EE 419/519: Industrial Control Systems

Lecture 6: Stability and Controllability

Dr. Luis Herrera
Dept. of Electrical Engineering
University at Buffalo

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Topic Outline

- **Internal Stability**
  - Solution by modal form
  - Stability definitions
- **Controllability**
  - Reachability subspace
  - Controllability, stabilizability
  - Controller canonical form
  - State feedback

State Space System Solution in Different Basis

- Let's consider again an **unforced** state space system

$$\dot{x} = Ax, \quad x(0) = x_0$$

where $x \in \mathbb{R}^n$ $(A \in \mathbb{R}^{n \times n})$.

- **What if the system is analyzed in a new basis, e.g.** $\mathcal{V} = \{v_1, \cdots, v_n\}$**?**
$\Rightarrow \quad \forall x \in \mathbb{R}^n, \;\; x = Vz$ or $z = V^{-1}x$

$$\Rightarrow \quad \dot{z} = \underbrace{V^{-1}AV}_{\Lambda} z = \Lambda z, \qquad z(0) = z_0 = V^{-1}x_0$$

- Which has a solution as follows:

$$z(t) = e^{\Lambda t} z_0 \quad \Rightarrow \quad x(t) = \underbrace{V e^{\Lambda t} V^{-1}}_{e^{At}} x(0)$$

- **This gives us another formula for computing** $\mathbf{e^{At}} = V e^{\Lambda t} V^{-1}$

Jordan Canonical Form Solution

- From the previous slides/lectures, we have discussed the existence of a basis given by the columns of $V = \begin{bmatrix} v_1 & \cdots & v_n \end{bmatrix} \in \mathbb{R}^{n \times n}$, such that $\Lambda = V^{-1}AV$ is a Jordan matrix:

$$\dot{x} = Ax \quad \Rightarrow \quad \dot{z} = \Lambda z, \quad \Lambda = \begin{pmatrix} J_{n_1}(\lambda_1) & 0 & \cdots & 0 \\ 0 & J_{n_2}(\lambda_2) & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & J_{n_q}(\lambda_q) \end{pmatrix}$$

  where each Jordan block is as follows:

$$J_k(\lambda) = \begin{pmatrix} \lambda & 1 & 0 & \cdots & 0 \\ 0 & \lambda & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda & 1 \\ 0 & 0 & \cdots & 0 & \lambda \end{pmatrix}$$

- To compute the matrix exponential in the standard basis: $e^{At} = Ve^{\Lambda t}V^{-1}$, we need to know $e^{\Lambda t}$

Jordan Canonical Form Solution (2)

- The matrix exponential of a matrix in Jordan form is:

$$\dot{z} = \Lambda z \quad \Rightarrow \quad z(t) = e^{\Lambda t} z_0, \quad \text{where} \;\; e^{\Lambda t} = \begin{pmatrix} e^{J_{n_1}(\lambda_1)t} & 0 & \cdots & 0 \\ 0 & e^{J_{n_2}(\lambda_2)t} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & e^{J_{n_q}(\lambda_q)t} \end{pmatrix}$$

where each block is in the following form:

$$J_k(\lambda) = \begin{pmatrix} \lambda & 1 & 0 & \cdots & 0 \\ 0 & \lambda & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda & 1 \\ 0 & 0 & \cdots & 0 & \lambda \end{pmatrix} \quad \Rightarrow \quad e^{J_k(\lambda)t} = \begin{pmatrix} e^{\lambda t} & e^{\lambda t} t & e^{\lambda t} \frac{t^2}{2!} & \cdots & e^{\lambda t} \frac{t^{k-1}}{(k-1)!} \\ 0 & e^{\lambda t} & e^{\lambda t} t & \cdots & e^{\lambda t} \frac{t^{k-2}}{(n-2)!} \\ 0 & 0 & e^{\lambda t} & \cdots & e^{\lambda t} \frac{t^{n-3}}{(k-3)!} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & e^{\lambda t} \end{pmatrix}$$

Modal Analysis – General Jordan Form Summary

- Consider the LTI unforced system $\dot{x} = Ax$, $x(0) = x_0$, and $x \in \mathbb{R}^n$
- Assume $\sigma(A) = \{\lambda_1, \lambda_2, \cdots, \lambda_q\} \subset \mathbb{C}$ (the eigenvalues of A) may be repeated and degenerate
- Solution to the original system is as follows:

$$x(t) = e^{At}x_0 = Ve^{\Lambda t}V^{-1}x_0 = Ve^{\Lambda t}z_0$$

  where Λ is in Jordan form

- The solution can then be given in terms of the **modes** of the system

$$x(t) = V\left[\sum_{i=1}^{q}\sum_{j=1}^{n_i} e^{\lambda_i t}\frac{t^{j-1}}{(j-1)!}\mathbf{w}_{ij}\right]$$

Example: RLC Circuit (1)

- Consider an RLC circuit:

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \begin{pmatrix} \frac{-r}{L} & \frac{-1}{L} \\ \frac{1}{C} & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

$$\sigma(A) = \{\lambda_1,\ \lambda_2\} = \{-456.15,\ -43.85\}$$

$$v_1 = \begin{pmatrix} -0.9158 \\ 0.4015 \end{pmatrix},\quad v_2 = \begin{pmatrix} 0.2141 \\ -0.9768 \end{pmatrix}$$

- Characterize the solutions for any initial condition, $x(0) = x_0 \in \mathbb{R}^2$, and $r = 5\ \Omega,\ L = 0.01\ \text{H},\ C = 0.005\ \text{F}$
  - Form a new basis given by the columns of $V = \begin{pmatrix} v_1 & v_2 \end{pmatrix}$
  - In the new basis, the system becomes:

$$\dot{z} = V^{-1}AVz \quad \Rightarrow \quad \begin{pmatrix} \dot{z}_1 \\ \dot{z}_2 \end{pmatrix} = \begin{pmatrix} -456.15 & 0 \\ 0 & -43.85 \end{pmatrix} \begin{pmatrix} z_1 \\ z_2 \end{pmatrix}$$

