EE 419/519: Industrial Control Systems

Lecture 7: Observability and Dynamic Feedback

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Topic Outline

- **Observability**
  - Observability Definition
  - Unobservable Subspace
  - Luenberger Observer
  - Detectability
- **Dynamic Feedback**

State Feedback Controller

- We will now consider a more complete LTI state space model:

$$\begin{aligned} \dot{x} &= Ax + Bu, \quad && x \in \mathbb{R}^n,\ u \in \mathbb{R}^m \\ y &= Cx, && y \in \mathbb{R}^p \end{aligned}$$

- If the system is **controllable**, then a state feedback $u = Kx$ can asymptotically stabilize the system (we can place the closed loop eigenvalues freely)

- **What if we cannot measure/obtain all of the states x?**

Dynamic Feedback Controller

- We will now consider a more complete LTI state space model:

$$\begin{aligned} \dot{x} &= Ax + Bu, \quad && x \in \mathbb{R}^n,\ u \in \mathbb{R}^m \\ y &= Cx, && y \in \mathbb{R}^p \end{aligned}$$

- If the system is **controllable**, then a state feedback $u = Kx$ can asymptotically stabilize the system (we can place the closed loop eigenvalues freely)
- **Can we develop an "observer" to obtain an estimate of x, defined as $\hat{x}$?**
- When is this possible?

Observability Definition

- Without loss of generality, let's consider a system without inputs

$$\begin{aligned} \dot{x} &= Ax, \quad x \in \mathbb{R}^n, \\ y &= Cx, \quad y \in \mathbb{R}^p \end{aligned} \qquad x(0) = x_0$$

- **Definition:** The pair $(C,\ A)$ is **observable** for any $t \in [0,\ T]$ where $T < \infty$, we can find $x(0)$ from $y(t)$

- Main idea: If $x(0)$ can be obtained, then we can **reconstruct** the states $x(t)$ for any $t \in [0,\ T]$ using only the outputs $y(t)$

Observability Definition (cont'd)

- Without loss of generality, let's consider a system without inputs

$$\begin{aligned} \dot{x} &= Ax, \quad x \in \mathbb{R}^n, \\ y &= Cx, \quad y \in \mathbb{R}^p \end{aligned} \qquad\qquad x(0) = x_0$$

- **Definition:** The pair $(C,\ A)$ is **observable** for any $t \in [0,\ T]$ where $T < \infty$, we can find $x(0)$ from $y(t)$

- How can we obtain $x(0)$? We can differentiate the output $(n-1)$ times (Cayley Hamilton):

Observability Matrix

- Without loss of generality, let's consider a system without inputs

$$\begin{aligned} \dot{x} &= Ax, \quad x \in \mathbb{R}^n, \\ y &= Cx, \quad y \in \mathbb{R}^p \end{aligned} \qquad x(0) = x_0$$

- **Definition:** The pair $(C,\ A)$ is **observable** for any $t \in [0,\ T]$ where $T < \infty$, we can find $x(0)$ from $y(t)$

- The pair $(C,\ A)$ is **observable** if and only if the observability matrix has full rank:

$$\operatorname{rank}\{\mathcal{O}\} = \operatorname{rank}\left\{\begin{pmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{pmatrix}\right\} = n$$

Observability Matrix – Unobservable Subspace

- Without loss of generality, let's consider a system without inputs

$$\begin{aligned} \dot{x} &= Ax, \quad x \in \mathbb{R}^n, \\ y &= Cx, \quad y \in \mathbb{R}^p \end{aligned} \qquad x(0) = x_0$$

- The **unobservable subspace** is given by:

$$\mathcal{N}\{\mathcal{O}\} = \mathcal{N}\left\{\begin{pmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{pmatrix}\right\}$$

  - The unobservable subspace is the **largest** $A-$**invariant subspace** contained in the $\mathcal{N}\{C\}$
  - The $\text{rank}\{\mathcal{O}\} + \dim\{\mathcal{N}\{\mathcal{O}\}\} = n$ (rank-nullity theorem)

Decomposition w.r.t. Unobservable Subspace

- Let's consider a general state space model:

$$\begin{aligned} \dot{x} &= Ax + Bu, & x \in \mathbb{R}^n,\ u \in \mathbb{R}^m \\ y &= Cx, & y \in \mathbb{R}^p \end{aligned}$$

- Assume the $\dim\{\mathcal{N}\{\mathcal{O}\}\}\, p \geq 1$ (the system is unobservable) and define a basis for $\mathcal{N}\{\mathcal{O}\} = \operatorname{span}\{v_1,\ \cdots,\ v_p\}$
- Find a complementary subspace for $\mathbb{R}^n = \mathcal{N}\{\mathcal{O}\} \oplus \mathcal{W}$
- What is the system structure w.r.t. basis given by $T = \big(\underbrace{v_1\ \cdots\ v_p}_{\mathcal{N}\{\mathcal{O}\}}\ \underbrace{w_1\ \cdots\ w_{n-p}}_{\mathcal{W}}\big)$

Decomposition w.r.t. Unobservable Subspace

- Let's consider a general state space model:

$$\dot{x} = Ax + Bu, \quad x \in \mathbb{R}^n, \; u \in \mathbb{R}^m$$
$$y = Cx, \quad y \in \mathbb{R}^p$$

- Assume the $\dim\{\mathcal{N}\{\mathcal{O}\}\}\, p \geq 1$ (the system is unobservable)

- What is the system structure w.r.t. basis given by $T = \big(\underbrace{v_1 \;\cdots\; v_p}_{\mathcal{N}\{\mathcal{O}\}} \; \underbrace{w_1 \;\cdots\; w_{n-p}}_{\mathcal{W}}\big)$

