EE 419/519: Industrial Control Systems

Lecture 4: State Space Systems

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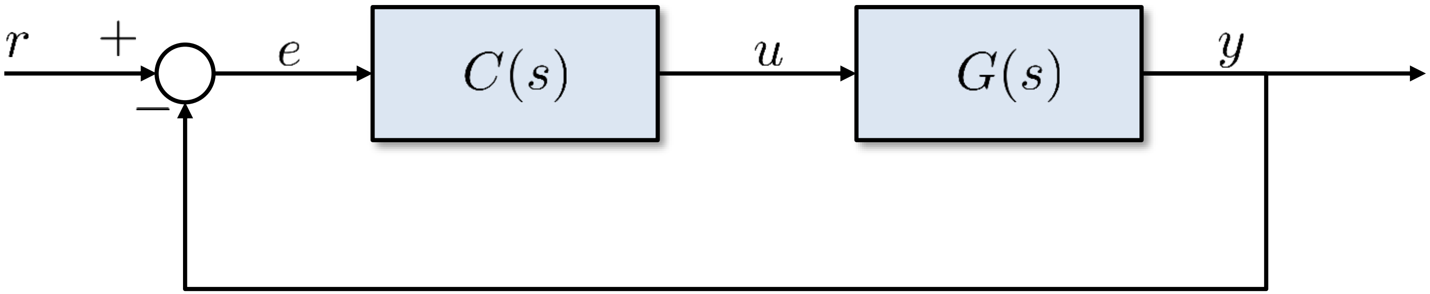
Topic Outline

- **Transfer Function vs State Space (Motivation)**
- **State Space Models and Examples**
- **From Nonlinear to Linear State Space Models**
- **Linear Algebra Review**
- **Jordan Canonical Form**

Transfer Function (SISO) Drawbacks

- We have discussed the controller design for Single Input Single Output Systems (SISO)

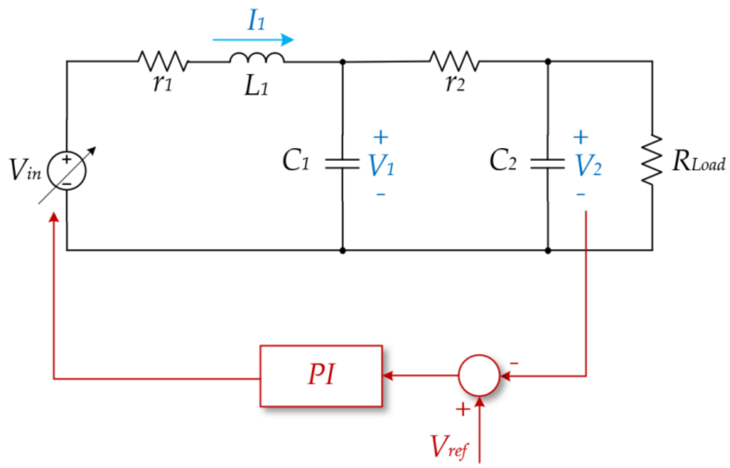


Observations

- What if we had more sensors/observations available?
- What if we had more "inputs"?
- Could these be used to improve the closed loop system?

Transfer Function (SISO) Drawbacks – Example 1

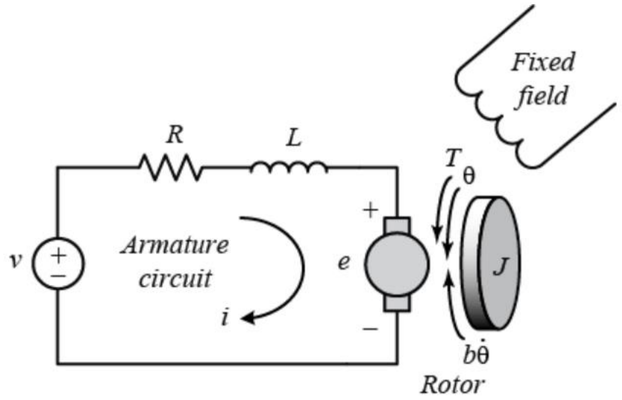
- In HW 2 problem 4, you will design a controller to regulate the C_2 voltage
- **Assumptions:** Single input (V_{in}) single output (V_2)



- What if we could also measure I_1 and V_1?