  - The solution of this system is easier to find:

$$\begin{pmatrix} z_1(t) \\ z_2(t) \end{pmatrix} = \begin{pmatrix} e^{-456.15t} & 0 \\ 0 & e^{-45.85t} \end{pmatrix} \begin{pmatrix} z_1(0) \\ z_2(0) \end{pmatrix}$$

Example: RLC Circuit (2)

- Consider an RLC circuit:

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \begin{pmatrix} \frac{-r}{L} & \frac{-1}{L} \\ \frac{1}{C} & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

- Characterize the solutions for any initial condition, $x(0) = x_0 \in \mathbb{R}^2$, and $r = 5,\ L = 0.01,\ C = 0.005$
  - Back in the original coordinates (standard basis), the solution is:

$$x(t) = V e^{\Lambda t} \underbrace{V^{-1} x(0)}_{z(0)} \Rightarrow \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} = \begin{pmatrix} v_1 & v_1 \end{pmatrix} \begin{pmatrix} e^{-456.15t} & 0 \\ 0 & e^{-43.85t} \end{pmatrix} \begin{pmatrix} z_1(0) \\ z_2(0) \end{pmatrix}$$

  - or:

$$\begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} = v_1 e^{-456.15t} z_1(0) + v_2 e^{-43.85t} z_2(0)$$

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  - Controllability, stabilizability
  - Controller canonical form
  - State feedback

Internal Stability - Definitions

- Consider the LTI unforced system $\dot{x} = Ax, \; x(0) = x_0$, and $x \in \mathbb{R}^n$
- Assume $\sigma(A) = \{\lambda_1, \; \lambda_2, \; \cdots, \; \lambda_q\} \subset \mathbb{C}$ (the eigenvalues of A) may be repeated and degenerate
- Solution to the original system is as follows:

$$x(t) = e^{At}x_0 = Ve^{\Lambda t}V^{-1}x_0 = Ve^{\Lambda t}z_0$$

  where Λ is in Jordan form
- The solution can then be given in terms of the **modes** of the system

$$x(t) = V\left[\sum_{i=1}^{q}\sum_{j=1}^{n_i} e^{\lambda_i t}\frac{t^{j-1}}{(j-1)!}\mathbf{w}_{ij}\right]$$

- As opposed to SISO systems, now we have an n-dimensional vector $x(t)$ which caracterizes the solution of the system
- **How can we define stability for this system?**
- **How can we assign a number (metric) for $x(t)$?**

Internal Stability - Norm

- A **norm** on $\mathbb{R}^n$ is a generalization of length. Any norm on $\mathbb{R}^n$ is a function/map $||\cdot|| : \mathbb{R}^n \to \mathbb{R}$ such that
    1. Non-negativity: $\forall x \in \mathbb{R}^n,\ ||x|| \geq 0$
    2. Positive definiteness: $\forall x \in \mathbb{R}^n,\ ||x|| = 0$ if and only if $x = 0$ (zero vector)
    3. Absolute homogeneity: $\forall \lambda \in \mathbb{R}$ and $x \in \mathbb{R}^n$, $||\lambda x|| = |\lambda|\ ||x||$
    4. Triangle inequality: $\forall x,\ y \in \mathbb{R}^n$, $||x + y|| \leq ||x|| + ||y||$

- The 2-norm on $\mathbb{R}^n$ is defined as $||x||_2 = \sqrt{x_1^2 + \cdots + x_n^2}$

- The 1-norm on $\mathbb{R}^n$ (taxicab norm) is defined as $||x||_1 = |x_1| + \cdots + |x_n|$

- In general, the p-norm on $\mathbb{R}^n$ $(p \geq 1)$ is defined as $||x||_p = (|x_1|^p + \cdots + |x_n|^p)^{\frac{1}{p}}$

- Lastly, the infinity norm is $||x||_\infty = \max_i |x_i|$

Internal Stability - Stable

- Consider an **unforced** LTI system $\dot{x} = Ax,\ x(0) = x_0$, and $x \in \mathbb{R}^n$
- **Definition:** The equilibrium point $x^* = 0$ of $\dot{x} = Ax$ is said to be:
  - ***Stable*** if for any $\epsilon > 0$, there exists a $\delta(\epsilon) > 0$ such that

$$||x(0)||_2 < \delta \quad \Rightarrow \quad ||x(t)||_2 < \epsilon, \quad \forall t \geq 0$$

Internal Stability – Asymptotically Stable

- Consider an **unforced** LTI system $\dot{x} = Ax, \; x(0) = x_0$, and $x \in \mathbb{R}^n$

- **Definition:** The equilibrium point $x^* = 0$ of $\dot{x} = Ax$ is said to be:
  - ***Asymptotically Stable*** if it is stable and a $\delta > 0$ can be chosen such that

$$||x(0)||_2 < \delta \quad \Rightarrow \quad \lim_{t \to \infty} ||x(t)||_2 = 0$$

Internal Stability – Unstable

- Consider an **unforced** LTI system $\dot{x} = Ax,\ x(0) = x_0$, and $x \in \mathbb{R}^n$
- **Definition:** The equilibrium point $x^* = 0$ of $\dot{x} = Ax$ is said to be:
  - ***Unstable*** if it is not stable