- The system is transformed ($x = Tz$):

$$\begin{pmatrix} \dot{z}_1 \\ \dot{z}_2 \end{pmatrix} = \begin{pmatrix} \tilde{A}_{11} & \tilde{A}_{12} \\ 0 & \tilde{A}_{22} \end{pmatrix} \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} + \begin{pmatrix} \tilde{B}_1 \\ \tilde{B}_2 \end{pmatrix} u, \qquad y = \begin{pmatrix} 0 & \tilde{C}_2 \end{pmatrix} \begin{pmatrix} z_1 \\ z_2 \end{pmatrix}$$

  - The pair $(\tilde{A}_{22},\, \tilde{C}_2)$ is **observable**
  - The state z_1 / modes given by $\tilde{A}_{11}$ are unobservable!
  - The system is **detectable** if $\forall \lambda \in \sigma(\tilde{A}_{11})$ the $\mathrm{Re}\{\lambda\} < 0$

Duality

- Let's consider a general state space model:

$$\begin{aligned} \dot{x} &= Ax + Bu, \quad && x \in \mathbb{R}^n,\ u \in \mathbb{R}^m \\ y &= Cx, && y \in \mathbb{R}^p \end{aligned}$$

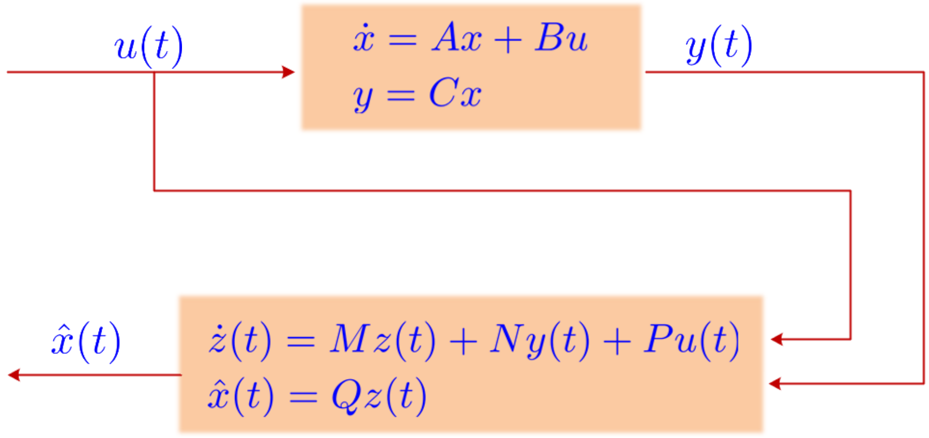
- The controllability and observability matrices are related $\mathcal{O}\left(C,\ A\right) = \mathcal{W}\left(A^T,\ C^T\right)^T$
- We can investigate observability of $(C,\ A)$ by studying controllability of $(A^T,\ C^T)$
- $(C,\ A)$ is observable if rank $\left\{\mathcal{W}(A^T,\ C^T)\right\} = n$
- *We can use this for observer design!*

Observer Definition

- Let's consider a general state space model:

$$\dot{x} = Ax + Bu, \quad x \in \mathbb{R}^n, \; u \in \mathbb{R}^m$$
$$y = Cx, \quad y \in \mathbb{R}^p$$

- **Definition:** A **state observer** is a (dynamical) system that provides an estimate, $\hat{x}(t)$, of the internal state, $x(t)$, of a real system using only the inputs, $u(t)$, and the outputs, $y(t)$.



- When can we design an observer? What should M, N, P, Q be?

- What are the dynamics of the error $e(t) = x(t) - \hat{x}(t)$? (Goal: $e(t) \to 0$)

Luenberger Observer

- Let's consider a general state space model:

$$\dot{x} = Ax + Bu, \quad x \in \mathbb{R}^n, \; u \in \mathbb{R}^m$$
$$y = Cx, \quad y \in \mathbb{R}^p$$

- We will consider the following Luenberger observer:

$$\dot{z} = Az + Bu + L(y - Cz), \quad \hat{x} = z$$

- What are the error dynamics $e(t) = x - z$?

Luenberger Observer Conditions

- Let's consider a general state space model:

$$\begin{aligned} \dot{x} &= Ax + Bu, \quad && x \in \mathbb{R}^n,\ u \in \mathbb{R}^m \\ y &= Cx, && y \in \mathbb{R}^p \end{aligned}$$

- We will consider the following Luenberger observer:

$$\begin{aligned} \dot{z} &= Az + Bu + L(y - Cz) \\ \hat{x} &= z \end{aligned} \quad \Rightarrow \quad \begin{aligned} \dot{z} &= (A - LC)z + Bu + Ly \\ \hat{x} &= z \end{aligned}$$

- The error dynamics are given by $\dot{e} = (A - LC)e$
  - The error will converge to zero, $e(t) \to 0$ or $\hat{x} \to x$, iff $\forall \lambda \in \sigma(A - LC)$ the $\mathrm{Re}\{\lambda\} < 0$
  - The eigenvalues or $\sigma(A - LC)$ is freely assignable (by L) iff $(C,\ A)$ is observable
  - By duality, we can use a similar procedure as state feedback design to obtain L:

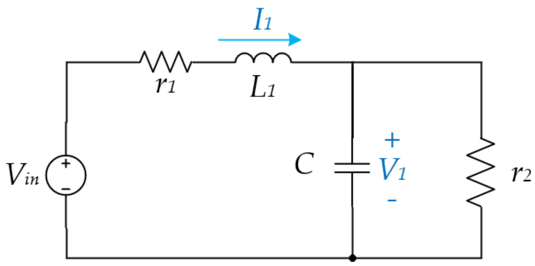
```
%% Obtain feedback matrix using place
L = place(A', C', lam_des)';
```

