\ p = 1$

- When we say $x \in \mathbb{R}^2$, how can we visualize this? What **basis** are we referring to? What if we used a different basis?

Linear Vector Spaces

- A *Linear Vector Space* is a *set*, $\mathcal{V}$, over a field, $\mathbb{R}$, in which having **two** operations:

	Example: $\mathcal{V} = \mathbb{R}^2$
1. Addition : $x, y \in \mathcal{V}, \quad z = x + y \in \mathcal{V}$	
2. Scalar multiplication : $x \in \mathcal{V}, \quad \alpha \in \mathbb{R}, \quad z = \alpha x \in \mathcal{V}$	

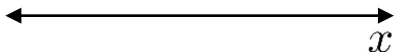
- (cont'd) it satisfies the following properties (Axioms): $x, y, z \in \mathcal{V}$

1. (Commutativity) $x + y = y + x$
2. (Associativity) $(x + y) + z = x + (y + z)$
3. (Zero vector) $\mathbf{0} + x = x, \quad x, \mathbf{0} \in \mathcal{V}$
4. (Additive inverse) $\forall x \in \mathcal{V}$ there exists $-x \in \mathcal{V}$ such that $x + (-x) = \mathbf{0}$
5. (Identity scalar) $1x = x$ for $1 \in \mathbb{R}$
6. (Compatibility of scalar mult.) $(ab)x = a(bx) \qquad a, b \in \mathbb{R}$
7. (Distributivity w.r.t addition) $a(x + y) = ax + ay \qquad a \in \mathbb{R}$
8. (Distributivity w.r.t scalar mult.) $(a + b)x = ax + bx \qquad a, b \in \mathbb{R}$

Linear Vector Spaces - Examples

- What are some familiar examples of **Vector Spaces?**

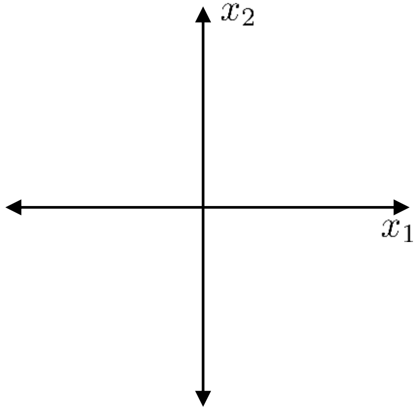
1. The real numbers $\mathbb{R}^1$



$x \in \mathbb{R}^1$

2. The 2 dimensional space, $\mathbb{R}^2$

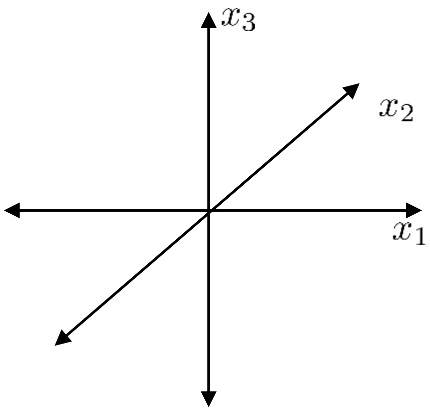
How to identify an element in $\mathbb{R}^2$?



$$x \in \mathbb{R}^2 \quad \Rightarrow \quad x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

3. The 3 dimensional space, $\mathbb{R}^3$

How to identify an element in $\mathbb{R}^3$?



$$x \in \mathbb{R}^3 \quad \Rightarrow \quad x = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

Linear Vector Spaces – Examples (cont'd)

- Other not so common Linear Vector spaces:
  - The set of **polynomials** of degree less than or equal to n, e.g. $n = 2$

$$\text{if } p, q \in \mathbb{P}_2 \quad \Rightarrow \quad p(x) = c_2 x^2 + c_1 x + c_0, \quad q(x) = d_2 x^2 + d_1 x + d_0$$

$$\text{e.g. } h(x) = 3x^2 + 1 \qquad m(x) = 2x^2 + x - 3$$

  - Define addition and scalar multiplication:

$$p(x) + q(x) = (c_2 + d_2)x^2 + (c_1 + d_1)x + (c_0 + d_0)$$

$$h(x) + m(x) =$$

2. Scalar multiplication:

$$\alpha \in \mathbb{R} \quad \alpha p(x) = \alpha c_2 x^2 + \alpha c_1 x + \alpha c_0$$

$$3h(x) =$$

  - **$\mathbb{P}_2$ and in general $\mathbb{P}_n$ satisfy all the axioms for a vector space**

Topic Outline

- Transfer Function vs State Space (Motivation)
- State Space Models and Examples
- From Nonlinear to Linear State Space Models
- **Linear Algebra Review**
  - Vector Space
  - **Subspace**
  - Span and Linear Independence
  - Basis
  - Change of Basis
  - Linear Maps
  - Column Space and Null Space

Subspace of a Vector Space

- Vector spaces provide us with a field to conduct extensive analysis
- **Definition:** $\mathcal{S}$ is called a *subspace* of a vector space $\mathcal{V}$ if $\mathcal{S}$ is a subset of V and $\mathcal{S}$ satisfies:

  (i) $\mathbf{0} \in \mathcal{S}$ ($\mathcal{S}$ is non-empty)

  (ii) For all $x, y \in \mathcal{S}$, $x + y \in \mathcal{S}$ ($\mathcal{S}$ is closed under addition)

  (iii) For all $x \in \mathcal{S}$, $\alpha x \in \mathcal{S}$, where $\alpha \in \mathbb{R}$ ($\mathcal{S}$ is closed under scalar multiplication)

- To verify if a subset $\mathcal{S}$ is a **subspace** of a vector space $\mathcal{V}$, we need to check the three conditions above

Remember them!

Example: What are the trivial subspaces of the vector space $\mathbb{R}^3$?

Subspace Example 1

- **Example:** Consider the following subset of $\mathbb{R}^3$:

$$\mathcal{S} = \left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} \in \mathbb{R}^3 \;\middle|\; x = 3y,\ z = -2y \right\}$$

- Is it a subspace of $\mathbb{R}^3$?

  a) $\mathbf{0} \in \mathcal{S}$?

  b) $\forall p, q \in \mathcal{S}$, is $p + q \in \mathcal{S}$?

  c) $\forall p \in \mathcal{S}$, is $\alpha p \in \mathcal{S}$?