Transfer Function (SISO) Drawbacks – Example 2

- We have also designed a controller to track the speed of a dc motor



- What if we could also measure the current i?

From Transfer Function to State Space Models – Case 1

- Let's consider a strictly proper transfer function of the form:

$$\frac{Y(s)}{U(s)} = \frac{1}{s^n + a_1 s^{n-1} + \cdots + a_{n-1}s + a_n}$$

From Transfer Function to State Space Models – Case 2

- Let's consider a **more general** strictly proper transfer function

$$\frac{Y(s)}{U(s)} = \frac{b_1 s^{n-1} + \cdots + b_{n-1} s + b_n}{s^n + a_1 s^{n-1} + \cdots + a_{n-1} s + a_n}$$

From Transfer Function to State Space Models – Case 2

- Let's consider a **more general** strictly proper transfer function

$$\frac{Y(s)}{U(s)} = \frac{b_1 s^{n-1} + \cdots + b_{n-1}s + b_n}{s^n + a_1 s^{n-1} + \cdots + a_{n-1}s + a_n}$$

- We can also use the command **tf2ss(num, den)**

```
%% Write the transfer function first
num = [b1 b2 ... bn];
den = [1  a1 ... an];
[A, B, C, D] = tf2ss(num, den)
```

$$A = \begin{pmatrix} -a_1 & -a_2 & \cdots & \cdots & -a_n \\ 1 & 0 & \cdots & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 \cdots & \cdots & 1 & 0 \end{pmatrix} \qquad B = \begin{pmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \qquad C = \begin{pmatrix} b_1 & b_2 & \cdots & \cdots & b_n \end{pmatrix}$$

$$D = 0$$

Example – Transfer Function to State Space

- Transform the following transfer function to state space:

$$\frac{Y(s)}{U(s)} = \frac{2s+1}{s^2+7s+10}$$

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- **State Space Models and Examples**
- **Linear Approximation of Nonlinear Systems**
- Linear Algebra Review
- Jordan Canonical Form

Review: Nonlinear State Space System Definition

- A (nonlinear) State Space model with outputs is of the form:

$$\dot{x} = f(x,\ u)$$
$$y = h(x)$$

- The vector $x \in \mathbb{R}^n$ are the **states**
- The vector $u \in \mathbb{R}^m$ are the **inputs**
- The vector $y \in \mathbb{R}^p$ are the **outputs**

Review: Linear State Space System Definition

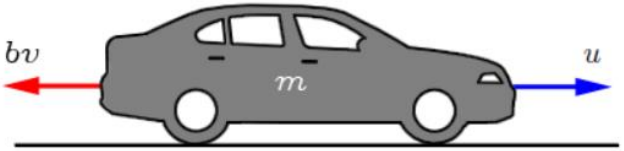
- An LTI State Space model with outputs is of the form:

$$\dot{x} = Ax + Bu, \quad x(0) = x_0$$
$$y = Cx$$

- The vector $x \in \mathbb{R}^n$ are the **states**
- The vector $u \in \mathbb{R}^m$ are the **inputs**
- The vector $y \in \mathbb{R}^p$ are the **outputs**

Example 1: Cruise Control

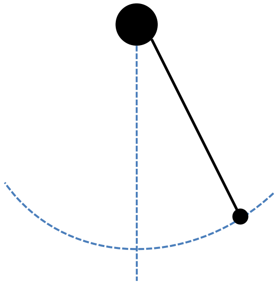
- Derive a state space model of a car as shown in the figure



Source: https://ctms.engin.umich.edu/CTMS/index.php?example=CruiseControl&section=SystemModeling

Example 2: Pendulum

- Derive the state space differential equations for a pendulum



Equilibrium Point Definition

- Many systems can be characterized by a nonlinear syste space system

$$\dot{x} = f(x,\ u)$$

- An **equilibrium point** $(x_e,\ u_e)$ is a point for which if the system starts there, it will remain there for all future time

- With respect to our state space model, this implies: $0 = f(x_e,\ u_e)$

Linear Approximation of Nonlinear Systems

- Many systems can be characterized by a nonlinear state space system

$$\dot{x} = f(x,\ u)$$

- However, close to an **equilibrium point**, many nonlinear systems can be **approximated** by a linear system

Linear Approximation of Nonlinear Systems (2)

- Many systems can be characterized by a nonlinear state space system

$$\dot{x} = f(x,\ u)$$

- How can we obtain the A and B matrices which approximate $f(x,\ u)$ close to $(x_e,\ u_e)$?