Luenberger Observer Example

- Let's consider the following circuit:

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} -\frac{r_1}{L_1} & \frac{-1}{L_1} \\ \frac{1}{C} & -\frac{1}{r_2 C} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} \frac{1}{L_1} \\ 0 \end{pmatrix} u$$

$$y = \begin{pmatrix} 1 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$



- Assume that we can only measure the inductor current x_1 and $r_1 = 1$ mΩ, $L_1 = 10$ mH, $C_1 = 10$ mF, and $r_2 = 1\ \Omega$
- We would like to build an observer to estimate the capacitor voltage
- Is the system observable?

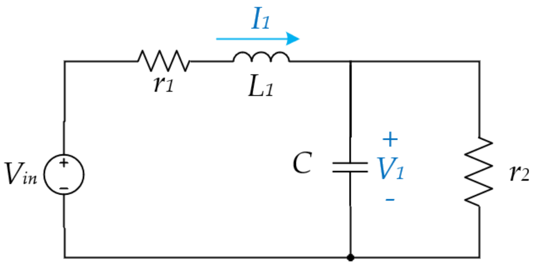
$$\mathcal{O} = \begin{pmatrix} C \\ CA \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ -0.1 & -100 \end{pmatrix} \Rightarrow \text{Rank}(\mathcal{O}) = 2 \qquad \text{System is observable!}$$

Luenberger Observer Example

- Let's consider the following circuit:

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} -\frac{r_1}{L_1} & \frac{-1}{L_1} \\ \frac{1}{C} & -\frac{1}{r_2 C} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} \frac{1}{L_1} \\ 0 \end{pmatrix} u$$

$$y = \begin{pmatrix} 1 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$



- Assume that we can only measure the inductor current x_1 and $r_1 = 1\ \text{m}\Omega$, $L_1 = 10\ \text{mH}$, $C_1 = 10\ \text{mF}$, and $r_2 = 1\ \Omega$

- Build a Luenberger observer with eigenvalues at $\{-1000,\ -900\}$
  - $A,\ B,\ C$ are known, we need to design L!
  - $L = \text{place}\,(A',\ C',\ [-1000,\ -900])'$

$$\dot{z} = (A - LC)z + Bu + Ly$$
$$\hat{x} = z$$

$$L = \begin{pmatrix} 1799.9 \\ -7100 \end{pmatrix}$$

$$\sigma(A - LC) = \{-1000,\ -900\}$$

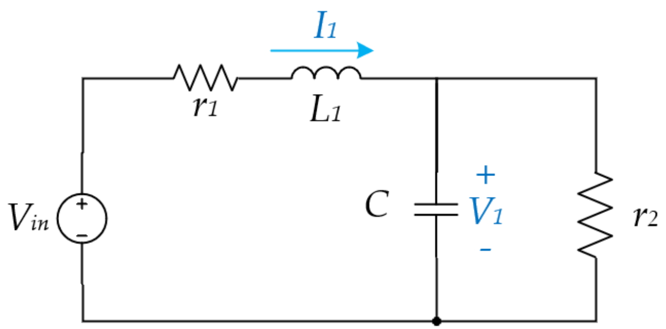
Luenberger Observer Example

- Let's consider the following circuit:

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} -\frac{r_1}{L_1} & \frac{-1}{L_1} \\ \frac{1}{C} & -\frac{1}{r_2 C} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} \frac{1}{L_1} \\ 0 \end{pmatrix} u$$

$$y = \begin{pmatrix} 1 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

- How does the observer work?



$$\dot{z} = (A - LC)z + Bu + Ly$$
$$\hat{x} = z$$

Luenberger Observer Example

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$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} -\frac{r_1}{L_1} & \frac{-1}{L_1} \\ \frac{1}{C} & -\frac{1}{r_2 C} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} \frac{1}{L_1} \\ 0 \end{pmatrix} u$$