Internal Stability – LTI Systems - Stable

- Consider an **unforced** LTI system $\dot{x} = Ax,\ x(0) = x_0$, and $x \in \mathbb{R}^n$
- The previous definitions of stability can be simplified for linear systems by analyzing the eigenvalues of A

- For LTI systems, the equilibrium point $x^* = 0$ of $\dot{x} = Ax$ is said to be:
  - **Stable** if and only if all $\lambda_i \in \sigma(A)$ satisfy $\mathrm{Re}\{\lambda_i\} \leq 0$ and for every eigenvalue with $\mathrm{Re}\{\lambda_i\} = 0$ its algebraic multiplicity = geometric multiplicity

- All eigenvalues of A must have real part less than or equal to 0, i.e. $\mathrm{Re}\{\lambda_i\} \leq 0$ for all i
- For the eigenvalues that have zero real part, they must be non-degenerate
- Therefore, using a Jordan canonical form, the Jordan blocks associated with the eigenvalues with zero real part are still diagonalizable, i.e.

$$J(\lambda_i) = \begin{pmatrix} \lambda_i & 0 & \cdots & 0 \\ 0 & \lambda_i & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda \end{pmatrix} \Rightarrow e^{J(\lambda_i)t} = \begin{pmatrix} e^{\lambda_i t} & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & e^{\lambda_i t} \end{pmatrix}$$

- We don't have any polynomial terms t^p and each $e^{\lambda_i t} = e^{(0+j\omega_i)t} = e^{j\omega_i t}$ or $e^{0t} = 1$ if $\omega_i = 0$

Internal Stability – LTI Systems – Asymptotically Stable

- Consider an **unforced** LTI system $\dot{x} = Ax,\ x(0) = x_0$, and $x \in \mathbb{R}^n$
- The previous definitions of stability can be simplified for linear systems by analyzing the eigenvalues of A
- For LTI systems, the equilibrium point $x^* = 0$ of $\dot{x} = Ax$ is said to be:
  - **Asymptotically stable** if and only if all $\lambda_i \in \sigma(A)$ satisfy $\mathrm{Re}\{\lambda_i\} < 0$

- All eigenvalues of A must have real part **less than** 0, i.e. $\mathrm{Re}\{\lambda_i\} < 0$ for all i

Internal Stability – LTI Systems –Unstable

- Consider an **unforced** LTI system $\dot{x} = Ax,\ x(0) = x_0$, and $x \in \mathbb{R}^n$
- The previous definitions of stability can be simplified for linear systems by analyzing the eigenvalues of A
- For LTI systems, the equilibrium point $x^* = 0$ of $\dot{x} = Ax$ is said to be:
  - **Unstable** if there exist at least one $\lambda_i \in \sigma(A)$ which satisfies $\mathrm{Re}\{\lambda_i\} > 0$ and/or if any λ_i s.t. $\mathrm{Re}\{\lambda_i\} = 0$ is degenerate

- Recall the general solution to a state space system: $x(t) = V\left[\sum_{i=1}^{q}\sum_{j=1}^{n_i} e^{\lambda_i t}\frac{t^{j-1}}{(j-1)!}\mathbf{w}_{ij}\right]$
- Unstable if there is at least one λ_i such that $\mathrm{Re}\{\lambda_i\} > 0$
- Unstable if there is at least one λ_i with $\mathrm{Re}\{\lambda_i\} = 0$ and degenerate, i.e. geom. mult. $<$ alg. mult
  For this particular eigenvalue, there will be polnomial terms

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Feedback Control Motivation

- Consider an **unforced** LTI system $\dot{x} = Ax, \ x(0) = x_0$, and $x \in \mathbb{R}^n$
- For physical systems, we would like the eq. point to be asymptotically stable or stable
- What if a LTI system is unstable?

Feedback Control Goals

- We will now consider an LTI system with input(s):

$$\dot{x} = Ax + Bu, \quad x \in \mathbb{R}^n, \; u \in \mathbb{R}^m$$

- We want to design a controller $u = \mu(x)$ in order to:

Feedback Control - Stabilization

- We will now consider an LTI system with input(s):

$$\dot{x} = Ax + Bu, \quad x \in \mathbb{R}^n,\ u \in \mathbb{R}^m$$

- We want to design a controller $u = \mu(x)$ in order to stabilize the system, i.e. $\dot{x} = Ax + Bu = Ax + B\mu(x)$ be an asymptotically stable system
- We begin by finding a control law of the form $u = \mu(x) = Kx$, where $K \in \mathbb{R}^{m \times n}$
- This type of control is known as **linear state feedback** since the input is a linear combination of the states:

for 1 input, (m = 1): $u = Kx \;\Rightarrow\; u = \begin{pmatrix} k_1 & k_2 & \cdots & k_n \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} \;\Rightarrow\; u = k_1 x_1 + k_2 x_2 + \cdots + k_n x_n$

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for 2 inputs, (m = 2): $\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = Kx \;\Rightarrow\; \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} k_{11} & k_{12} & \cdots & k_{1n} \\ k_{21} & k_{22} & \cdots & k_{2n} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} \;\Rightarrow\; \begin{matrix} u_1 = k_{11} x_1 + k_{12} x_2 + \cdots + k_{1n} x_n \\ u_2 = k_{21} x_1 + k_{22} x_2 + \cdots + k_{2n} x_n \end{matrix}$

$\vdots$

Feedback Control – Stabilization Summary

- We will now consider an LTI system with input(s):

$$\dot{x} = Ax + Bu, \quad x \in \mathbb{R}^n, \; u \in \mathbb{R}^m$$

- We want to design a controller $u = \mu(x)$ in order to stabilize the system, i.e. $\dot{x} = Ax + Bu = Ax + B\mu(x)$ be an asymptotically stable system
- We begin by finding a control law of the form $u = \mu(x) = Kx$ (linear state feedback)