Linear Approximation of Nonlinear Systems (3)

- Many systems can be characterized by a nonlinear state space system

$$\dot{x} = f(x,\ u)$$

- A linear approximation of the system 'close' to an equilibrium point is given as follows

- Define $\tilde{x} = x - x_e$ and $\tilde{u} = u - u_e$

$$f(x,\ u) \approx \underbrace{\left.\frac{\partial f}{\partial x}\right|_{x_e,\ u_e}}_{\triangleq A}(x - x_e) + \underbrace{\left.\frac{\partial f}{\partial u}\right|_{x_e,\ u_e}}_{\triangleq B}(u - u_e)$$

- Then the behavior of the original nonlinear system close to the equilibrium point can be modeled by:

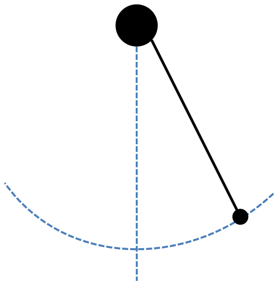
$$\dot{\tilde{x}} = A\tilde{x} + B\tilde{u}$$

Example: Pendulum Linearization

- The nonlinear state space model of a pendulum is

$$\dot{x}_1 = x_2$$
$$\dot{x}_2 = -\frac{g}{l}\sin(x_1) - \frac{k}{m}x_2$$

- Obtain a linear approximation at $x_e = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$



Example: Pendulum Linearization (2)

- The nonlinear state space model of a pendulum is

$$\dot{x}_1 = x_2$$
$$\dot{x}_2 = -\frac{g}{l}\sin(x_1) - \frac{k}{m}x_2$$

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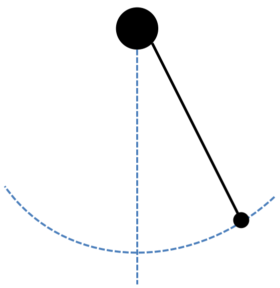

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Motivation: Linear State Space Systems

- We will analyze linear time invariant (LTI) models in state space form:

$$\dot{x} = Ax + Bu, \quad x(0) = x_0$$
$$y = Cx$$

$$x \in \mathbb{R}^n, \; u \in \mathbb{R}^m, \; y \in \mathbb{R}^p$$

- Let's consider for example $n = 2, \; m = 1, \; p = 1$

- When we say $x \in \mathbb{R}^2$, how can we visualize this? What **basis** are we referring to? What if we used a different basis?

Linear Vector Spaces

- A *Linear Vector Space* is a *set*, $\mathcal{V}$, over a field, $\mathbb{R}$, in which having **two** operations:

1. Addition : $x, y \in \mathcal{V}, \quad z = x + y \in \mathcal{V}$	Example: $\mathcal{V} = \mathbb{R}^2$
2. Scalar multiplication : $x \in \mathcal{V}, \quad \alpha \in \mathbb{R}, \quad z = \alpha x \in \mathcal{V}$	

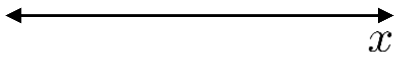
- (cont'd) it satisfies the following properties (Axioms): $x, y, z \in \mathcal{V}$

1. (Commutativity) $x + y = y + x$
2. (Associativity) $(x + y) + z = x + (y + z)$
3. (Zero vector) $\mathbf{0} + x = x, \quad x, \mathbf{0} \in \mathcal{V}$
4. (Additive inverse) $\forall x \in \mathcal{V}$ there exists $-x \in \mathcal{V}$ such that $x + (-x) = \mathbf{0}$
5. (Identity scalar) $1x = x$ for $1 \in \mathbb{R}$
6. (Compatibility of scalar mult.) $(ab)x = a(bx) \qquad a, b \in \mathbb{R}$
7. (Distributivity w.r.t addition) $a(x + y) = ax + ay \qquad a \in \mathbb{R}$
8. (Distributivity w.r.t scalar mult.) $(a + b)x = ax + bx \qquad a, b \in \mathbb{R}$