Subspace Example 2

- **Example:** Consider the following subset of $\mathbb{R}^2$:

$$\mathcal{S} = \left\{ \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \in \mathbb{R}^2 \;\middle|\; x_2 = 3x_1 \right\}$$

- Is it a subspace of $\mathbb{R}^2$?

a) $\mathbf{0} \in \mathcal{S}$? $\quad \mathbf{0} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad x_2 = 3x_1 \;\Rightarrow\; 0 = 3(0)?$

b) $\forall p, q \in \mathcal{S}$, is $p + q \in \mathcal{S}$? $\quad p + q = \begin{pmatrix} p_1 + q_1 \\ 3p_1 + 3q_1 \end{pmatrix} = \begin{pmatrix} (p_1 + q_1) \\ 3(p_1 + q_1) \end{pmatrix} \in \mathcal{S}?$

$$p = \begin{pmatrix} p_1 \\ p_2 \end{pmatrix} \in \mathcal{S} \;\Rightarrow\; p = \begin{pmatrix} p_1 \\ 3p_1 \end{pmatrix}, \qquad q = \begin{pmatrix} q_1 \\ q_2 \end{pmatrix} \in \mathcal{S} \;\Rightarrow\; q = \begin{pmatrix} q_1 \\ 3q_1 \end{pmatrix}$$

c) $\forall p \in \mathcal{S}$, is $\alpha p \in \mathcal{S}$?

$$\alpha p = \alpha \begin{pmatrix} p_1 \\ 3p_1 \end{pmatrix} = \begin{pmatrix} (\alpha p_1) \\ 3(\alpha p_1) \end{pmatrix} \in \mathcal{S}?$$

Subspace Example 3

- **Example:** Consider the following subset of $\mathbb{R}^2$:

$$\mathcal{S} = \left\{ \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \in \mathbb{R}^2 \;\middle|\; x_1 \geq 0,\; x_2 \geq 0 \right\}$$

- Is it a subspace of $\mathbb{R}^2$?

a) $\mathbf{0} \in \mathcal{S}$?

is the first coordinate greater than or equal to 0?

is the second coordinate greater than or equal to 0?

$$\mathbf{0} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \in \mathcal{S}?$$

b) $\forall p, q \in \mathcal{S}$, is $p + q \in \mathcal{S}$?

$$p = \begin{pmatrix} p_1 \\ p_2 \end{pmatrix} \in \mathcal{S}, \qquad q = \begin{pmatrix} q_1 \\ q_2 \end{pmatrix} \in \mathcal{S} \qquad\qquad p + q = \begin{pmatrix} p_1 + q_1 \\ p_2 + q_2 \end{pmatrix} \in \mathcal{S}?$$

$$p_1 \geq 0,\; p_2 \geq 0 \qquad q_1 \geq 0,\; q_2 \geq 0 \qquad\qquad p_1 + q_1 \geq 0,\; p_2 + q_2 \geq 0?$$

c) $\forall p \in \mathcal{S}$, is $\alpha p \in \mathcal{S}$?

is $\alpha p = \alpha \begin{pmatrix} \alpha p_1 \\ \alpha p_2 \end{pmatrix} \in \mathcal{S},\; \forall \alpha \in \mathbb{R}$? $\qquad \alpha p_1 \geq 0,\; \alpha p_2 \geq 0$?

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Span of a Set of Vectors

- Suppose $v_1,\ v_2,\ \ldots,\ v_n$ are vectors defined in a vector space $\mathcal{V}$
- A *Linear Combination* of these is a vector:

$$\alpha_1 v_1 + \alpha_2 v_2 + \cdots + \alpha_n v_n \in \mathcal{V} \qquad \alpha_1, \ldots, \alpha_n \in \mathbb{R}$$

- The set of all linear combinations of $v_1,\ v_2,\ \ldots,\ v_n$ is called **span** $\{v_1,\ v_2,\ \cdots,\ v_n\}$

◦ Example: in $\mathcal{V} = \mathbb{R}^3$, let $v_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$, $v_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$, what is the span $\{v_1,\ v_2\}$?

$$\text{span}\,\{v_1,\ v_2\} = \text{span}\left\{\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}\right\} = \left\{ x \in \mathbb{R}^3 \;\middle|\; x = \alpha_1 \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + \alpha_2 \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \right\}$$

Span of a Set of Vectors – Example 1

- Determine if $v = \begin{pmatrix} 2 \\ -2 \\ -1 \end{pmatrix} \in \text{span}\left\{ \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}, \begin{pmatrix} 2 \\ -3 \\ 1 \end{pmatrix}, \begin{pmatrix} 5 \\ -4 \\ 0 \end{pmatrix} \right\}$

Problem: Determine if v can be written as a linear combination of the three vectors

$$v = \begin{pmatrix} 2 \\ -1 \\ -1 \end{pmatrix} = \alpha_1 \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} + \alpha_2 \begin{pmatrix} 2 \\ -3 \\ 1 \end{pmatrix} + \alpha_3 \begin{pmatrix} 5 \\ -4 \\ 0 \end{pmatrix} \quad \Leftrightarrow \quad \begin{pmatrix} 1 & 2 & 5 \\ 0 & -3 & -4 \\ -1 & 1 & 0 \end{pmatrix} \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{pmatrix} = \begin{pmatrix} 2 \\ -2 \\ -1 \end{pmatrix}$$

- This is a system of linear equations $Ax = b$
  - Gaussian elimination - Row Echelon Form (REF)
  - Gauss-Jordan - Reduced Row Echelon Form (RREF)
  - If A^{-1} exists, then $x = A^{-1}b$

- Using RREF, build augmented matrix $\begin{bmatrix} A & b \end{bmatrix}$ and use $\text{rref}\left(\begin{bmatrix} A & b \end{bmatrix}\right)$

$$\text{rref}\left(\begin{pmatrix} 1 & 2 & 5 & 2 \\ 0 & -3 & -4 & -2 \\ -1 & 1 & 0 & -1 \end{pmatrix}\right) = \begin{pmatrix} 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & -1 \end{pmatrix} \quad \Rightarrow \quad \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{pmatrix} = \begin{pmatrix} 3 \\ 2 \\ -1 \end{pmatrix}$$

Span of a Set of Vectors – Example 2

- Determine if $v = \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix} \in \text{span}\left\{ \begin{pmatrix} 1 \\ -2 \\ 5 \end{pmatrix}, \begin{pmatrix} 2 \\ 3 \\ -1 \end{pmatrix} \right\}$

Problem: Determine if v can be written as a linear combination of the two vectors

$$v = \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix} = \alpha_1 \begin{pmatrix} 1 \\ -2 \\ 5 \end{pmatrix} + \alpha_2 \begin{pmatrix} 2 \\ 3 \\ -1 \end{pmatrix} \quad \Leftrightarrow \quad \begin{pmatrix} 1 & 2 \\ -2 & 3 \\ 5 & -1 \end{pmatrix} \begin{pmatrix} \alpha_1 \\ \alpha_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix}$$

- Using RREF, build augmented matrix $\begin{bmatrix} A & b \end{bmatrix}$ and use $\text{rref}\left(\begin{bmatrix} A & b \end{bmatrix}\right)$

$$\text{rref}\left\{ \begin{pmatrix} 1 & 2 & 1 \\ -2 & 3 & 2 \\ 5 & -1 & -1 \end{pmatrix} \right\} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad \Rightarrow \quad \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} \alpha_1 \\ \alpha_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

- There does not exists constants $\alpha_1,\ \alpha_2$ such that $v = \alpha_1 v_1 + \alpha_2 v_2 \quad \Rightarrow \quad v \notin \text{span}\{v_1,\ v_2\}$

Span of a Set of Vectors - Subspace

- **Theorem:** if $v_1, \cdots, v_n$ are elements of a vector space $\mathcal{V}$, then the $\mathcal{S} = \text{span}\{v_1, v_2, \cdots, v_n\}$ is a subspace of $\mathcal{V}$

Proof:

- Need to show the 3 conditions for a subspace are satisfied:
  - $0 \in \mathcal{S} = \text{span}\{v_1, v_2, \cdots, v_n\}$?
  - $p, q \in \mathcal{S}$ is $p + q \in \mathcal{S}$?
  - $p \in \mathcal{S}$ is $\gamma p \in \mathcal{S}$?