- Our goal will be to find a feedback matrix $K \in \mathbb{R}^{m \times n}$ such that for $u = Kx$, the closed loop system:

$$\dot{x} = Ax + Bu = Ax + BKx = (A + BK)x$$

is asymptotically stable, i.e. all $\lambda_i \in \sigma(A + BK)$ satisfy $\mathrm{Re}\{\lambda_i\} < 0$

- **When is this possible?**

Cayley Hamilton Theorem

- Consider the LTI system $\dot{x} = Ax + Bu$
- The eigenvalues of the matrix A are the roots of its **characteristic polynomial**:

$$p_A(t) = \det(tI - A) = t^n + a_{n-1}t^{n-1} + \cdots + a_1 t + a_0 = 0$$

Cayley Hamilton Theorem: A matrix $A \in \mathbb{R}^{n\times n}$ satisfies its own characteristic polynomial:

$$p_A(A) = A^n + a_{n-1}A^{n-1} + \cdots + a_1 A + a_0 I = 0$$

Applications

- Computing powers of A, since $A^n = -a_0 I - a_1 A - \cdots - a_{n-1}A^{n-1}$
- **Matrix exponential** $e^{At} = \sum_{i=0}^{\infty} \frac{(At)^i}{i!} = c_0(t)I + c_1(t)A + \cdots + c_{n-1}(t)A^{n-1}$

Reachable States

- We will now consider an LTI system with input(s):

$$\dot{x} = Ax + Bu, \quad x \in \mathbb{R}^n, \; u \in \mathbb{R}^m$$

which has solution $x(t) = e^{At}x(0) + \int_0^t e^{A(t-\tau)}Bu(\tau)d\tau$

- **Definition:** A state $x_S \in \mathbb{R}^n$ is said to be reachable from $x(0) = x_0$ if there exists a finite T and input $u(t)$ from $0 \leq t \leq T$ such that $x(T) = x_S$
- How can we obtain $u(t)$?
- At a finite time $t = T$, the solution satisfies:

$$x(T) = x_S = e^{AT}x(0) + \int_0^T e^{A(T-\tau)}Bu(\tau)d\tau$$

$$\Rightarrow \; x_S - e^{AT}x(0) = \int_0^T e^{A(T-\tau)}Bu(\tau)d\tau, \qquad \text{let } u(\tau) = B^T e^{A^T(T-\tau)}z, \text{ for some unknown vector } z \in \mathbb{R}^n$$

$$\Rightarrow \; x_S - e^{AT}x(0) = \int_0^T e^{A(T-\tau)}BB^T e^{A^T(T-\tau)}d\tau z = \int_0^T M(\tau)M^T(\tau)d\tau \;\; z = X(\tau)z$$

- We can solve for z if $X(\tau)$ is invertible, i.e. $z = X^{-1}(\tau)\left(x_s - e^{AT}x(0)\right)$
- When is $X(\tau)$ **invertible**?

Reachable States

- We will now consider an LTI system with input(s):

$$\dot{x} = Ax + Bu, \quad x \in \mathbb{R}^n, \; u \in \mathbb{R}^m$$

which has solution $x(t) = e^{At}x(0) + \int_0^t e^{A(t-\tau)}Bu(\tau)d\tau$

- **Definition:** A state $x_S \in \mathbb{R}^n$ is said to be reachable from $x(0) = x_0$ if there exists a finite T and input $u(t)$ from $0 \leq t \leq T$ such that $x(T) = x_S$

- How can we obtain $u(t)$?

- At a finite time $t = T$, the solution satisfies:

$$\Rightarrow \; x_S - e^{AT}x(0) = \int_0^T e^{A(T-\tau)}BB^T e^{A^T(T-\tau)}d\tau z = \int_0^T M(\tau)M^T(\tau)d\tau \;\; z = X(\tau)z$$

$$\Rightarrow \; z = X^{-1}(\tau)\left(x_s - e^{AT}x(0)\right)$$

- $X(\tau)$ is invertible iff $M(\tau) = e^{A(T-\tau)}B$ has full rank

- Lastly, the rank $(M(\tau))$ = rank $\left(\begin{bmatrix} B & AB & A^2B & \cdots & A^{n-1}B \end{bmatrix}\right)$
  - $\operatorname{Im}\left\{e^{A(T-\tau)}B\right\} = \operatorname{Im}\left\{\left[c_0(T-\tau)I + c_1(T-\tau)A + \cdots + c_{n-1}(T-\tau)A^{n-1}\right]B\right\}$ by C.H.
  - $\operatorname{Im}\left\{e^{A(T-\tau)}B\right\} = \operatorname{Im}\left\{\begin{bmatrix} B & AB & \cdots & A^{n-1}B \end{bmatrix}\right\}$

Reachable Subspace

- We will now consider an LTI system with input(s):

$$\dot{x} = Ax + Bu, \quad x \in \mathbb{R}^n, \; u \in \mathbb{R}^m$$

  which has solution $x(t) = e^{At}x(0) + \int_0^t e^{A(t-\tau)}Bu(\tau)d\tau$

- **Fact:** The set of all reachable states, $\mathcal{R}_0 \triangleq \{x \in \mathbb{R}^n \mid x(0) = 0, \; \exists T, \; u(t) \;\text{ s.t. }\; x(T) = x\}$ (from from $x(0) = 0$), is a linear subspace of $\mathbb{R}^n$