Linear Vector Spaces - Examples

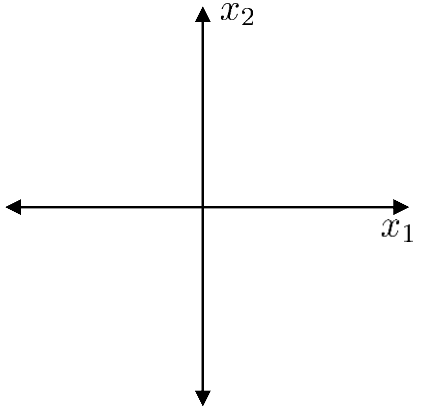
- What are some familiar examples of **Vector Spaces?**

1. The real numbers $\mathbb{R}^1$



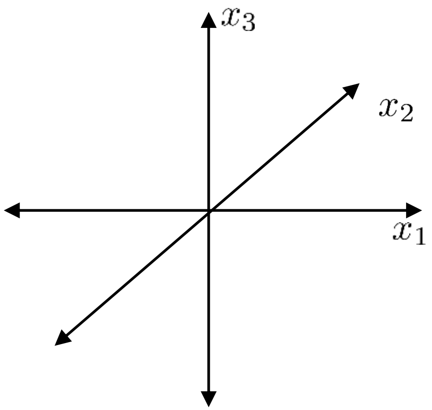
2. The 2 dimensional space, $\mathbb{R}^2$

How to identify an element in $\mathbb{R}^2$?



3. The 3 dimensional space, $\mathbb{R}^3$

How to identify an element in $\mathbb{R}^3$?



Linear Vector Spaces – Examples (cont'd)

- Other not so common Linear Vector spaces:
  - The set of **polynomials** of degree less than or equal to n, e.g. $n = 2$

$$\text{if } p, q \in \mathbb{P}_2 \quad \Rightarrow \quad p(x) = c_2 x^2 + c_1 x + c_0, \quad q(x) = d_2 x^2 + d_1 x + d_0$$

$$\text{e.g. } h(x) = 3x^2 + 1 \qquad m(x) = 2x^2 + x - 3$$

  - Define addition and scalar multiplication:

$$p(x) + q(x) = (c_2 + d_2)x^2 + (c_1 + d_1)x + (c_0 + d_0)$$

$$h(x) + m(x) =$$

2. Scalar multiplication:

$$\alpha \in \mathbb{R} \quad \alpha p(x) = \alpha c_2 x^2 + \alpha c_1 x + \alpha c_0$$

$$3h(x) =$$

  - **$\mathbb{P}_2$ and in general $\mathbb{P}_n$ satisfy all the axioms for a vector space**

Subspace of a Vector Space

- Vector spaces provide us with a field to conduct extensive analysis
- **Definition:** $\mathcal{S}$ is called a *subspace* of a vector space $\mathcal{V}$ if $\mathcal{S}$ is a subset of V and $\mathcal{S}$ satisfies:

  (i) $\mathbf{0} \in \mathcal{S}$ ($\mathcal{S}$ is non-empty)

  (ii) For all $x, y \in \mathcal{S}$, $x + y \in \mathcal{S}$ ($\mathcal{S}$ is closed under addition)

  (iii) For all $x \in \mathcal{S}$, $\alpha x \in \mathcal{S}$, where $\alpha \in \mathbb{R}$ ($\mathcal{S}$ is closed under scalar multiplication)
- To verify if a subset, $\mathcal{S}$, of a vector space, $\mathcal{V}$, is a subspace we need to check the three conditions above **Remember them!**

Subspace Example 1

- **Example:** Consider the following subset of $\mathbb{R}^3$:

$$\mathcal{S} = \left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} \in \mathbb{R}^3 \;\middle|\; x = 3y,\ z = -2y \right\}$$

- Is it a subspace of $\mathbb{R}^3$?

Subspace Example 2

- **Example:** Consider the following subset of $\mathbb{R}^2$:

$$\mathcal{S} = \left\{ \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \in \mathbb{R}^2 \;\middle|\; x_2 = 3x_1 \right\}$$

- Is it a subspace of $\mathbb{R}^2$?

Subspace Example 3

- **Example:** Consider the following subset of $\mathbb{R}^2$:

$$\mathcal{S} = \left\{ \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \in \mathbb{R}^2 \;\middle|\; x_1 \geq 0,\; x_2 \geq 0 \right\}$$

- Is it a subspace of $\mathbb{R}^2$?