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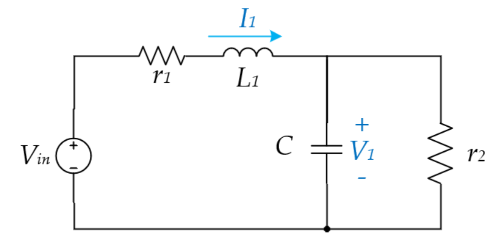
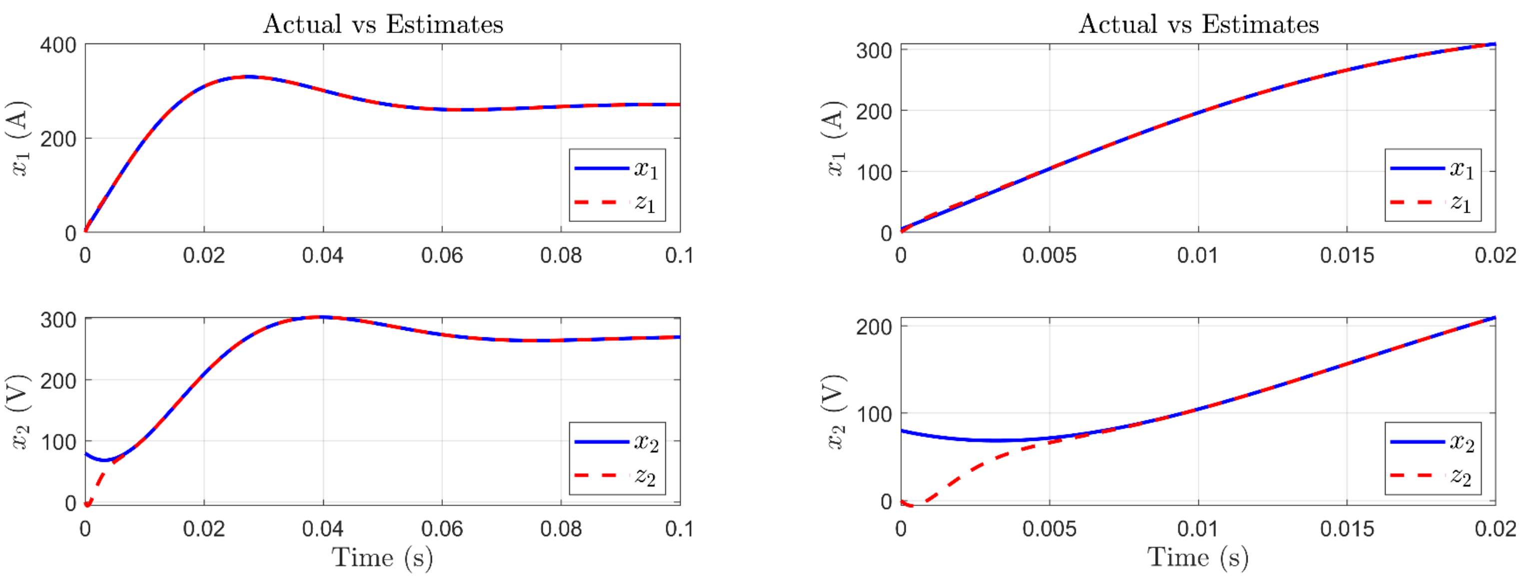


- Simulation results with $u = 270$ V and $i_1(0) = 5$ A and $v_1(0) = 80$ V



Unobservable Systems

- Let's consider a general state space model: $\dot{x} = Ax + Bu, \quad y = Cx$
- What if the pair $(C,\ A)$ is not observable?
- There exists a basis matrix $T = \begin{pmatrix} \underbrace{v_1 \ \cdots \ v_p}_{\mathcal{N}\{\mathcal{O}\}} & \underbrace{w_1 \ \cdots \ w_{n-p}}_{\mathcal{W}} \end{pmatrix}$

  such that the system can be transformed to:

$$\begin{pmatrix} \dot{\tilde{x}}_1 \\ \dot{\tilde{x}}_2 \end{pmatrix} = \begin{pmatrix} \tilde{A}_{11} & \tilde{A}_{12} \\ 0 & \tilde{A}_{22} \end{pmatrix} \begin{pmatrix} \tilde{x}_1 \\ \tilde{x}_2 \end{pmatrix} + \begin{pmatrix} \tilde{B}_1 \\ \tilde{B}_2 \end{pmatrix} u, \qquad y = \begin{pmatrix} 0 & \tilde{C}_2 \end{pmatrix} \begin{pmatrix} \tilde{x}_1 \\ \tilde{x}_2 \end{pmatrix}$$

- Design a Luenberger for the system in the new basis

Unobservable Systems

- Let's consider a general state space model: $\dot{x} = Ax + Bu, \quad y = Cx$
- What if the pair $(C, \ A)$ is not observable?
- There exists a basis matrix $T = \big(\underbrace{v_1 \ \cdots \ v_p}_{\mathcal{N}\{\mathcal{O}\}} \ \underbrace{w_1 \ \cdots \ w_{n-p}}_{\mathcal{W}}\big)$

  such that the system can be transformed to:

$$\begin{pmatrix} \dot{\tilde{x}}_1 \\ \dot{\tilde{x}}_2 \end{pmatrix} = \begin{pmatrix} \tilde{A}_{11} & \tilde{A}_{12} \\ 0 & \tilde{A}_{22} \end{pmatrix} \begin{pmatrix} \tilde{x}_1 \\ \tilde{x}_2 \end{pmatrix} + \begin{pmatrix} \tilde{B}_1 \\ \tilde{B}_2 \end{pmatrix} u, \qquad y = \begin{pmatrix} 0 & \tilde{C}_2 \end{pmatrix} \begin{pmatrix} \tilde{x}_1 \\ \tilde{x}_2 \end{pmatrix}$$

- A Luenberger observer with feedback matrix $\tilde{L} = \begin{pmatrix} \tilde{L}_1 \\ \tilde{L}_2 \end{pmatrix}$ is of the form:

$$\begin{pmatrix} \dot{\tilde{z}}_1 \\ \dot{\tilde{z}}_2 \end{pmatrix} = \begin{pmatrix} \tilde{A}_{11} & \tilde{A}_{12} - \tilde{L}_1\tilde{C}_2 \\ 0 & \tilde{A}_{22} - \tilde{L}_2\tilde{C}_2 \end{pmatrix} \begin{pmatrix} \tilde{z}_1 \\ \tilde{z}_2 \end{pmatrix} + \begin{pmatrix} \tilde{B}_1 \\ \tilde{B}_2 \end{pmatrix} u + \begin{pmatrix} \tilde{L}_1 \\ \tilde{L}_2 \end{pmatrix} y$$