Spanning Set for a Vector Space (1)

- For a certain set of vectors $v_1, \cdots, v_n \in \mathcal{V}$, we know that $\text{span}\{v_1, \cdots, v_n\} \subset \mathcal{V}$
- Is it possible that instead $\text{span}\{v_1, \cdots, v_n\} = \mathcal{V}$?

a) Assume $\mathcal{V} = \mathbb{R}^2$, and let $v_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, $v_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$, we know that $\text{span}\{v_1, v_2\} \subseteq \mathbb{R}^2$.

However, is $\text{span}\{v_1, v_2\} = \mathbb{R}^2$?

- We need to show that any vector in $\mathbb{R}^2$ can be written as a linear combination of v_1, v_2
- Let $x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \in \mathbb{R}^2$ be any vector, is $x \in \text{span}\left\{\begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix}\right\}$?

$$x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \alpha_1 \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \alpha_2 \begin{pmatrix} 0 \\ 1 \end{pmatrix} \Rightarrow \alpha_1 = x_1, \ \alpha_2 = x_2$$

- Therefore, $\text{span}\left\{\begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix}\right\} = \mathbb{R}^2$

- v_1, v_2 form a **spanning set** of $\mathbb{R}^2$

Spanning Set for a Vector Space (2)

- For a certain set of vectors $v_1, \cdots, v_n \in \mathcal{V}$, we know that span $\{v_1, \cdots, v_n\} \subset \mathcal{V}$
- Is it possible that instead span $\{v_1, \cdots, v_n\} = \mathcal{V}$?

b) Assume $\mathcal{V} = \mathbb{R}^2$, and let $v_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, $v_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$, $v_3 = \begin{pmatrix} 3 \\ 4 \end{pmatrix}$, we know that span $\{v_1, v_2, v_3\} \subseteq \mathbb{R}^2$. However, is span $\{v_1, v_2, v_3\} = \mathbb{R}^2$?

– From the previous slide, we know that span $\{v_1, v_2\} = \mathbb{R}^2$. Thus, it should be clear that any $x \in \mathbb{R}^2$ is also defined in the span $\{v_1, v_2, v_3\}$

– v_1, v_2, v_3 also forms a **spanning set** of $\mathbb{R}^2$.

– *However, this spanning set has 3 vectors (one more vector than the previous one)*

Spanning Set for a Vector Space (3)

- For a certain set of vectors $v_1, \cdots, v_n \in \mathcal{V}$, we know that $\text{span}\{v_1, \cdots, v_n\} \subset \mathcal{V}$
- Is it possible that instead $\text{span}\{v_1, \cdots, v_n\} = \mathcal{V}$?

c) Assume $\mathcal{V} = \mathbb{R}^3$, and let $v_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$, $v_2 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$. We know that $\text{span}\{v_1, v_2\} \subset \mathbb{R}^3$

Does the $\text{span}\{v_1, v_2\} = \mathbb{R}^3$?

$$x \in \mathbb{R}^3 \quad \Rightarrow \quad x = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \alpha_1 \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + \alpha_2 \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} ?$$

$$\begin{pmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \alpha_1 \\ \alpha_2 \end{pmatrix} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \quad \Rightarrow \quad \begin{aligned} \alpha_1 &= x_1 \\ 0 &= x_2 \\ \alpha_2 &= x_3 \end{aligned}$$

if $x_2 \neq 0$ then $x \notin \text{span}\{v_1, v_2\}$

– v_1, v_2 is **NOT a spanning set** for $\mathbb{R}^3$

Spanning Set for a Vector Space (2)

- Let $\mathcal{V}$ be a vector space
- For a certain set of vectors $v_1, \cdots, v_n \in \mathcal{V}$, we know that span$\{v_1, \cdots, v_n\} \subset \mathcal{V}$
- Is it possible that instead span$\{v_1, \cdots, v_n\} = \mathcal{V}$

- **Definition:** The set v_1, v_2, ..., $v_n \in \mathcal{V}$ is a *spanning set* for $\mathcal{V}$ if and only if every vector in $\mathcal{V}$ can be written as a linear combination of v_1, v_2, ..., v_n

Spanning Set for a Vector Space - Example

- **Example:** Determine if $\text{span}\left\{\begin{pmatrix}3\\-5\\2\end{pmatrix}, \begin{pmatrix}8\\4\\1\end{pmatrix}, \begin{pmatrix}0\\3\\0\end{pmatrix}\right\} = \mathbb{R}^3$

- If $v_1,\ v_2,\ v_3$ forms a spanning set for $\mathcal{V} = \mathbb{R}^3$, then $\forall x \in \mathbb{R}^3 \quad \Rightarrow \quad x \in \text{span}\{v_1,\ v_2,\ v_3\}$
- Therefore, $\forall x \in \mathbb{R}^3,\ x = \begin{pmatrix}x_1\\x_2\\x_3\end{pmatrix} = \alpha_1\begin{pmatrix}3\\-5\\2\end{pmatrix} + \alpha_2\begin{pmatrix}8\\4\\1\end{pmatrix} + \alpha_3\begin{pmatrix}0\\3\\0\end{pmatrix}.$

  How can we show this?

$$\underbrace{\begin{pmatrix}3 & 8 & 0\\-5 & 4 & 3\\2 & 1 & 0\end{pmatrix}}_{A}\underbrace{\begin{pmatrix}\alpha_1\\\alpha_2\\\alpha_3\end{pmatrix}}_{\alpha} = \underbrace{\begin{pmatrix}x_1\\x_2\\x_3\end{pmatrix}}_{x} \quad \Leftrightarrow \quad A\alpha = x$$

- Is there always a solution vector $\begin{pmatrix}\alpha_1\\\alpha_2\\\alpha_3\end{pmatrix}$ for any arbitrary $x \in \mathbb{R}^3$?
  - If A matrix is invertible, i.e. A^{-1} exists
  - If A has three linearly independent columns (more on this later)
  - ...