Span of a Set of Vectors

- Suppose $v_1,\ v_2,\ \ldots,\ v_n$ are vectors defined in a vector space $\mathcal{V}$
- A *Linear Combination* of these is a vector:

$$\alpha_1 v_1 + \alpha_2 v_2 + \cdots + \alpha_n v_n \in \mathcal{V} \qquad \alpha_1, \ldots, \alpha_n \in \mathbb{R}$$

- The set of all linear combinations of $v_1,\ v_2,\ \ldots,\ v_n$ is called **span** $\{v_1,\ v_2,\ \cdots,\ v_n\}$

◦ Example:

Span of a Set of Vectors - Example

- Determine if $v = \begin{pmatrix} 2 \\ -2 \\ -1 \end{pmatrix} \in \text{span}\left\{ \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}, \begin{pmatrix} 2 \\ -3 \\ 1 \end{pmatrix}, \begin{pmatrix} 5 \\ -4 \\ 0 \end{pmatrix} \right\}$

Span of a Set of Vectors - Subspace

- **Theorem:** if $v_1, \cdots, v_n$ are elements of a vector space $\mathcal{V}$, then the $\mathcal{S} = \text{span}\{v_1, v_2, \cdots, v_n\}$ is a subspace of $\mathcal{V}$

Proof:

Spanning Set for a Vector Space

- Let $\mathcal{V}$ be a vector space
- For a certain set of vectors $v_1, \cdots, v_n \in \mathcal{V}$, we know that span$\{v_1, \cdots, v_n\} \subset \mathcal{V}$
- Is it possible that instead span$\{v_1, \cdots, v_n\} = \mathcal{V}$

Spanning Set for a Vector Space (2)

- Let $\mathcal{V}$ be a vector space
- For a certain set of vectors $v_1, \cdots, v_n \in \mathcal{V}$, we know that span $\{v_1, \cdots, v_n\} \subset \mathcal{V}$
- Is it possible that instead span $\{v_1, \cdots, v_n\} = \mathcal{V}$

- **Definition:** The set $v_1, v_2, \ldots, v_n \in \mathcal{V}$ is a *spanning set* for $\mathcal{V}$ if and only if every vector in $\mathcal{V}$ can be written as a linear combination of $v_1, v_2, \ldots, v_n$

Spanning Set for a Vector Space - Example

- **Example:** Determine if $\operatorname{span}\left\{\begin{pmatrix}3\\-5\\2\end{pmatrix}, \begin{pmatrix}8\\4\\1\end{pmatrix}, \begin{pmatrix}0\\3\\0\end{pmatrix}\right\} = \mathbb{R}^3$

Spanning Set and Linearly Dependent Set of Vectors

- Consider two spanning sets for $\mathbb{R}^2$

  - Let $\{v_1, v_2, \cdots, v_n\}$ be a **linearly dependent** set of vectors

    $\Rightarrow$ There is at least one vector, say v_1, that can be written as a sum of the others

  - Then, the span $\{v_1, v_2, \cdots, v_n\} = \text{span}\{v_2, v_3, \cdots, v_n\}$

Linearly Dependent Vectors

- In general, given $\{v_1, \ v_2, \ \cdots, \ v_n\}$, it is possible to write one of the vectors as a linear combination of the others $n-1$ vectors **if and only if** there exists $c_1, \ c_2, \ \cdots, \ c_n \in \mathbb{R}$ *not all zero* such that $c_1v_1 + c_2v_2 + \cdots + c_nv_n = 0$

Linear Independence

- The vectors $v_1,\ v_2,\ \cdots,\ v_n$ in a vector space $\mathcal{V}$ are said to be *Linearly Independent* if

$$c_1 v_1 + c_2 v_2 + \cdots + c_n v_n = 0 \quad \Rightarrow \quad c_1 = c_2 = \cdots = c_n = 0$$

- If $\{v_1,\ v_2,\ \cdots,\ v_n\}$ is a *minimal spanning* set for $\mathcal{V}$ then $v_1,\ v_2,\ \cdots,\ v_n$ are *linearly independent*!