  - Assume $\tilde{L}_2$ chosen such that $\sigma(\tilde{A}_{22} - \tilde{L}_2\tilde{C}_2)$ have negative real part
  - $\dot{e}(t) = (A - LC)e \quad \Rightarrow \quad \sigma(A - LC) = \sigma(\tilde{A} - \tilde{L}\tilde{C}) = \sigma(\tilde{A}_{11}) \cup \sigma(\tilde{A}_{22} - \tilde{L}_2\tilde{C}_2)$
  - The error $e(t) \to 0$ iff $\sigma(\tilde{A}_{11})$ have negative real part (system is detectable)
  - We can compute $L = T\tilde{L}$, $\tilde{L}_1$ can be anything, e.g. $\tilde{L}_1 = 0$

Topic Outline

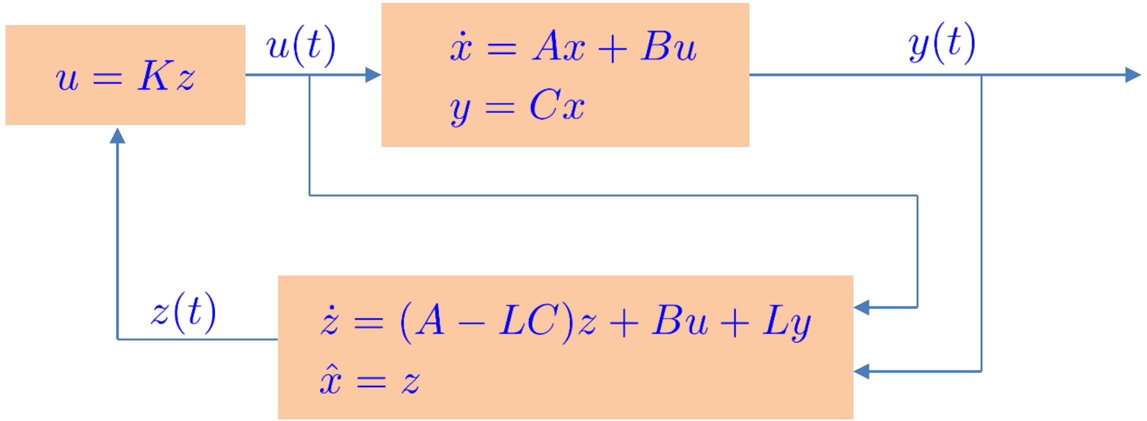
- **Observability**
  - Observability Definition
  - Unobservable Subspace
  - Luenberger Observer
  - Detectability
- **Dynamic Feedback**

Motivation

- We will now consider a more complete LTI state space model:

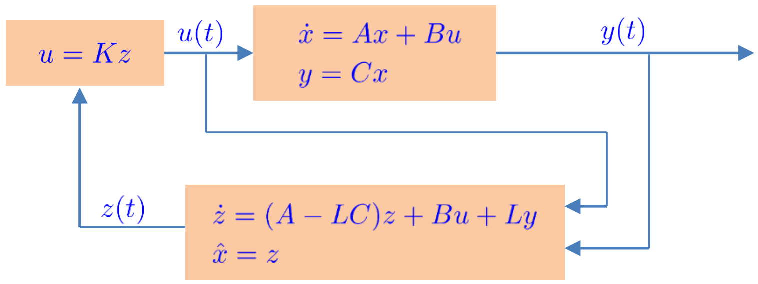
$$\begin{aligned} \dot{x} &= Ax + Bu, & x \in \mathbb{R}^n,\ u \in \mathbb{R}^m \\ y &= Cx, & y \in \mathbb{R}^p \end{aligned}$$

- **Goal:** design a controller such that the closed loop system is asymptotically stable
- Approach: combine **state observer** with **state feedback**



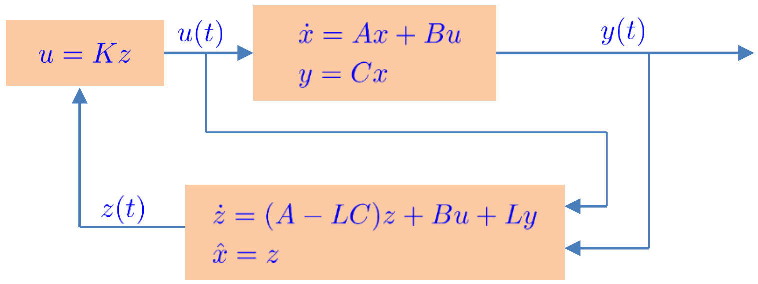
Closed Loop Analysis

- We will now consider a more complete LTI state space model: $\dot{x} = Ax + Bu, \quad y = Cx$
- **Goal:** design a controller such that the closed loop system is asymptotically stable
- The observer is of the form: $\dot{z} = (A - LC)z + Bu + Ly$
- What are the closed loop system dynamics?



Closed Loop Analysis (cont'd)

- We will now consider a more complete LTI state space model: $\dot{x} = Ax + Bu, \quad y = Cx$
- **Goal:** design a controller such that the closed loop system is asymptotically stable
- The observer is of the form: $\dot{z} = (A - LC)z + Bu + Ly$



- The closed loop system is of the form:

$$\begin{pmatrix} \dot{x} \\ \dot{z} \end{pmatrix} = \begin{pmatrix} A & BK \\ LC & A - LC + BK \end{pmatrix} \begin{pmatrix} x \\ z \end{pmatrix} \Rightarrow \dot{x}_{cl} = A_{cl} x_{cl}$$

- The closed loop system is asymp. stable iff $\forall \lambda \in \sigma(A_c l)$, $\mathrm{Re}\{\lambda\} < 0$
- What are the closed loop eigenvalues, i.e. $\sigma(A_{cl})$?

