EE 419/519: Industrial Control Systems

Lecture 5: State Space Solution and Jordan Canonical Form

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Topic Outline

- **LTI State Space System Solution**
  - **Laplace vs Time domain**
  - **Matrix Exponential**
- **Jordan Canonical Form**
  - **Change of Basis**
  - **Matrix Diagonalization**
  - **Degenerate case (repeated poles)**

LTI State Space Model

- Let's consider an LTI state space system of the form:

$$\dot{x} = Ax + Bu, \quad x(0) = x_0$$
$$y = Cx$$

where $A \in \mathbb{R}^{n \times n}$, $B \in \mathbb{R}^{n \times m}$, $C \in \mathbb{R}^{p \times n}$

LTI State Space Model – Unforced System

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- Let's first analyze the solution to the **unforced** system $(u = 0)$

$$\dot{x} = Ax, \quad x(0) = x_0$$

State Space Unforced System Solution

- Let's first analyze the solution to the **unforced** system ($u = 0$)

$$\dot{x} = Ax, \quad x(0) = x_0$$

where $A \in \mathbb{R}^{n \times n}$, $x \in \mathbb{R}^n$ (n^{th} order system)

- What if $n = 1$?

SS Unforced System Solution - Laplace

- Let's first analyze the solution to the **unforced** system $(u = 0)$

$$\dot{x} = Ax, \quad x(0) = x_0, \qquad x \in \mathbb{R}^n, \; A \in \mathbb{R}^{n \times n}$$

- What if $n > 1$?

  Solution by Laplace transform:

SS Unforced System Solution – Time Domain

- Let's first analyze the solution to the **unforced** system $(u = 0)$

$$\dot{x} = Ax, \quad x(0) = x_0, \qquad x \in \mathbb{R}^n, \; A \in \mathbb{R}^{n \times n}$$

- What if $n > 1$?

Solution in time domain: $\mathcal{L}^{-1}\left\{(sI - A)^{-1}\right\} = I + At + \frac{1}{2!}A^2t^2 + \cdots = e^{At}$

SS Unforced System Solution - Summary

- Let's first analyze the solution to the **unforced** system ($u = 0$)

$$\dot{x} = Ax, \quad x(0) = x_0, \qquad x \in \mathbb{R}^n, \; A \in \mathbb{R}^{n \times n}$$

- **Definition:** The **matrix exponential** is given by:

$$e^{At} = \sum_{k=0}^{\infty} \frac{1}{k!}(At)^k = I + At + \frac{1}{2}A^2t^2 + \cdots = \mathcal{L}^{-1}\left\{(sI - A)^{-1}\right\}$$

- The **solution** to an unforced LTI SS model, $\dot{x} = Ax$ with $x(0) = x_0$, is of the form:

$$x(t) = e^{At}x_0$$

where $x \in \mathbb{R}^n$.

Example 1

- Find a solution to the following second order system:

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -2 & -3 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, \qquad x(0) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

SS System Solution – Laplace Domain

- Now let's analyze a more general state space system:

$$\dot{x} = Ax + Bu, \qquad x \in \mathbb{R}^n,\ u \in \mathbb{R}^m,\ x(0) = x_0$$

- The solution we obtained from **Laplace transform** is:

SS System Solution – Time Domain

- Now let's analyze a more general state space system:

$$\dot{x} = Ax + Bu, \qquad x \in \mathbb{R}^n,\ u \in \mathbb{R}^m,\ x(0) = x_0$$

- The general **solution** to this system is of the form:

$$x(t) = e^{At}x_0 + \int_0^t e^{A(t-\tau)}Bu(\tau)d\tau$$

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Motivation for Change of Basis

- Let's first begin with an unforced state space model:

$$\dot{x} = Ax, \qquad x \in \mathbb{R}^n,\ x(0) = x_0$$

- As discussed previously, $A : \mathbb{R}^n \to \mathbb{R}^n$, w.r.t. the **standard basis**
- What happens to our system if we were to change basis?

Motivation for Change of Basis

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- As discussed previously, $A : \mathbb{R}^n \to \mathbb{R}^n$, w.r.t. the **standard basis**
- What happens to the **solution** if we were to change basis?

What basis should we use?

- Let's first begin with an unforced state space model:

$$\dot{x} = Ax, \qquad x \in \mathbb{R}^n, \; x(0) = x_0$$

- As discussed previously, $A : \mathbb{R}^n \to \mathbb{R}^n$, w.r.t. the **standard basis**

- Define a new basis $\mathscr{V} = \{v_1, \cdots, v_n\}$

- Then $\forall x \in \mathbb{R}^n$ can be written as $x = V x_{\mathscr{V}}$, where $V = \begin{bmatrix} v_1 & \cdots & v_n \end{bmatrix} \in \mathbb{R}^{n \times n}$

- The state space system is then changed to:

$$\dot{x}_{\mathscr{V}} = V^{-1} A V x_{\mathscr{V}}, \qquad x_{\mathscr{V}}(0) = V^{-1} x_0$$

$$\Rightarrow \quad \dot{x}_{\mathscr{V}} = \tilde{A} x_{\mathscr{V}}$$

- **But, what is a good basis to try?**

Eigenvalues and Eigenvectors

- Consider an unforced state space model: $\dot{x} = Ax, \qquad x \in \mathbb{R}^n, \; x(0) = x_0$

- An eigenvalue and eigenvector pair $\lambda_1 \in \mathbb{C}$, $v_1 \in \mathbb{R}^n$ where $v_1 \neq 0$, of the A matrix satisfy:

$$Av_1 = \lambda_1 v_1$$

- How can we find the eigenvalues/eigenvectors?