Spanning Set and Linearly Dependent Set of Vectors

- Consider two spanning sets for $\mathcal{V} = \mathbb{R}^2$

  - $\text{span}\left\{\begin{pmatrix}1\\0\end{pmatrix}, \begin{pmatrix}0\\1\end{pmatrix}\right\} = \mathbb{R}^2$ (see prev. slide)

  - $\text{span}\left\{\begin{pmatrix}1\\0\end{pmatrix}, \begin{pmatrix}0\\1\end{pmatrix}, \begin{pmatrix}1\\1\end{pmatrix}\right\} = \mathbb{R}^2$

Spanning Set and Linearly Dependent Set of Vectors

- Let $\{v_1, v_2, \cdots, v_n\}$ be a **linearly dependent** set of vectors

  $\Rightarrow$ There is at least one vector, say v_1, that can be written as a sum of the others
- Then, the span $\{v_1, v_2, \cdots, v_n\} = \text{span}\{v_2, v_3, \cdots, v_n\}$

- Without loss of generality, assume that $v_1 = c_2v_2 + \cdots + c_nv_n$, show that span $\{v_1, v_2, \cdots, v_n\} =$ span $\{v_2, \cdots, v_n\}$

- Assume that $x \in \text{span}\{v_1, \cdots, v_n\} \Rightarrow x = d_1v_1 + d_2v_2 + \cdots + d_nv_n$. This implies that $x = d_1(c_2v_2 + \cdots + c_nv_n) + d_2v_2 + \cdots + d_nv_n$

$$\Rightarrow \quad x = (d_1c_2 + d_2)\, v_2 + \cdots + (d_1c_n + d_n)\, v_n \quad \Rightarrow \quad x \in \text{span}\{v_2, \cdots, v_n\}$$

Linearly Dependent Vectors

- In general, given $\{v_1,\ v_2,\ \cdots,\ v_n\}$, it is possible to write one of the vectors as a linear combination of the others $n-1$ vectors **if and only if** there exists $c_1,\ c_2,\ \cdots,\ c_n \in \mathbb{R}$ *not all zero* such that $c_1v_1 + c_2v_2 + \cdots + c_nv_n = 0$

- **Statement A:** given $\{v_1,\ v_2,\ \cdots,\ v_n\}$, we can write $v_1 = d_2v_2 + \cdots + d_nv_n$
- **Statement B:** $\exists c_1,\ c_2,\ \cdots,\ c_n \in \mathbb{R}$ **not all zero** such that $c_1v_1 + \cdots + c_nv_n = 0$

Need to show $A \Rightarrow B$ and $B \Rightarrow A$ (if and only if)

$B \Rightarrow A$

- Assume B is true, $\exists c_1,\ c_2,\ \cdots,\ c_n \in \mathbb{R}$ not all zero s.t. $c_1v_1 + c_2v_2 + \cdots + c_nv_n = 0$
- Assume $c_1 \neq 0$ without loss of generality
- $\Rightarrow\ c_1v_1 = -c_2v_2 - \cdots - c_nv_n$
- $\Rightarrow\ v_1 = -\frac{c_2}{c_1}v_2 - \cdots - \frac{c_n}{c_1}v_n$
- $\Rightarrow\ v_1 = d_2v_2 + \cdots + d_nv_n,\ \ d_i = -\frac{c_i}{c_1}$, for $i = 2,\ \cdots,\ n$ (A)

$A \Rightarrow B$

- ?

Linear Independence

- The vectors $v_1,\ v_2,\ \cdots,\ v_n$ in a vector space $\mathcal{V}$ are said to be *Linearly Independent* if

$$c_1 v_1 + c_2 v_2 + \cdots + c_n v_n = 0 \quad \Rightarrow \quad c_1 = c_2 = \cdots = c_n = 0$$

- If $\{v_1,\ v_2,\ \cdots,\ v_n\}$ is a *minimal spanning* set for $\mathcal{V}$ then $v_1,\ v_2,\ \cdots,\ v_n$ are *linearly independent*!

- If $\{v_1,\ v_2,\ \cdots,\ v_n\}$ is a *linearly independent* set of vectors, then $\text{span}\{v_1,\ v_2,\ \cdots,\ v_n\} = \mathcal{V}$ is a *minimal spanning* set for $\mathcal{V}$!

$$\textit{Minimal Spanning Set} \equiv \textit{Basis}$$

Linear Independence – Example 1

- Determine if $\left\{ \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \end{pmatrix} \right\}$ are linearly independent

- To show linear independence: $c_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} + c_2 \begin{pmatrix} 1 \\ 2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ only for $c_1 = c_2 = 0$

$$\begin{aligned} 1c_2 + 1c_2 = 0 \\ 1c_1 + 2c_2 = 0 \end{aligned} \quad \Rightarrow \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\text{(Using RREF)} \quad \Rightarrow \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\Rightarrow c_1 = 0, \; c_2 = 0$$

- Since $c_1 = c_2 = 0$, the vectors $\begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \end{pmatrix}$ are **linearly independent**

Linear Independence – Example 2

- Determine if $\left\{ \begin{pmatrix} 1 \\ 2 \\ 4 \end{pmatrix}, \begin{pmatrix} 2 \\ 1 \\ 3 \end{pmatrix}, \begin{pmatrix} 4 \\ -1 \\ 1 \end{pmatrix} \right\}$ are linearly independent
- To show linear independence: $c_1 \begin{pmatrix} 1 \\ 2 \\ 4 \end{pmatrix} + c_2 \begin{pmatrix} 2 \\ 1 \\ 3 \end{pmatrix} + c_3 \begin{pmatrix} 4 \\ -1 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$

$$\begin{pmatrix} 1 & 2 & 4 \\ 2 & 1 & -1 \\ 4 & 3 & 1 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \Rightarrow \left(\begin{array}{ccc|c} 1 & 2 & 4 & 0 \\ 2 & 1 & -1 & 0 \\ 4 & 3 & 1 & 0 \end{array} \right) \Rightarrow \text{(using rref)} \left(\begin{array}{ccc|c} 1 & 0 & -2 & 0 \\ 0 & 1 & 3 & 0 \\ 0 & 0 & 0 & 0 \end{array} \right)$$

The set of all solutions are of the form:

$$\left\{ \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} \in \mathbb{R}^3 \;\middle|\; c_1 = 2c_3,\ c_2 = -3c_3,\ \text{for any } c_3 \in \mathbb{R} \right\} = \left\{ x \in \mathbb{R}^3 \;\middle|\; x = \begin{pmatrix} 2c_3 \\ -3c_3 \\ c_3 \end{pmatrix} = c_3 \begin{pmatrix} 2 \\ -3 \\ 1 \end{pmatrix} \right\}$$

- Or we can also say all of the solutions are in the span $\left\{ \begin{pmatrix} 2 \\ -3 \\ 1 \end{pmatrix} \right\}$, e.g. $c_1 = 2,\ c_2 = -3,\ c_3 = 1$
- The vectors are linearly dependent!

Linear Independence – Example 3

- Determine if $\left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} \right\}$ are linearly independent

- Linear independent if: $c_1 \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + c_2 \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} + c_3 \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} + c_4 \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$ only for $c_1 = c_2 = c_3 = c_4 = 0$

$$\Rightarrow \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 3 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \quad \Rightarrow \begin{cases} c_1 = -c_4 \\ c_2 = -2c_4 \\ c_3 = -3c_4 \end{cases}$$

- There are infinite many solutions of the form:

$$\left\{ \begin{pmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \end{pmatrix} \in \mathbb{R}^4 \,\middle|\, c_1 = -c_4,\ c_2 = -2c_4,\ c_3 = -3c_4,\ \text{for } c_4 \in \mathbb{R} \right\} = \left\{ x \in \mathbb{R}^4 \,\middle|\, x = \begin{pmatrix} -c_4 \\ -2c_4 \\ -3c_4 \\ c_4 \end{pmatrix} = -c_4 \begin{pmatrix} -1 \\ -2 \\ -3 \\ 1 \end{pmatrix} \right\}$$

- We can also set that the solutions are as: $\operatorname{span} \left\{ \begin{pmatrix} -1 \\ -2 \\ -3 \\ 1 \end{pmatrix} \right\}$ The vectors are lin. dependent!