Closed Loop Analysis – New Basis

- We will now consider a more complete LTI state space model: $\dot{x} = Ax + Bu, \quad y = Cx$
- The observer is of the form: $\dot{z} = (A - LC)z + Bu + Ly$
- The closed loop system is of the form:

$$\begin{pmatrix} \dot{x} \\ \dot{z} \end{pmatrix} = \begin{pmatrix} A & BK \\ LC & A - LC + BK \end{pmatrix} \begin{pmatrix} x \\ z \end{pmatrix} \quad \Rightarrow \quad \dot{x}_{cl} = A_{cl} x_{cl}$$

- What are the closed loop eigenvalues, i.e. $\sigma(A_{cl})$?

Separation Principle

- We will now consider a more complete LTI state space model: $\dot{x} = Ax + Bu, \quad y = Cx$
- The observer is of the form: $\dot{z} = (A - LC)z + Bu + Ly$
- The closed loop system is of the form:

$$\begin{pmatrix} \dot{x} \\ \dot{z} \end{pmatrix} = \begin{pmatrix} A & BK \\ LC & A - LC + BK \end{pmatrix} \begin{pmatrix} x \\ z \end{pmatrix} \quad \Rightarrow \quad \dot{x}_{cl} = A_{cl} x_{cl}$$

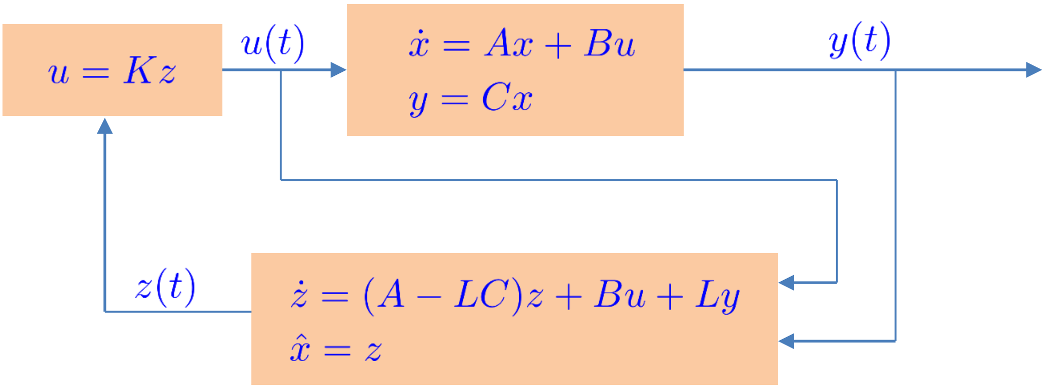
- Analyzing the system in a basis defined by $\begin{pmatrix} x \\ z \end{pmatrix} = \begin{pmatrix} I & 0 \\ I & -I \end{pmatrix} \begin{pmatrix} x \\ e \end{pmatrix}$:

$$\begin{pmatrix} \dot{x} \\ \dot{e} \end{pmatrix} = \begin{pmatrix} A + BK & -BK \\ 0 & A - LC \end{pmatrix} \begin{pmatrix} x \\ e \end{pmatrix} \quad \Rightarrow \quad \dot{\tilde{x}}_{cl} = \tilde{A}_{cl} \tilde{x}_{cl}$$

- We can see the closed loop eigenvalues are $\sigma(A_{cl}) = \sigma(\tilde{A}_{cl}) = \sigma(A + BK) \cup \sigma(A - LC)$
  - We can simply design the observer and state feedback separately!
  - This is known as the **Separation Principle**

Separation Principle – Design

- We will now consider a more complete LTI state space model: $\dot{x} = Ax + Bu, \quad y = Cx$
- The observer is of the form: $\dot{z} = (A - LC)z + Bu + Ly$
- The controller is of the form: $u = Kz$
- The closed loop eigenvalues are $\sigma(A_{cl}) = \sigma(\tilde{A}_{cl}) = \sigma(A + BK) \cup \sigma(A - LC)$
- What location of the poles give us good performance?
  - Several ways to approach this
  - One technique is for the observer eigenvalues to be much faster (less than) the controller eigenvalues:
    $\lambda_{obs} \ll \lambda_{ctr} \quad$ for all $\lambda_{obs} \in \sigma(A - LC),\ \lambda_{ctr} \in \sigma(A + BK)$

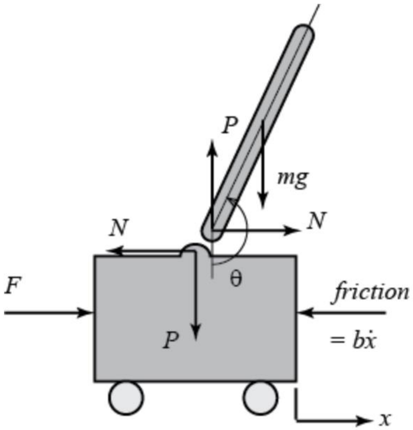


Inverted Pendulum Example

- Let's consider an inverted pendulum pushed by a cart, the **linearized dynamics** are as follows:

$$\begin{pmatrix}\dot{x}\\ \dot{v}\\ \dot{\phi}\\ \dot{\omega}\end{pmatrix} = \begin{pmatrix}0 & 1 & 0 & 0\\ 0 & \frac{-(I+ml^2)b}{I(M+m)+Mml^2} & \frac{m^2gl^2}{I(M+m)+Mml^2} & 0\\ 0 & 0 & 0 & 1\\ 0 & \frac{-mlb}{I(M+m)+Mml^2} & \frac{mgl(M+m)}{I(M+m)+Mml^2} & 0\end{pmatrix}\begin{pmatrix}x\\ v\\ \phi\\ \omega\end{pmatrix} + \begin{pmatrix}0\\ \frac{I+ml^2}{I(M+m)+Mml^2}\\ 0\\ \frac{ml}{I(M+m)+Mml^2}\end{pmatrix} u$$

$$y = \begin{pmatrix}x\\ \phi\end{pmatrix} = \begin{pmatrix}1 & 0 & 0 & 0\\ 0 & 0 & 1 & 0\end{pmatrix}\begin{pmatrix}x\\ v\\ \phi\\ \omega\end{pmatrix}$$