- If $\{v_1,\ v_2,\ \cdots,\ v_n\}$ is a *linearly independent* set of vectors, then $\text{span}\,\{v_1,\ v_2,\ \cdots,\ v_n\} = \mathcal{V}$ is a *minimal spanning* set for $\mathcal{V}$!

$$\textit{Minimal Spanning Set} \equiv \textit{Basis}$$

Linear Independence – Example 1

- Determine if $\left\{ \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \end{pmatrix} \right\}$ are linearly independent

Linear Independence – Example 2

- Determine if $\left\{ \begin{pmatrix} 1 \\ 2 \\ 4 \end{pmatrix}, \begin{pmatrix} 2 \\ 1 \\ 3 \end{pmatrix}, \begin{pmatrix} 4 \\ -1 \\ 1 \end{pmatrix} \right\}$ are linearly independent

Linear Independence – Example 3

- Determine if $\left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} \right\}$ are linearly independent

Basis of a Vector Space

- **Theorem:** Suppose $v_1,\ v_2,\ \cdots,\ v_n$ are linearly independent vectors in a vector space $\mathcal{V}$ and let $v \in \operatorname{span}\{v_1,\ v_2,\ \ldots,\ v_n\}$, **then** v can be written **uniquely** as a linear combination of $v_1,\ v_2,\ \cdots,\ v_n$

- **Definition:** The vectors $v_1,\ v_2,\ \cdots,\ v_n$ form a *basis* for a vector space $\mathcal{V}$ iff:

    1. $\{v_1,\ v_2,\ \cdots,\ v_n\}$ are linearly independent

    2. $\operatorname{span}\{v_1,\ v_2,\ \cdots,\ v_n\} = \mathcal{V}$

Basis of a Vector Space – Example 1

- In $\mathbb{R}^3$ the *standard or natural* basis set is

Basis of a Vector Space – Example 2

- Does the set $v_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$, $v_2 = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$, $v_3 = \begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix}$ form a basis for $\mathbb{R}^3$?

Basis of a Vector Space – Example 3

- Does the set $v_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$, $v_2 = \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix}$, $v_3 = \begin{pmatrix} 0 \\ 2 \\ 4 \end{pmatrix}$ form a basis for $\mathbb{R}^3$?

1. Are they linearly independent?

$$c_1 v_1 + c_2 v_2 + c_3 v_3 = 0 \;\Rightarrow\; \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 2 \\ 1 & 2 & 4 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \xrightarrow{\text{RREF}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

2. Do they span $\mathbb{R}^3$? can any vector $x = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \in \mathbb{R}^3$ be written as a combination of $v_1,\ v_2,\ v_3$?

$$\alpha_1 v_1 + \alpha_2 v_2 + \alpha_3 v_3 = x$$

$$\Rightarrow \underbrace{\begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 2 \\ 1 & 2 & 4 \end{pmatrix}}_{=A} \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{pmatrix} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \xrightarrow{\text{RREF}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{pmatrix} \;\Rightarrow\; \det(A) = 0$$

Dimension of a Vector Space and Basis Summary

- **Theorem:** Let $\mathcal{V}$ is a vector space of dimension $n > 0$. Then:
  1. Any set of n linearly independent vectors span $\mathcal{V}$
  2. Any set of n vectors that span $\mathcal{V}$ are linearly independent
  3. No set of less than n vectors can span $\mathcal{V}$
  4. Any subset of less than n linearly indep. vectors can be extended to form a basis for $\mathcal{V}$
  5. Any spanning set of $> n$ vectors can be parted down to form a basis for $\mathcal{V}$

- $\mathbb{R}^3$ is a vector space with dimension 3 since any basis must have 3 vectors, e.g. $\{e_1,\ e_2,\ e_3\}$

  No set of less than 3 vectors can span $\mathbb{R}^3$!

Change of Basis - Motivation

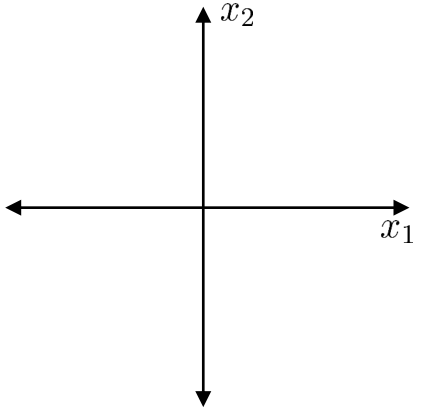
Motivation:

- Many applied problems can be simplified by changing from one coordinate system to another, e.g. when integrating the volume of a solid is better to use spherical coordinates, $\{r,\ \theta,\ \phi\}$, rather than rectangular, $\{x,\ y,\ z\}$