Characteristic Polynomial of a Matrix

- Consider an unforced state space model: $\dot{x} = Ax, \qquad x \in \mathbb{R}^n,\ x(0) = x_0$

- **Definition:** The **spectrum** of $A \in \mathbb{R}^{n \times n}$ is the set of all $\lambda \in \mathbb{C}$ that are eigenvalues of A. The spectrum is denoted by $\sigma(A)$

- **Definition:** The **characteristic polynomial** of $A \in \mathbb{R}^{n \times n}$, denoted as $p_A(t)$ is defined as follows:

$$p_A(t) = \det(tI - A)$$

Algebraic Multiplicity

- Consider an unforced state space model: $\dot{x} = Ax, \qquad x \in \mathbb{R}^n, \; x(0) = x_0$
- What if an eigenvalue is repeated multiple times?
- **Definition:** Let $A \in \mathbb{R}^{n \times n}$, the **algebraic multiplicity** of an eigenvalue $\lambda \in \sigma(A)$ is its multiplicity as a zero of the characteristic polynomial $p_A(t)$

Eigenspace and Geometric Multiplicity

- Consider an unforced state space model: $\dot{x} = Ax, \qquad x \in \mathbb{R}^n,\ x(0) = x_0$

- **Definition:** Let $A \in \mathbb{R}^{n\times n}$, for a given $\lambda \in \sigma(A)$, the set of all eigenvectors $x \in \mathbb{C}^n$, $x \neq 0$, i.e. satisfying $Ax = \lambda x$, is known as the **eigenspace of A associated with λ**

Eigenspace and Geometric Multiplicity

- Consider an unforced state space model: $\dot{x} = Ax, \qquad x \in \mathbb{R}^n, \; x(0) = x_0$

- **Definition:** Let $A \in \mathbb{R}^{n \times n}$ and $\lambda \in \sigma(A)$. The dimension of the eigenspace of A associated with λ is known as the **geometric multiplicity of** λ

Matrix Diagonalization (1) – Distinct Eigenvalues

- Consider an unforced state space model: $\dot{x} = Ax, \qquad x \in \mathbb{R}^n,\ x(0) = x_0$
- **Assume** that all $\lambda \in \sigma(A)$ are **distinct**

Matrix Diagonalization (2) – Change of Basis

- Consider an unforced state space model: $\dot{x} = Ax, \qquad x \in \mathbb{R}^n,\ x(0) = x_0$
- **Assume** that all $\lambda \in \sigma(A)$ are **distinct**

Matrix Diagonalization (3) – Summary

- Consider an unforced state space model: $\dot{x} = Ax, \qquad x \in \mathbb{R}^n,\ x(0) = x_0$
- **Assume** that all $\lambda_i \in \sigma(A)$ are **distinct**
- Define a new basis $\mathscr{V} = \{v_1,\ \cdots,\ v_n\}$, where each $(\lambda_i,\ v_i)$ is an eigenvalue/eigenvector pair
- Analyzing the state space model in a new basis defined by $\mathcal{V}$, the system becomes:

$$z = V^{-1}AVz, \quad z(0) = V^{-1}x_0$$

$$\Rightarrow \quad \dot{z} = \Lambda z, \qquad \text{where} \qquad \Lambda = \begin{pmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_n \end{pmatrix}$$

- We can conclude then that matrix is diagonalizable if all $\lambda_i \in \sigma(A)$ are distinct
- **Actually, a matrix is diagonalizable if the geometric multiplicty of each eigenvalue is equal to its algebraic multiplicity**

Example

- Consider an unforced state space model: $\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \begin{pmatrix} -2 & 3 \\ 0 & 4 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$

- Obtained a new basis such that the system is transformed to $\dot{z} = \Lambda z$, where Λ is a diagonal matrix

Example (cont'd)

- Consider an unforced state space model: $\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \begin{pmatrix} -2 & 3 \\ 0 & 4 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$
- Obtained a new basis such that the system is transformed to $\dot{z} = \Lambda z$, where Λ is a diagonal matrix

Eigenvalues/Eigenvectors in Matlab

- Consider an unforced state space model: $\dot{x} = Ax, \qquad x \in \mathbb{R}^n,\ x(0) = x_0$
- For a given $A \in \mathbb{R}^{n\times n}$, we can obtain its eigenvalues and eigenvectors in **Matlab** using the following command:

```
e = eig(A);            % Returns a column vector containing the eigenvalues of A

[V, D] = eig(A);       % D is a diagonal matrix with eigenvealues in diagonal term
                       % V is a matrix containing the corresponding eigenvectors
```

Complex Eigenvalues

- Consider an unforced state space model: $\dot{x} = Ax, \qquad x \in \mathbb{R}^n,\ x(0) = x_0$
- The previous procedure for matrix diagonalization can be implemented even if an eigenvalue is complex

Complex Eigenvalues (cont'd)

- Consider an unforced state space model: $\dot{x} = Ax, \qquad x \in \mathbb{R}^n,\ x(0) = x_0$
- The previous procedure for matrix diagonalization can be implemented even if an eigenvalue is complex

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What if a Matrix is not Diagonalizable?