M (cart mass)	0.5 kg	m (pendulum mass)	0.2 kg
b (cart friction)	0.1 N/m/s	l (pend. length to center)	0.3 m
I (pendulum inertia)	0.006 kgm2		

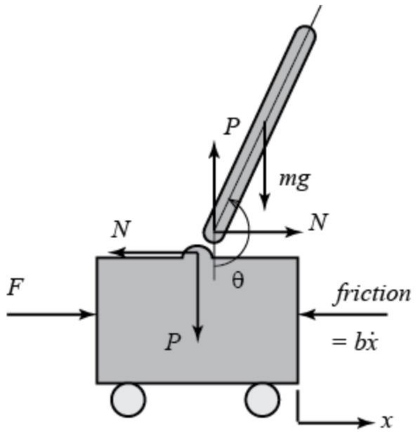
https://ctms.engin.umich.edu/CTMS/index.php?example=InvertedPendulum&section=SystemModeling

Inverted Pendulum Example – Obs. And Controllability

- Let's consider an inverted pendulum pushed by a cart, the **linearized dynamics** are as follows:

$$\begin{pmatrix} \dot{x} \\ \dot{v} \\ \dot{\phi} \\ \dot{\omega} \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & -0.1818 & 2.6727 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & -0.4545 & 31.182 & 0 \end{pmatrix} \begin{pmatrix} x \\ v \\ \phi \\ \omega \end{pmatrix} + \begin{pmatrix} 0 \\ 1.8182 \\ 0 \\ 4.5455 \end{pmatrix} u$$

$$y = \begin{pmatrix} x \\ \phi \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} x \\ v \\ \phi \\ \omega \end{pmatrix}$$



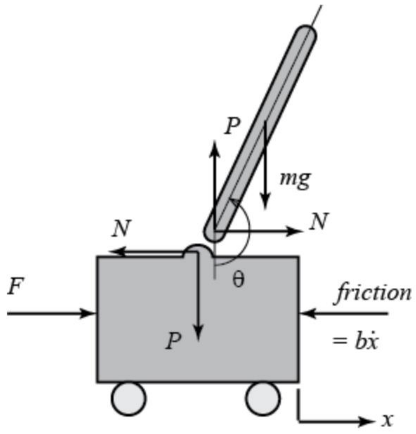
- The system is observable, i.e. $\operatorname{rank}\{\mathcal{O}\} = \operatorname{rank}\left\{\begin{pmatrix} C \\ CA \\ CA^2 \\ CA^3 \end{pmatrix}\right\} = 4$

- The system is controllable, i.e. $\operatorname{rank}\{\mathcal{W}\} = \operatorname{rank}\left\{\begin{pmatrix} B & AB & A^2B & A^3B \end{pmatrix}\right\} = 4$

https://ctms.engin.umich.edu/CTMS/index.php?example=InvertedPendulum&section=SystemModeling

Inverted Pendulum Example – Controller Design

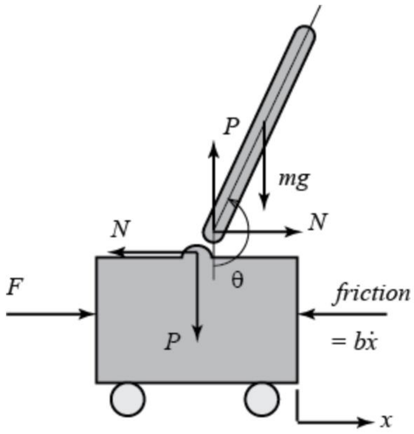
- Let's consider an inverted pendulum pushed by a cart
- The controller is composed of an observer and a regulator:

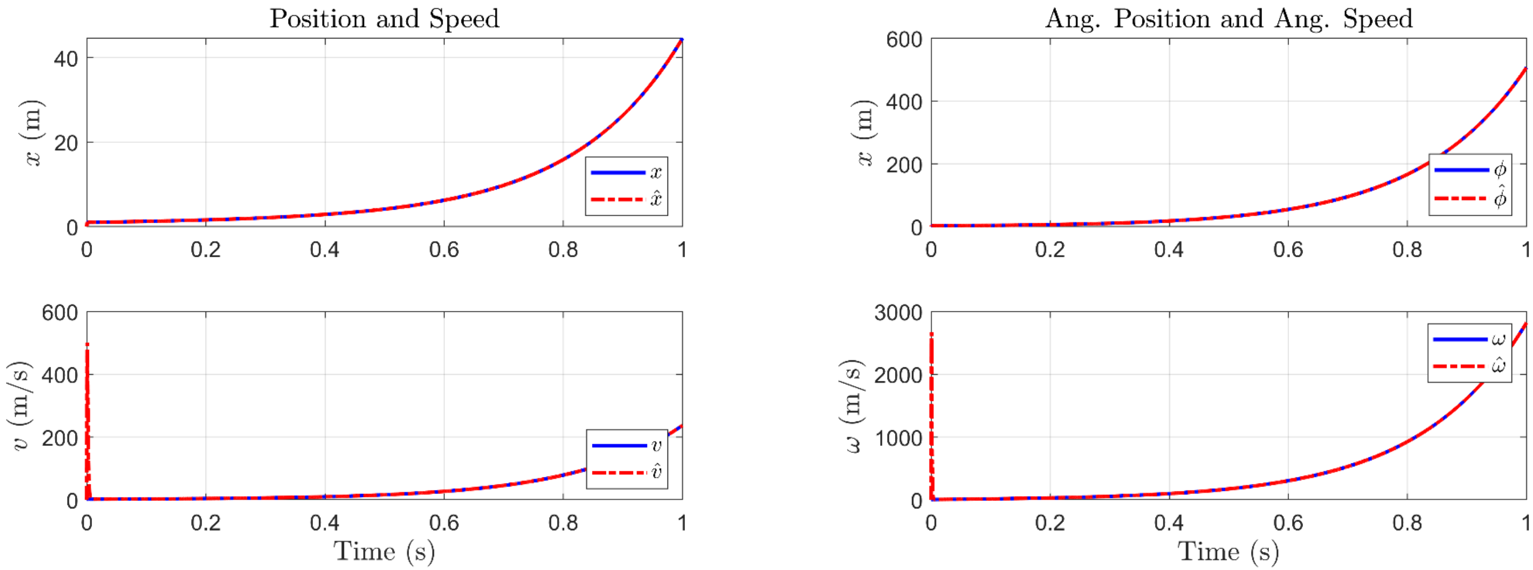


https://ctms.engin.umich.edu/CTMS/index.php?example=InvertedPendulum&section=SystemModeling

Inverted Pendulum Example – Open Loop Results

- Let's consider an inverted pendulum pushed by a cart
- The controller is composed of an observer and a regulator:

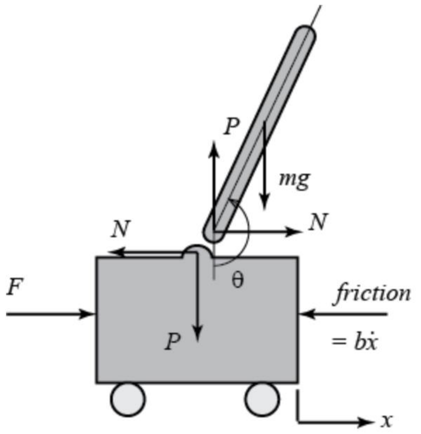


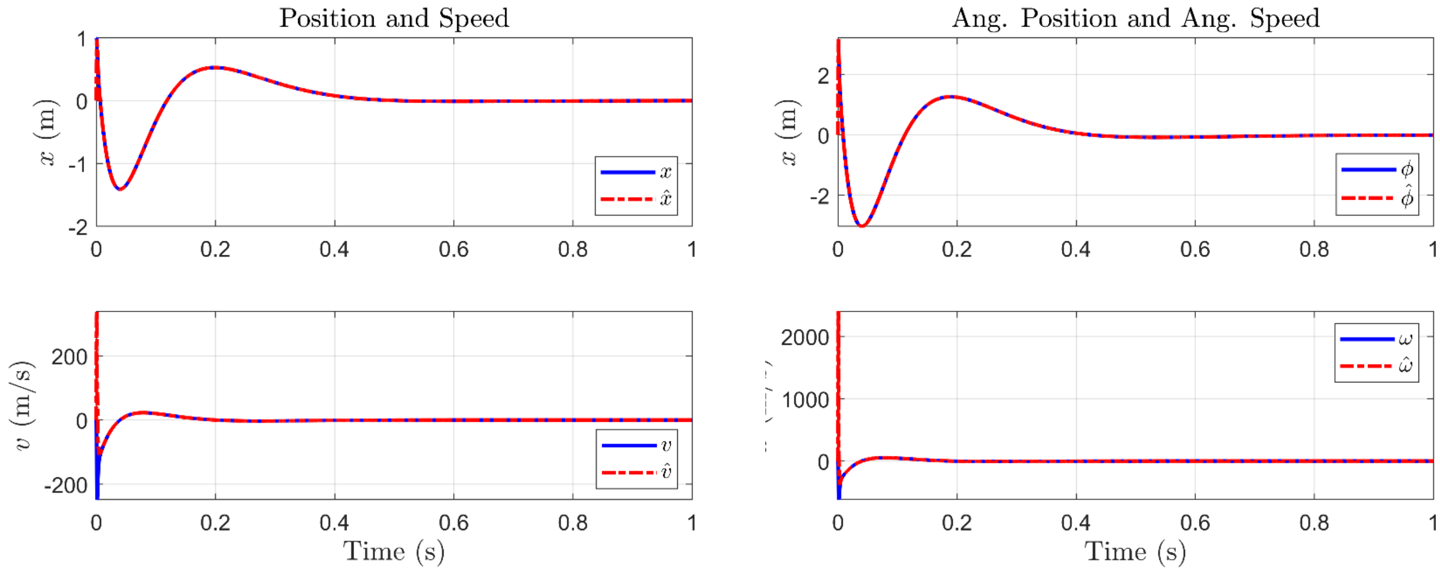


https://ctms.engin.umich.edu/CTMS/index.php?example=InvertedPendulum&section=SystemModeling

Inverted Pendulum Example – Closed Loop Results

- Let's consider an inverted pendulum pushed by a cart
- The controller is composed of an observer and a regulator:





Summary

- We have discussed state feedback design for state space systems with the goal of asymptotically stabilizing a system (controllability and stabilizability)
- When the states are not available for measurement, we can develop a state estimator/observer (observability and detectability)
- We can then combine the observer and controller to develop a dynamic feedback for asymptotic stabilization
- By the **separation principle** the observer and controller can be designed separately