Reachable Subspace – Computation

- We will now consider an LTI system with input(s):

$$\dot{x} = Ax + Bu, \quad x \in \mathbb{R}^n, \; u \in \mathbb{R}^m$$

which has solution $x(t) = e^{At}x(0) + \int_0^t e^{A(t-\tau)}Bu(\tau)d\tau$

- How can we find $\mathcal{R}_0$?
- The reachable subspace can be obtained as follows:

$$\mathcal{R}_0 \triangleq \langle A \mid B \rangle = \operatorname{Im}\begin{bmatrix} B & AB & A^2B & A^{n-1}B \end{bmatrix}$$

Controllability

- Consider an LTI system with input(s):

$$\dot{x} = Ax + Bu, \quad x \in \mathbb{R}^n,\ u \in \mathbb{R}^m$$

- **Controllability:** An LTI system is said to be **controllable** if and only if the rank of the matrix

$$\mathcal{W} = \begin{bmatrix} B & AB & \cdots & A^{n-1}B \end{bmatrix}$$

is n, i.e. it has n linearly independent columns

Controllability Canonical Form

- Consider an LTI system with input(s) $\dot{x} = Ax + Bu, \quad x \in \mathbb{R}^n, \; u \in \mathbb{R}^m$

- If a system is **controllable** ($\text{Rank}\,(\mathcal{W}) = n$) with $m = 1$, there exists a change of basis matrix T ($x = Tz$) such that:

$$\dot{z} = T^{-1}ATz + T^{-1}Bu = A_c z + B_c u$$

where A_c and B_c are of the form:

$$A_c = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ -\alpha_0 & -\alpha_1 & -\alpha_2 & \cdots & -\alpha_{n-1} \end{pmatrix}, \quad B_c = \begin{pmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix}$$

where α_i are the coefficients of $p_A(t) = t^n + \alpha_{n-1}t^{n-1} + \cdots + \alpha_1 t + \alpha_0$

Controllability Canonical Form – Transformation Matrix

- Consider an LTI system with input(s) $\dot{x} = Ax + Bu, \quad x \in \mathbb{R}^n,\ u \in \mathbb{R}^m$

- If a system is **controllable** ($\text{Rank}\,(\mathcal{W}) = n$) with $m = 1$, there exists a change of basis matrix T ($x = Tz$) such that:

$$\dot{z} = T^{-1}ATz + T^{-1}Bu = A_c z + B_c u$$

where A_c and B_c are in **controllability canonical form**

- The transformation matrix is given by:

$$T = \underbrace{\begin{pmatrix} B & AB & \cdots & A^{n-1}N \end{pmatrix}}_{\mathcal{W}} T_\alpha, \qquad T_\alpha = \begin{pmatrix} \alpha_1 & \alpha_2 & \alpha_3 & \cdots & \alpha_{n-1} & 1 \\ \alpha_2 & \alpha_3 & \vdots & \cdots & 1 & 0 \\ \alpha_3 & \vdots & \alpha_{n-1} & \cdots & 0 & 0 \\ \vdots & \alpha_{n-1} & 1 & \vdots & \vdots & \vdots \\ \alpha_{n-1} & 1 & 0 & \cdots & 0 & 0 \\ 1 & 0 & 0 & \cdots & 0 & 0 \end{pmatrix}$$

where α_i are the coefficients of $p_A(t) = t^n + \alpha_{n-1}t^{n-1} + \cdots + \alpha_1 t + \alpha_0$

Implications for Full State Feedback

- Consider an LTI system with input(s) $\dot{x} = Ax + Bu, \quad x \in \mathbb{R}^n,\ u \in \mathbb{R}^m$
- Assume $m = 1$, the eigenvalues of the closed loop matrix $(A+BK)$ are **freely assignable** if and only if $(A,\ B)$ is controllable

Proof $(\Leftarrow)$

- Simply find K such that $\sigma(A+BK)$ is any desired set of eigenvalues
- If (A, B) is controllable, then there exists a change of coordinates T such that:

$$\dot{z} = T^{-1}ATz + T^{-1}Bu \quad \Rightarrow \quad \dot{z} = \begin{pmatrix} 0 & 1 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \\ -\alpha_0 & -\alpha_1 & \cdots & -\alpha_{n-1} \end{pmatrix} z + \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix} u$$

- let $u = \tilde{K}z = \begin{pmatrix} \tilde{k}_1 & \tilde{k}_2 & \cdots & \tilde{k}_n \end{pmatrix} \begin{pmatrix} z_1 \\ z_2 \\ \vdots \\ z_n \end{pmatrix}$

Implications for Full State Feedback

- Consider an LTI system with input(s) $\dot{x} = Ax + Bu, \quad x \in \mathbb{R}^n, \ u \in \mathbb{R}^m$

- Assume $m = 1$, the eigenvalues of the closed loop matrix $(A+BK)$ are **freely assignable** if and only if $(A, \ B)$ is controllable

Proof (cont'd)

- what is the closed loop system?

$$\dot{z} = (A_c + B_c\tilde{K})z = \begin{pmatrix} 0 & 1 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ -\alpha_0 + \tilde{k}_1 & -\alpha_1 + \tilde{k}_2 & \cdots & -\alpha_{n-1}\tilde{k}_n \end{pmatrix} z$$

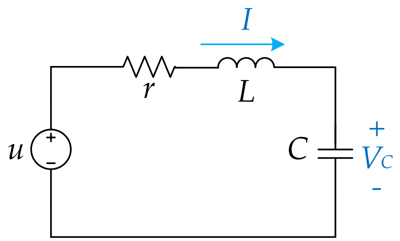
- The characteristic polynomial of $(A_c + B_c\tilde{K})$ is:

$$p_{(A_c+B_c\tilde{K})}(t) = t^n + \beta_{n-1}t^{n-1} + \cdots + \beta_1 t + \beta_0$$

Example: RLC Circuit with Input

- Consider an RLC circuit with input:

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \begin{pmatrix} \frac{-r}{L} & \frac{-1}{L} \\ \frac{1}{C} & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} \frac{1}{L} \\ 0 \end{pmatrix} u$$



- Let $r = 0.005,\ L = 0.01,\ C = 0.005$, is the system controllable?

$$\mathcal{W} = \begin{pmatrix} B & AB \end{pmatrix} = \begin{pmatrix} 100 & -50 \\ 0 & 20000 \end{pmatrix} \Rightarrow \text{Rank}\,(\mathcal{W}_c) = 2$$

- Find T such that $x = Tz$ the system is in controllability canonical form

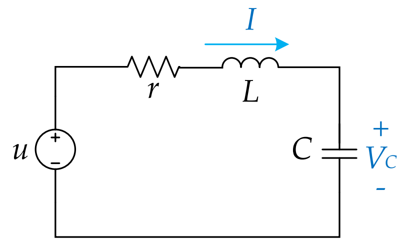
$$p_A(t) = t^2 + \alpha_1 t + \alpha_0 = t^2 + 0.5t + 20000 \qquad T = \begin{pmatrix} B & AB \end{pmatrix} \begin{pmatrix} 0.5 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 100 \\ 20000 & 0 \end{pmatrix}$$

$$\Rightarrow \quad \dot{z} = T^{-1}ATz + T^{-1}Bu \quad \Rightarrow \quad \begin{pmatrix} \dot{z}_1 \\ \dot{z}_2 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -20000 & -0.5 \end{pmatrix} \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \end{pmatrix} u$$

Example: RLC Circuit with Input (2)

- Consider an RLC circuit with input:

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \begin{pmatrix} \frac{-r}{L} & \frac{-1}{L} \\ \frac{1}{C} & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} \frac{1}{L} \\ 0 \end{pmatrix} u$$



- Let $r = 0.005$, $L = 0.01$, $C = 0.005$, find $u = Kx$ such that $\sigma(A + BK) = \{-1000,\ -2000\}$

$$\begin{pmatrix} \dot{z}_1 \\ \dot{z}_2 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -20000 & -0.5 \end{pmatrix} \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \end{pmatrix} u, \qquad u = \tilde{K} z = \begin{pmatrix} \tilde{K}_1 & \tilde{K}_2 \end{pmatrix} \begin{pmatrix} z_1 \\ z_2 \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} \dot{z}_1 \\ \dot{z}_2 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -20000 + \tilde{K}_1 & -0.5 + \tilde{K}_2 \end{pmatrix} \begin{pmatrix} z_1 \\ z_2 \end{pmatrix}$$

Desired eigenvalues imply $\Rightarrow \ p_{A+BK}(t) = (t + 1000)(t + 2000) = t + 3000t + 2000000$

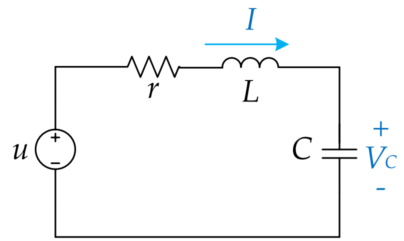
$$\Rightarrow \ \tilde{K}_1 = 20000 - 2000000$$
$$\Rightarrow \ \tilde{K}_2 = 0.5 - 3000$$

$$\Rightarrow \ \tilde{K} = \begin{pmatrix} -1980000 & -2999.5 \end{pmatrix} \qquad \Rightarrow \ K = \tilde{K} T^{-1} = \begin{pmatrix} -29.995 & -99 \end{pmatrix}$$

Example: RLC Circuit with Input (3)

- Consider an RLC circuit with input:

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \begin{pmatrix} \frac{-r}{L} & \frac{-1}{L} \\ \frac{1}{C} & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} \frac{1}{L} \\ 0 \end{pmatrix} u$$

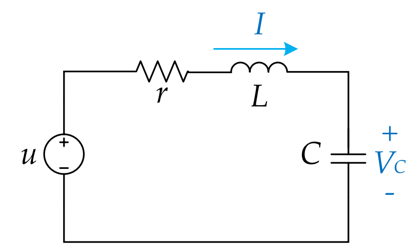


- Let $r = 0.005$, $L = 0.01$, $C = 0.005$, find $u = Kx$ such that $\sigma(A + BK) = \{-1000, -2000\}$

Example: RLC Circuit with Input (4)

- Consider an RLC circuit with input:

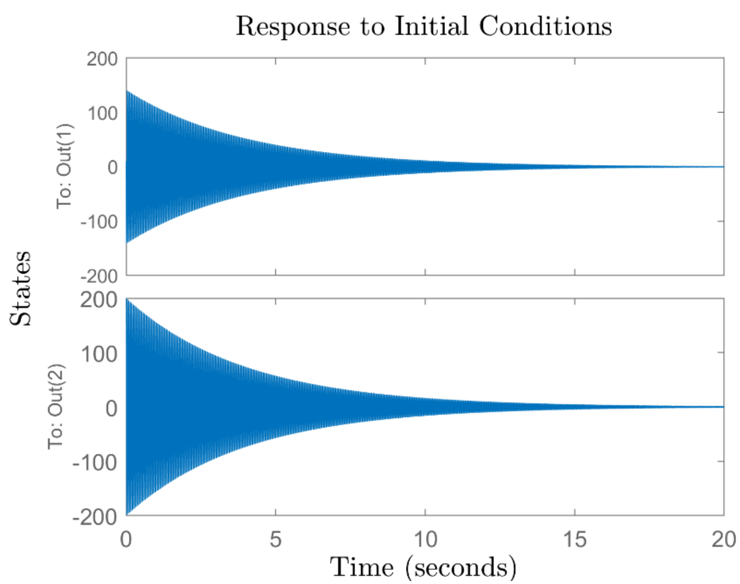
$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \begin{pmatrix} \frac{-r}{L} & \frac{-1}{L} \\ \frac{1}{C} & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} \frac{1}{L} \\ 0 \end{pmatrix} u$$