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Basis of a Vector Space

- **Theorem:** Suppose $v_1,\ v_2,\ \cdots,\ v_n$ are linearly independent vectors in a vector space $\mathcal{V}$ and let $v \in \text{span}\{v_1,\ v_2,\ ...,\ v_n\}$, **then** v can be written **uniquely** as a linear combination of $v_1,\ v_2,\ \cdots,\ v_n$

Proof (by contradiction $\neg B \Rightarrow \neg A$)

- Assume $\neg B$, i.e. there are two ways to write write $v \in \text{span}\{v_1,\ v_2,\ \cdots,\ v_n\}$, e.g. $v = \alpha_1 v_1 + \alpha_2 v_2 + \cdots + \alpha_n v_n$, and $v = \beta_1 v_1 + \beta_2 v_2 + \cdots + \beta_n v_n$
- Need to show that this implies $\neg A$, i.e. $\{v_1,\ v_2,\ \cdots,\ v_n\}$ are linearly dependent
- Consider the following:

$$v - v = \alpha_1 v_1 + \alpha_2 v_2 + \cdots + \alpha_n v_n - (\beta_1 v_1 + \beta_2 v_2 + \cdots + \beta_n v_n) = 0$$
$$\Rightarrow \quad (\alpha_1 - \beta_1) v_1 + (\alpha_2 - \beta_2)\, v_2 + \cdots + (\alpha_n - \beta_n)\, v_n = 0$$

- Since there is at least one $\alpha_i \neq \beta_i$ (by assumption), then the vectors $v_1,\ \cdots,\ v_n$ are linearly dependent.

- **Definition:** The vectors $v_1,\ v_2,\ \cdots,\ v_n$ form a ***basis*** for a vector space $\mathcal{V}$ iff:
  1. $\{v_1,\ v_2,\ \cdots,\ v_n\}$ are linearly independent
  2. $\text{span}\{v_1,\ v_2,\ \cdots,\ v_n\} = \mathcal{V}$

Basis of a Vector Space – Example 1

- In $\mathbb{R}^3$ the *standard or natural* basis set is $e_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$, $e_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$, and $e_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$

Show that the set of vectors $\{e_1,\ e_2,\ e_3\}$ for a basis for $\mathbb{R}^3$

- **Linear independence**:

$$c_1 \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + c_2 \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} + c_3 \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

only for $c_1 = c_2 = c_3 = 0$

- **span** $\{e_1,\ e_2,\ e_3\} = \mathbb{R}^3$:

$$\forall x \in \mathbb{R}^3, \quad \text{is } x = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \in \operatorname{span}\{e_1,\ e_2,\ e_3\}?$$

$$\alpha_1 \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + \alpha_2 \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} + \alpha_3 \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{pmatrix} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

Yes! for $\alpha_1 = x_1,\ \alpha_2 = x_2,\ \alpha_3 = x_3$

Basis of a Vector Space – Example 2

- Does the set $v_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$, $v_2 = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$, $v_3 = \begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix}$ form a basis for $\mathbb{R}^3$?

- **Linear independence:**

$$c_1 \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} + c_2 \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} + c_3 \begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 2 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} = (\text{rref}) \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix}$$

only for $c_1 = c_2 = c_3 = 0$

- $\textbf{span}\,\{v_1,\ v_2,\ v_3\} = \mathbb{R}^3$:

$$\forall x \in \mathbb{R}^3, \ \text{ is } x = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \in \text{span}\,\{v_1,\ v_2,\ v_3\}?$$

$$\begin{pmatrix} 1 & 0 & 2 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{pmatrix} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

Yes! since A is invertible (note: we can also use rref and find the actual $\alpha_1,\ \alpha_2,\ \alpha_3$)

Basis of a Vector Space – Example 3

- Does the set $v_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$, $v_2 = \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix}$, $v_3 = \begin{pmatrix} 0 \\ 2 \\ 4 \end{pmatrix}$ form a basis for $\mathbb{R}^3$?

1. Are they linearly independent?

$$c_1 v_1 + c_2 v_2 + c_3 v_3 = 0 \;\Rightarrow\; \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 2 \\ 1 & 2 & 4 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \xrightarrow{\text{RREF}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

No and we can stop here!

2. Do they span $\mathbb{R}^3$? can any vector $x = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \in \mathbb{R}^3$ be written as a combination of $v_1,\ v_2,\ v_3$?

$\alpha_1 v_1 + \alpha_2 v_2 + \alpha_3 v_3 = x$

$$\Rightarrow \underbrace{\begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 2 \\ 1 & 2 & 4 \end{pmatrix}}_{=A} \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{pmatrix} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

Dimension of a Vector Space and Basis Summary

- **Theorem:** Let $\mathcal{V}$ is a vector space of dimension $n > 0$. Then:
  1. Any set of n linearly independent vectors span $\mathcal{V}$
  2. Any set of n vectors that span $\mathcal{V}$ are linearly independent
  3. No set of less than n vectors can span $\mathcal{V}$
  4. Any subset of less than n linearly indep. vectors can be extended to form a basis for $\mathcal{V}$
  5. Any spanning set of $> n$ vectors can be parted down to form a basis for $\mathcal{V}$

- $\mathbb{R}^3$ is a vector space with dimension 3 since any basis must have 3 vectors, e.g. $\{e_1,\ e_2,\ e_3\}$

  No set of less than 3 vectors can span $\mathbb{R}^3$!

  Any set of more than 3 vectors in $\mathbb{R}^3$ have to be linearly dependent

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Change of Basis - Motivation

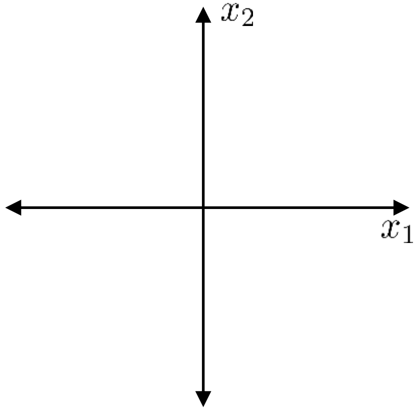
Motivation:

- Many applied problems can be simplified by changing from one coordinate system to another, e.g. when integrating the volume of a solid is better to use spherical coordinates, $\{r,\ \theta,\ \phi\}$, rather than rectangular, $\{x,\ y,\ z\}$

- When we specify a vector, we typically assume it is with respect to the standard basis

- For example in $\mathbb{R}^2$, a vector $x = \begin{bmatrix} \alpha \\ \beta \end{bmatrix}$ is implied to be $x = \alpha \begin{bmatrix} 1 \\ 0 \end{bmatrix} + \beta \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

Standard basis for $\mathbb{R}^2 = \operatorname{span}\{e_1,\ e_2\}$



Change of Basis – General Basis

Definition:

- Let $\mathcal{V}$ be a finite dimensional vector space, and let $\mathscr{B} = \{b_1,\ b_2,\ \ldots,\ b_k\}$ be a basis for $\mathcal{V}$. Then any $v \in \mathcal{V}$, can be written uniquely as: $v = \alpha_1 b_1 + \alpha_2 b_2 + \cdots \alpha_k b_k$
- The elements $\alpha_1,\ \alpha_2,\ \cdots,\ \alpha_k$ are called the **coordinates of** v **w.r.t. to** $\mathscr{B}$
- $v_{\mathscr{B}} = \begin{bmatrix} \alpha_1 \\ \vdots \\ \alpha_k \end{bmatrix}$ is called the **coordinate vector of** v **w.r.t.** $\mathscr{B}$
- $v = \begin{bmatrix} v_1 \\ \vdots \\ v_k \end{bmatrix}$ is called the **coordinate vector of** v **w.r.t. the standard basis**

- Notice that $v = v_1 \begin{pmatrix} 1 \\ \vdots \\ 0 \end{pmatrix} + \cdots + v_n \begin{pmatrix} 0 \\ \vdots \\ 1 \end{pmatrix} = \alpha_1 b_1 + \cdots + \alpha_n b_n \;\Rightarrow\; v = \underbrace{\begin{pmatrix} \vdots & & \vdots \\ b_1 & \cdots & b_n \\ \vdots & & \vdots \end{pmatrix}}_{B} v_{\mathscr{B}}$

- Therefore $v = B v_{\mathscr{B}}$ or $v_{\mathscr{B}} = B^{-1} v$

Change of Basis – Coordinate Vector

- Given a vector $x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ (w.r.t standard basis), we would like to find the coordinates w.r.t another basis $\mathscr{B}$
- For example, $\mathscr{B} = \left\{ \begin{bmatrix} 3 \\ 5 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \end{bmatrix} \right\}$, then $x_{\mathscr{B}} = \begin{bmatrix} \alpha \\ \beta \end{bmatrix}$ implies that $x = Bx_{\mathscr{B}}$, i.e.

$$x = Bx_{\mathscr{B}} \quad \Rightarrow \quad \underbrace{\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}}_{x} = \underbrace{\begin{pmatrix} 3 & 1 \\ 5 & -1 \end{pmatrix}}_{B} \underbrace{\begin{pmatrix} \alpha \\ \beta \end{pmatrix}}_{x_{\mathscr{B}}}$$

- Therefore, we can use B to change basis, i.e. $x = Bx_{\mathscr{B}}$ or $x_{\mathscr{B}} = B^{-1}x$

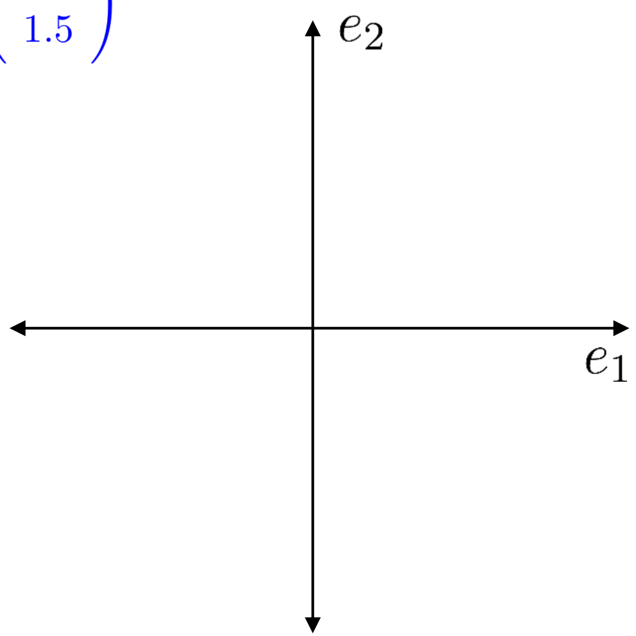
Change of Basis – Example (1)

- Example, given $x = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ find its coordinates w.r.t $\mathscr{B} = \left\{ \begin{pmatrix} 1 \\ -1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \end{pmatrix} \right\}$, i.e. $x_{\mathscr{B}}$
- Given that the set $\mathscr{B}$ is a basis for $\mathbb{R}^2$, then:

$$x = \begin{pmatrix} 1 \\ 2 \end{pmatrix} = \alpha_1 \begin{pmatrix} 1 \\ -1 \end{pmatrix} + \alpha_2 \begin{pmatrix} 1 \\ 1 \end{pmatrix} \Rightarrow \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} \alpha_1 \\ \alpha_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

The coordinates w.r.t. $\mathscr{B}$ are thus:

$$x_{\mathscr{B}} = B^{-1} x = \left(\begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} \right)^{-1} \begin{pmatrix} 1 \\ 2 \end{pmatrix} = \begin{pmatrix} -0.5 \\ 1.5 \end{pmatrix}$$



Change of Basis – Example (2)

- Given two basis for $\mathbb{R}^2$: $\mathscr{B} = \left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \end{bmatrix} \right\}$ $\mathscr{C} = \left\{ \begin{bmatrix} 3 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ -2 \end{bmatrix} \right\}$
- Let $x_{\mathscr{B}} = \begin{bmatrix} -1 \\ 2 \end{bmatrix}$, find $x_{\mathscr{C}} = \begin{bmatrix} \alpha_1 \\ \alpha_2 \end{bmatrix}$
- How can we solve for $x_{\mathscr{C}}$? We can first find x and then find $x_{\mathscr{C}}$:

$$\begin{aligned} x &= Bx_{\mathscr{B}} \\ x &= Cx_{\mathscr{C}} \end{aligned} \quad \Rightarrow \quad Bx_{\mathscr{B}} = Cx_{\mathscr{C}} \quad \Rightarrow \quad x_{\mathscr{C}} = C^{-1}Bx_{\mathscr{B}}$$

- Therefore:

$$\begin{pmatrix} 1 & 1 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} -1 \\ 2 \end{pmatrix} = \begin{pmatrix} 3 & 1 \\ 2 & -2 \end{pmatrix} \begin{pmatrix} \alpha_1 \\ \alpha_2 \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} \alpha_1 \\ \alpha_2 \end{pmatrix} = \left(\begin{pmatrix} 3 & 1 \\ 2 & -2 \end{pmatrix} \right)^{-1} \begin{pmatrix} 1 & 1 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} -1 \\ 2 \end{pmatrix}$$

$$\Rightarrow x_{\mathscr{C}} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

- **A different way to find $C^{-1}B$ is using rref**$\{(C \mid B)\}$ (last 2 rows/columns)

Change of Basis – Transition Matrix

- For $\mathbb{R}^n$, let $B = \begin{bmatrix} b_1 & b_2 & \dots & b_n \end{bmatrix}$ be a matrix whose columns form a basis for $\mathbb{R}^n$.
  Let $v \in \mathbb{R}^n$, $v = \alpha_1 b_1 + \cdots + \alpha_n b_n \quad \Rightarrow \quad v = Bv_{\mathscr{B}}$