- Let's look at the following system/matrix: $$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{pmatrix} = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 3 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

- The matrix A has the following eigenvalues/eigenvector pairs:
  - $\lambda_1 = 1$ (alg. mult. 2), $v_1 = \begin{pmatrix} 1 & 1 & 0 \end{pmatrix}^T$ (geom. mult. 1)
  - $\lambda_2 = 3$ (alg. mult. 1), $v_2 = \begin{pmatrix} 1 & 2 & 2 \end{pmatrix}^T$ (geom. mult. 1)

Jordan Canonical Form

- Let's look at the following system/matrix: $\dot{x} = Ax, \qquad x \in \mathbb{R}^n, \; x(0) = x_0$

- Let $A \in \mathbb{R}^{n\times n}$, there exists a change of coordinates defined by the columns of $V \in \mathbb{R}^{n\times n}$ such that

$$J = \begin{pmatrix} J_{n_1}(\lambda_1) & 0 & \cdots & 0 \\ 0 & J_{n_2}(\lambda_2) & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & J_{n_q}(\lambda_q) \end{pmatrix} = V^{-1}AV$$

where $n_1 + n_2 + \cdots + n_q = n$ and $\sigma(A) = \{\lambda_1,\ \lambda_2,\ \cdots,\ \lambda_q\}$. The matrix J is known as the **Jordan matrix** and each *Jordan block* is as follows:

$$J_k(\lambda) = \begin{pmatrix} \lambda & 1 & & & \\ & \lambda & 1 & & \\ & & \ddots & \ddots & \\ & & & \lambda & 1 \\ & & & & \lambda \end{pmatrix}, \quad \text{e.g. } J_1(\lambda) = \begin{pmatrix} \lambda \end{pmatrix}, \quad J_2(\lambda) = \begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix}$$

Jordan Canonical Form – Distinct Eigenvalues

- Let's look at the following system/matrix: $\dot{x} = Ax, \qquad x \in \mathbb{R}^n, \; x(0) = x_0$
- What is the Jordan matrix when $A \in \mathbb{R}^{n \times n}$ has distinct eigenvalues?

Jordan Canonical Form – Degenerate Eigenvalues (1)

- What should we do when $A \in \mathbb{R}^{n \times n}$ has some degenerate eigenvalues?
- E.g. say $A \in \mathbb{R}^{3 \times 3}$ and $\sigma(A) = \{\lambda_1,\ \lambda_2\}$ where λ_2 is degenerate

Jordan Canonical Form – Degenerate Eigenvalues (2)

- What should we do when $A \in \mathbb{R}^{n \times n}$ has some degenerate eigenvalues?
- E.g. say $A \in \mathbb{R}^{3 \times 3}$ and $\sigma(A) = \{\lambda_1,\ \lambda_2\}$ where λ_2 is degenerate

Jordan Canonical Form – Generalized Eigenvectors

- Assume that $A \in \mathbb{R}^{n\times n}$ has an eigenvalue λ which is degenerate with algebraic multiplicity of $k > 1$ and geometric multiplicty of 1

- Then for this eigenvalue λ, we can form a chain of linearly independent **generalized eigenvectors** $\{v_1,\ v_2,\ \cdots,\ v_k\}$ satisfying:

$$\begin{aligned}
(A - \lambda I)v_1 &= 0 \\
(A - \lambda I)v_2 &= v_1 \\
&\vdots \\
(A - \lambda I)v_k &= v_{k-1}
\end{aligned}$$

- This gives an algorithm for computing the **generalized eigenvectors** of a degenerate eigenvalue

- We can then use these generalized eigenvectors to obtain the Jordan canonical form

Example Jordan Canonical Form (1)

- Find a change of basis V such that the system shown on the right is transformed to $\dot{z} = Jz$, where J is a Jordan matrix

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{pmatrix} = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 3 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

Example Jordan Canonical Form (2)

- Find a change of basis V such that the system shown on the right is transformed to $\dot{z} = Jz$, where J is a Jordan matrix

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{pmatrix} = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 3 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

Final Thoughts

- We are interested in simplifying our analysis of the state space system $\dot{x} = Ax, \ x \in \mathbb{R}^n$
- In a new basis defined by the columns of $V \in \mathbb{R}^{n \times n}$, the system is transformed to $\dot{z} = V^{-1}AVz$
- When V is formed using the generalized eigenvectors of A, the system becomes $\dot{z} = Jz$ where J is a Jordan matrix
- What is the solution to $\dot{z} = Jz$?, how is it related to $x(t)$?