- Let $r = 0.005$, $L = 0.01$, $C = 0.005$, find $u = Kx$ such that $\sigma(A + BK) = \{-1000, -2000\}$

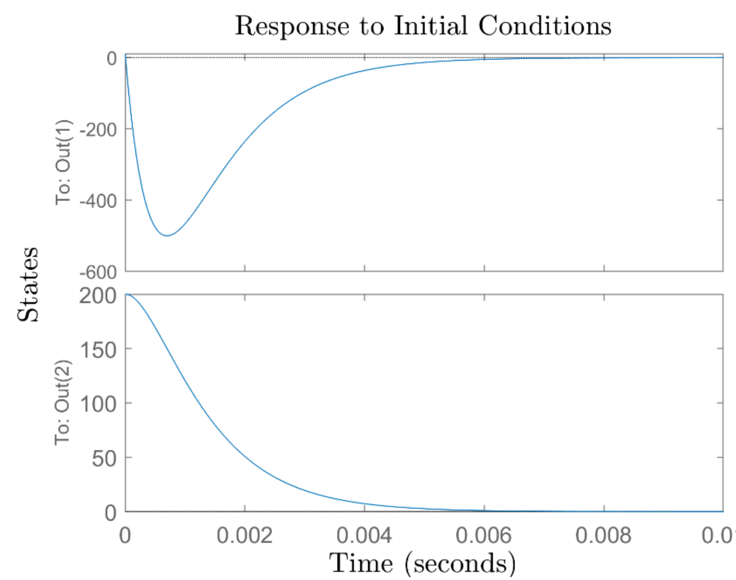
Open loop response

$\sigma(A) = \{-0.25 + j141.4, \ -0.25 - j141.4\}$



Closed loop response

$\sigma(A + BK) = \{-1000, \ -2000\}$

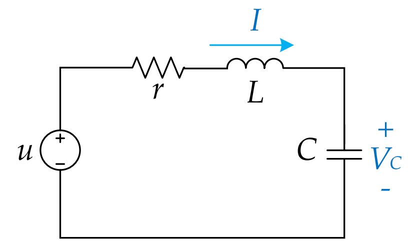


Example: RLC Circuit - Matlab code

- Consider an RLC circuit with input:

```matlab

%% RLC Circuit
r = 0.005; C = 5e-3; L = 1e-2;

% Write the system matrix
A = [-r/L -1/L; 1/C 0];
B = [1/L; 0];

%% Build the controllability matrix
Wc = [B A*B];                  % System is controllable
rank(Wc);

%% Build the change of basis matrix
% First obtain the char polynomial of A
syms t
pA = det(t*eye(2)-A);

% Extract the coefficients and build the matrix
CoeffsA = fliplr(coeffs(pA));             % Coefficients
Talph    = [CoeffsA(2) 1; 1 0];
% Change of coordinate matrix
T = Wc*Talph;

% Change coordinates
Anew = inv(T)*A*T;
Bnew = inv(T)*B;

%% Do pole placement using T
% Desired eigenvalues
lam1 = -1000; lam2 = -2000;
pAnew = (t-lam1)*(t-lam2)

CoeffsAnew = fliplr(coeffs(pAnew));

Ktil_all = fliplr(CoeffsA-CoeffsAnew)
Ktil = Ktil_all(1:2);

K = Ktil*inv(T);
double(K);

%% Check the closed loop eigenvalues
Acl = double(A+B*K);
eig(Acl)
```

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \begin{pmatrix} \frac{-r}{L} & \frac{-1}{L} \\ \frac{1}{C} & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} \frac{1}{L} \\ 0 \end{pmatrix} u$$



Pole Placement/Feedback Matrix Using Matlab

- Consider an LTI system with input(s) $\dot{x} = Ax + Bu, \quad x \in \mathbb{R}^n, \; u \in \mathbb{R}^m$
- If the system is controllable, we can place the poles in Matlab using the **place** command:

```
%% Obtain feedback matrix using place
K = -place(A, B, lam_des);
```

- OR the **acker** command, which uses Ackermann's formula

```
%% Obtain feedback matrix using acker
K = -acker(A, B, lam_des)
```

What if the system is not controllable?

- Consider an LTI system with input(s) $\dot{x} = Ax + Bu, \quad x \in \mathbb{R}^n,\ u \in \mathbb{R}^m$
- The system defined by $(A,\ B)$ is controllable iff $\text{Rank}\,(\mathcal{W}) = n$
- Assume $\text{Rank}\,(\mathcal{W}) = p < n$, what does this imply? will it be possible asymptoticaly stabilize an unstable open loop system?

Reachable Subspace – A Invariance

- Consider an LTI system with input(s) $\dot{x} = Ax + Bu, \quad x \in \mathbb{R}^n, \; u \in \mathbb{R}^m$
- The reachable subspace $\mathcal{R}_0 \triangleq \langle A \mid B \rangle = \mathrm{Im}\begin{bmatrix} B & AB & A^2B & A^{n-1}B \end{bmatrix}$ is the *smallest* A-*invariant subspace containing the* $\mathrm{Im}[B]$
- Assume $\mathrm{Rank}\,(\mathcal{W}) = p < n$, A-invariance of $\mathcal{R}_0$ implies the following:

Reachable Subspace – *A* Invariance (cont'd)