- Similarly, let $C = \begin{bmatrix} c_1 & \cdots & c_n \end{bmatrix}$ be a matrix whose columns form a basis for $\mathbb{R}^n$,
  then $v = \gamma_1 c_1 + \cdots + \gamma_n c_n \quad \Rightarrow \quad v = Cv_{\mathscr{C}}$

$$v = Bv_{\mathscr{B}} = Cv_{\mathscr{C}}$$

$$v_{\mathscr{B}} = B^{-1}Cv_c$$

$$v_{\mathscr{C}} = C^{-1}Bv_{\mathscr{B}}$$

- Therefore, we can find the following transition matrices:

$$Cv_{\mathscr{C}} = Bv_{\mathscr{B}} \quad \Rightarrow \quad v_{\mathscr{C}} = \overbrace{C^{-1}B}^{\text{Transition matrix from } \mathscr{B} \to \mathscr{C}} v_{\mathscr{B}}$$

$$\Rightarrow \quad v_{\mathscr{B}} = \underbrace{B^{-1}C}_{\text{Transition matrix from } \mathscr{C} \to \mathscr{B}} v_{\mathscr{C}}$$

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Linear Maps

- Let $\mathcal{V}$ be a vector space of dimension n
- and $\mathcal{W}$ be a vector space of dimension m over the field of $\mathbb{R}$

- **Definition:** A function $T : \mathcal{V} \to \mathcal{W}$ is called *linear* if

$$T(u+v) = T(u) + T(v) \qquad \forall \ u, \ v \in \mathcal{V}$$

$$T(\alpha v) = \alpha T(v) \qquad \forall \alpha \in \mathbb{R}, \ \ v \in \mathcal{V}$$

- Or we can check the following equivalent condition:

$$T\left(\alpha u + \beta v\right) = \alpha T(u) + \beta T(v)$$

Linear Mapping Examples

- Show whether the transformation, $L : \mathbb{R}^2 \to \mathbb{R}^3$, is linear or not

$$L\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = \begin{bmatrix} 2x_1 \\ x_1 - 2x_2 \\ 3x_1 + 4x_2 \end{bmatrix}$$

1. $x, y \in \mathbb{R}^2$, is $L(x+y) = L(x) + L(y)$? Let $z = x + y = \begin{pmatrix} x_1 + y_1 \\ x_2 + y_2 \end{pmatrix}$

$$L(z) = L(x+y) = \begin{pmatrix} 2(x_1 + y_1) \\ (x_1 + y_1) - 2(x_2 + y_2) \\ 3(x_1 + y_1) + 4(x_2 + y_2) \end{pmatrix} = \begin{pmatrix} 2x_1 \\ x_1 - 2x_2 \\ 3x_1 + 4x_2 \end{pmatrix} + \begin{pmatrix} 2y_1 \\ y_1 - 2y_2 \\ 3y_1 + 4y_2 \end{pmatrix} = L(x) + L(y)$$

2. $\alpha \in \mathbb{R},\ x \in \mathbb{R}^2$ is $L(\alpha x) = \alpha L(x)$?

$$L(\alpha x) = \begin{pmatrix} 2(\alpha x_1) \\ \alpha x_1 - 2\alpha x_2 \\ 3\alpha x_1 + 4\alpha x_2 \end{pmatrix} = \alpha \begin{pmatrix} 2x_1 \\ x_1 - 2x_2 \\ 3x_1 + 4x_2 \end{pmatrix} = \alpha L(x_1)$$

Linear Mapping Examples

- Show whether the transformation, $L : \mathbb{R}^2 \to \mathbb{R}^2$, is linear or not

$$L\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = \begin{bmatrix} 2x_1 \\ x_1 x_2 \end{bmatrix}$$

1. $x, y \in \mathbb{R}^2$, is $L(x+y) = L(x) + L(y)$? Let $z = x + y = \begin{pmatrix} x_1 + y_1 \\ x_2 + y_2 \end{pmatrix}$

$$L(z) = L(x+y) = \begin{pmatrix} 2(x_1 + y_1) \\ (x_1 + y_1)(x_2 + y_2) \end{pmatrix} = \begin{pmatrix} 2x_1 + 2y_1 \\ x_1 x_2 + x_1 y_2 + y_1 x_2 + y_1 y_2 \end{pmatrix}$$
$$\neq \begin{pmatrix} 2x_1 \\ x_1 x_2 \end{pmatrix} + \begin{pmatrix} 2y_1 \\ y_1 y_2 \end{pmatrix}$$

Therefore, this transformation **is not linear.**

Matrix as a Linear Map

- **Theorem:** Let $A \in \mathbb{R}^{m\times n}$. The mapping $L_A(x) : \mathbb{R}^n \to \mathbb{R}^m$ defined by:

$$L_A(x) = Ax, \quad \forall x \in \mathbb{R}^n$$

  is a linear transformation.

Show that $L_A(x)$ is a linear transformation:

1. $L(x+y) = A(x+y) = Ax + Ay = L(x) + L(y)$
2. $L(\alpha x) = A(\alpha x) = \alpha A x = \alpha L(x)$

Matrix of a Linear Map

- **Theorem:** If L is a *linear* transformation from $\mathbb{R}^n$ to $\mathbb{R}^m$, then there is an $m \times n$ matrix A such that :

$$L(x) = Ax$$

for all $x \in \mathbb{R}^n$. The j^{th} column of A is given by:

$$a_j = L\left(e_j\right), \quad \text{for } j = 1, \cdots, n$$

That is $L = L_A$:

$$A = \begin{bmatrix} L(e_1) & L(e_2) & \cdots & L(e_n) \end{bmatrix}$$

The matrix A is called the **standard** matrix of a linear transformation L.

Linear Mapping – Matrix Examples

- Find the standard matrix of the following linear transformation:

$$L\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = \begin{bmatrix} 2x_1 \\ x_1 - 2x_2 \\ 3x_1 + 4x_2 \end{bmatrix}$$

- To obtain the standard matrix, i.e. $L(x) = L_A(x) = Ax$ we can apply L to the standard basis:

$$A = \begin{pmatrix} a_1 & a_2 \end{pmatrix} \Rightarrow a_1 = L(e_1) = L\left(\begin{pmatrix} 1 \\ 0 \end{pmatrix}\right) = \begin{pmatrix} 2(1) \\ (1) - 2(0) \\ 3(1) + 4(0) \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \\ 3 \end{pmatrix}$$

$$a_2 = L(e_2) = L\left(\begin{pmatrix} 0 \\ 1 \end{pmatrix}\right) = \begin{pmatrix} 2(0) \\ (0) - 2(1) \\ 3(0) + 4(1) \end{pmatrix} = \begin{pmatrix} 0 \\ -2 \\ 4 \end{pmatrix}$$

$$A = \begin{pmatrix} 2 & 0 \\ 1 & -2 \\ 3 & 4 \end{pmatrix}$$

- Therefore $L(x) = L_A(x) = Ax = \begin{pmatrix} 2 & 0 \\ 1 & -2 \\ 3 & 4 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$