- Consider an LTI system with input(s) $\dot{x} = Ax + Bu, \quad x \in \mathbb{R}^n, \; u \in \mathbb{R}^m$
- The reachable subspace $\mathcal{R}_0 \triangleq \langle A \mid B \rangle = \operatorname{Im}\begin{bmatrix} B & AB & A^2B & A^{n-1}B \end{bmatrix}$ is the *smallest* A*-invariant subspace containing the* $\operatorname{Im}[B]$
- Assume $\operatorname{Rank}(\mathcal{W}) = p < n$, A-invariance of $\mathcal{R}_0$ implies the following:

Decomposition w.r.t. Reachable Subspace

- Consider an LTI system with input(s) $\dot{x} = Ax + Bu, \quad x \in \mathbb{R}^n,\ u \in \mathbb{R}^m$
- Assume that the dimension of $\mathcal{R}_0 = \dim\{\mathrm{Im}\,(\mathcal{W})\} = p < n$
- Find a complementary subspace $\mathcal{W}$ (e.g. $\mathcal{R}_0^\perp$), such that $\mathbb{R}^n = \mathcal{R}_0 \oplus \mathcal{W}$
- Form a basis for $\mathbb{R}^n$ s.t. $T = \big[\underbrace{v_1 \ \cdots \ v_p}_{\mathcal{R}_0} \ \underbrace{w_1 \ \cdots \ w_{n-p}}_{\mathcal{W}}\big]$
- Analyzing the system in the new basis, i.e. $x = Tz$, we can obtain:

$$\dot{z} = \tilde{A}z + \tilde{B}u, \quad \tilde{A} = T^{-1}AT, \quad \tilde{B} = T^{-1}B$$

  where $\tilde{A}$ and $\tilde{B}$ are of the form:

$$\tilde{A} = \begin{pmatrix} \tilde{A}_{11} & \tilde{A}_{12} \\ 0 & \tilde{A}_{22} \end{pmatrix}, \quad \tilde{B} = \begin{pmatrix} \tilde{B}_1 \\ 0 \end{pmatrix}$$

1. The eigenvalues of A are $\sigma(A) = \sigma(\tilde{A}) = \sigma(\tilde{A}_{11}) \cup \sigma(\tilde{A}_{22})$
2. The pair $(\tilde{A}_{11},\ \tilde{B}_1)$ is controllable
3. The system is **stabilizable** if the eigenvalues of $\tilde{A}_{22}$ have negative real part

Feedback Design for Stabilizable Systems

- Consider an LTI system with input(s) $\dot{x} = Ax + Bu, \quad x \in \mathbb{R}^n, \ u \in \mathbb{R}^m$
- Assume that the dimension of $\mathcal{R}_0 = \dim \{\mathrm{Im}\,(\mathcal{W})\} = p < n$
- Form a basis for $\mathbb{R}^n$ s.t. $T = \left[\underbrace{v_1 \quad \cdots \quad v_p}_{\mathcal{R}_0} \ \underbrace{w_1 \quad \cdots \quad w_{n-p}}_{\mathcal{W}}\right]$
- Analyzing the system in the new basis, i.e. $x = Tz$, we can obtain:

$$\begin{pmatrix} \dot{z}_1 \\ \dot{z}_2 \end{pmatrix} = \begin{pmatrix} \tilde{A}_{11} & \tilde{A}_{12} \\ 0 & \tilde{A}_{22} \end{pmatrix} \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} + \begin{pmatrix} \tilde{B}_1 \\ 0 \end{pmatrix} u$$

Feedback Design for Stabilizable Systems

- Consider an LTI system with input(s) $\dot{x} = Ax + Bu, \quad x \in \mathbb{R}^n,\ u \in \mathbb{R}^m$
- Assume that the dimension of $\mathcal{R}_0 = \dim\{\mathrm{Im}(\mathcal{W})\} = p < n$
- Form a basis for $\mathbb{R}^n$ s.t. $T = \big[\underbrace{v_1 \ \cdots \ v_p}_{\mathcal{R}_0} \ \underbrace{w_1 \ \cdots \ w_{n-p}}_{\mathcal{W}}\big]$
- Analyzing the system in the new basis, i.e. $x = Tz$, we can obtain:

$$\begin{pmatrix} \dot{z}_1 \\ \dot{z}_2 \end{pmatrix} = \begin{pmatrix} \tilde{A}_{11} & \tilde{A}_{12} \\ 0 & \tilde{A}_{22} \end{pmatrix} \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} + \begin{pmatrix} \tilde{B}_1 \\ 0 \end{pmatrix} u$$

- Let $\tilde{K} = \begin{pmatrix} \tilde{K}_1 & 0 \end{pmatrix}$ s.t. the eigenvalues of $\tilde{A}_{11} + \tilde{B}_1\tilde{K}_1$ are as desired (e.g. pole placement)
- In the standard basis, $K = \tilde{K}T^{-1}$, s.t. $u = Kx$ and the closed loop poles are $\sigma(A+BK) = \sigma(\tilde{A}_{11} + \tilde{B}_1\tilde{K}_1) \cup \sigma(\tilde{A}_{22})$

Final Thoughts

- Consider an LTI system with input(s) $\dot{x} = Ax + Bu, \quad x \in \mathbb{R}^n, \; u \in \mathbb{R}^m$
- We have discussed the concepts of internal stability (without inputs) and characterize them in terms of the eigenvealues of A
- When the system is **controllable**, a state feedback controller of the form $u = Kx$ can be used to modify/place all of the closed loop eigenvalues (i.e. $\sigma(A + BK)$)
- When the system is **not controllable**, we can still use a feedback controller $u = Kx$ to alter the controllable modes only
- The system is **stabilizable** if the uncontrollable modes have negative real part
- So far we have assume that all of the states are available for measurement, what if this is not possible? What if we only have the outputs $y = Cx$ available to design our controller?