Linear Mapping – Different Basis

- We now know a linear transformation from two finite dimensional vector spaces, i.e. $L : \mathbb{R}^n \to \mathbb{R}^m$, has a **standard** matrix representation
  - $L(x) = y \;\Leftrightarrow\; Ax = y, \;\; x \in \mathbb{R}^n, \; y \in \mathbb{R}^m$ — x w.r.t. standard basis for $\mathbb{R}^n$; y w.r.t. standard basis for $\mathbb{R}^m$
- What if we were to use a different (not standard) basis for $\mathbb{R}^n$ and $\mathbb{R}^m$? What is the matrix representation of L w.r.t. these different basis?
- $\mathbb{R}^n$, we have basis given by the columns of $B = \begin{pmatrix} b_1 & \cdots & b_n \end{pmatrix}$
- $\mathbb{R}^m$, we have basis given by the columns of $C = \begin{pmatrix} c_1 & \cdots & c_m \end{pmatrix}$

$$Ax = y \text{ (w.r.t. standard basis) and } y = Cy_{\mathscr{C}}, \quad x = Bx_{\mathscr{B}}$$

$$\Rightarrow \quad ABx_{\mathscr{B}} = Cy_{\mathscr{C}} \quad \Rightarrow \quad y_{\mathscr{C}} = C^{-1}ABx_{\mathscr{B}} = \tilde{A}x_{\mathscr{B}}$$

- Let $\tilde{A} = C^{-1}AB$, then $\tilde{A}$ is the matrix representation of L w.r.t. basis B for $\mathbb{R}^n$ and C for $\mathbb{R}^m$

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Null Space of a Linear Map

- **Definition:** Let $A \in \mathbb{R}^{m\times n}$, then $L_A(x) = Ax : \mathbb{R}^n \to \mathbb{R}^m$, then the set

$$\mathcal{N}(A) = \{x \in \mathbb{R}^n \mid \ Ax = 0\}$$

is a subspace of R^n (domain) and is called the **null space** of A

- The null space of A is a subspace of the domain, i.e. $\mathcal{N}(A) \subseteq \mathbb{R}^n$
- Let $A = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & -1 \end{pmatrix}$ be the standard matrix of a linear transformation $L_A(x) : \mathbb{R}^3 \to \mathbb{R}^2$
- What is the $\mathcal{N}(A)$? $\mathcal{N}(A) = \{x \in \mathbb{R}^3 \mid Ax = 0\}$

$$Ax = 0 \ \Rightarrow \ \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & -1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \ \Rightarrow \ (\text{rref}(A)) \ \begin{pmatrix} 1 & 0 & 2 \\ 0 & 1 & -1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\Rightarrow \ \begin{matrix} x_1 + 2x_3 = 0 \\ x_2 - x_3 = 0 \end{matrix} \ \Rightarrow \ \begin{matrix} x_1 = -2x_3 \\ x_2 = x_3 \end{matrix}$$

- $\mathcal{N}(A) = \{x \in \mathbb{R}^3 \mid x_1 = -2x_3, \ x_2 = x_3\} = \left\{ x \in \mathbb{R}^3 \mid x = \begin{pmatrix} -2x_3 \\ x_3 \\ x_3 \end{pmatrix} = x_3 \begin{pmatrix} -2 \\ 1 \\ 1 \end{pmatrix} \right\} = \text{span}\left\{ \begin{pmatrix} -2 \\ 1 \\ 1 \end{pmatrix} \right\}$

Range/Column Space of a Linear Map

- **Definition:** Let $A \in \mathbb{R}^{m\times n}$, i.e. $A : \mathbb{R}^n \to \mathbb{R}^m$, then the set

$$\mathcal{R}(A) = \{y \in \mathbb{R}^m \mid \exists x \in \mathbb{R}^n, \;\; Ax = y\} = \{y = Ax \in \mathbb{R}^m \mid x \in \mathbb{R}^n\}$$

  is a subspace of R^m and is called the **range/column space** of A

- The range space for this matrix is also called the **column** space of A since:

$$\mathcal{R}(A) = \text{span}\,\{a_1, \; a_2, \; \cdots, \; a_n\}$$

  where a_i are the columns of A, only drawback is that $\{a_1, \; \cdots, \; a_n\}$ may not be lin. indep.

- Let $A = \begin{pmatrix} 1 & 3 & 4 \\ 2 & 6 & 8 \end{pmatrix}$, $L_A(x) : \mathbb{R}^3 \to \mathbb{R}^2$, what is $\mathcal{R}(A) \subseteq \mathbb{R}^2$? Simply find linearly independent columns
- To find linearly indep. columns: rref $\{A\}$ and the leading ones show the location of linearly independent columns of A
- In this case, the 2 and 3 column are multiples of the first, therefore $\mathcal{R}(A) = \text{span}\left\{\begin{pmatrix} 1 \\ 2 \end{pmatrix}\right\}$

Null/Range Space Example

- Find a **basis for** the null and column space of the following matrix:

$$A = \begin{bmatrix} 1 & 1 & 1 & 0 \\ 2 & 1 & 0 & 1 \end{bmatrix}$$

- $L_A : \mathbb{R}^4 \to \mathbb{R}^2, \quad$ The null space is $\mathcal{N}(A) = \{x \in \mathbb{R}^4 \mid Ax = 0\} \subseteq \mathbb{R}^4$:

$$Ax = 0 \quad \Rightarrow \quad \begin{pmatrix} 1 & 1 & 1 & 0 \\ 2 & 1 & 0 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \Rightarrow \quad \text{rref(A)} \begin{pmatrix} 1 & 0 & -1 & 1 \\ 0 & 1 & 2 & -1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\Rightarrow \quad \begin{matrix} x_1 - x_3 + x_4 = 0 \\ x_2 + 2x_3 - x_4 = 0 \end{matrix} \quad \Rightarrow \quad \begin{matrix} x_1 = x_3 - x_4 \\ x_2 = -2x_3 + x_4 \end{matrix}$$

- The $\mathcal{N}(A) = \left\{ x \in \mathbb{R}^4 \mid x = \begin{pmatrix} x_3 - x_4 \\ -2x_3 + x_4 \\ x_3 \\ x_4 \end{pmatrix} = x_3 \begin{pmatrix} 1 \\ -2 \\ 1 \\ 0 \end{pmatrix} + x_4 \begin{pmatrix} -1 \\ 1 \\ 0 \\ 1 \end{pmatrix} \right\} = \text{span} \left\{ \begin{pmatrix} 1 \\ -2 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ 1 \\ 0 \\ 1 \end{pmatrix} \right\}$
- The dimension of the null space is 2, i.e. $\dim(\mathcal{N}(A)) = 2$

Null/Range Space Example

- Find a **basis for** the null and column space of the following matrix:

$$A = \begin{bmatrix} 1 & 1 & 1 & 0 \\ 2 & 1 & 0 & 1 \end{bmatrix}$$

- $L_A : \mathbb{R}^4 \to \mathbb{R}^2$, The range/column space of $A = \text{span}\{a_1, a_2, a_3, a_4\}$
- Find a basis for the column space:

$$A = \begin{pmatrix} 1 & 1 & 1 & 0 \\ 2 & 1 & 0 & 1 \end{pmatrix} \Rightarrow \text{rref}\{A\} = \begin{pmatrix} 1 & 0 & -1 & 1 \\ 0 & 1 & 2 & -1 \end{pmatrix}$$

- Therefore, the first and second column are linearly independent and form a basis for $\mathcal{R}(A)$
- $\mathcal{R}(A) = \text{span}\left\{\begin{pmatrix} 1 \\ 2 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \end{pmatrix}\right\}$