- When we specify a vector, we typically assume it is with respect to the standard basis

- For example in $\mathbb{R}^2$, a vector $x = \begin{bmatrix} \alpha \\ \beta \end{bmatrix}$ is implied to be $x = \alpha \begin{bmatrix} 1 \\ 0 \end{bmatrix} + \beta \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

Standard basis for $\mathbb{R}^2 = \text{span}\{e_1,\ e_2\}$



Change of Basis – General Basis

Definition:

- Let $\mathcal{V}$ be a finite dimensional vector space, and let $\mathscr{B} = \{b_1,\ b_2,\ \ldots,\ b_k\}$ be a basis for $\mathcal{V}$. Then any $v \in \mathcal{V}$, can be written uniquely as: $v = \alpha_1 b_1 + \alpha_2 b_2 + \cdots \alpha_k b_k$

- The elements $\alpha_1,\ \alpha_2,\ \cdots,\ \alpha_k$ are called the **coordinates of** v **w.r.t. to** $\mathscr{B}$

- $v_{\mathscr{B}} = \begin{bmatrix} \alpha_1 \\ \vdots \\ \alpha_k \end{bmatrix}$ is called the **coordinate vector of** v **w.r.t.** $\mathscr{B}$

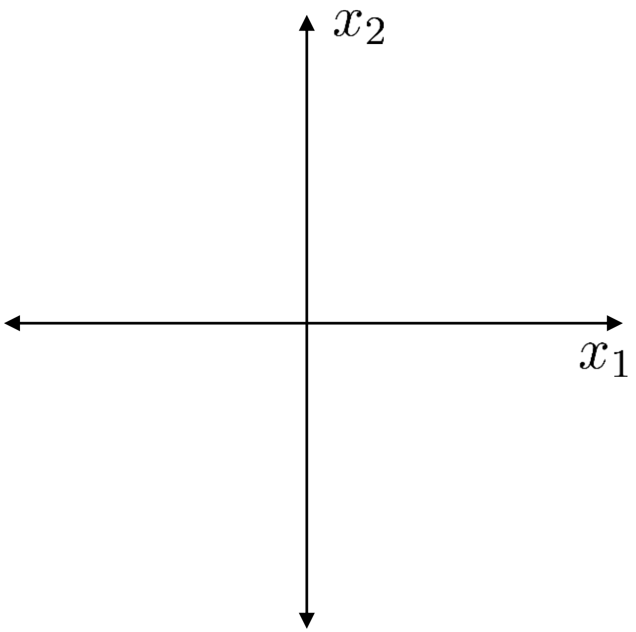
- $v = \begin{bmatrix} v_1 \\ \vdots \\ v_k \end{bmatrix}$ is called the **coordinate vector of** v **w.r.t. the standard basis**

Change of Basis – Coordinate Vector

- Given a vector $x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ (w.r.t standard basis), we would like to find the coordinates w.r.t another basis $\mathscr{B}$
- For example, $\mathscr{B} = \left\{ \begin{bmatrix} 3 \\ 5 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \end{bmatrix} \right\}$, then $x_{\mathscr{B}} = \begin{bmatrix} \alpha \\ \beta \end{bmatrix}$ implies that

Change of Basis – Example (1)

- Example, given $x = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ find its coordinates w.r.t $\mathscr{B} = \left\{ \begin{pmatrix} 1 \\ -1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \end{pmatrix} \right\}$, i.e. $x_{\mathscr{B}}$



Change of Basis – Example (2)

- Given two basis for $\mathbb{R}^2$: $\mathscr{B} = \left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \end{bmatrix} \right\}$ $\mathscr{C} = \left\{ \begin{bmatrix} 3 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ -2 \end{bmatrix} \right\}$
- Let $x_{\mathscr{B}} = \begin{bmatrix} -1 \\ 2 \end{bmatrix}$, find $x_{\mathscr{C}} = \begin{bmatrix} \alpha_1 \\ \alpha_2 \end{bmatrix}$

Change of Basis – Transition Matrix

- For $\mathbb{R}^n$, let $B = \begin{bmatrix} b_1 & b_2 & \ldots & b_n \end{bmatrix}$ be a matrix whose columns form a basis for $\mathbb{R}^n$. Let $v \in \mathbb{R}^n$, $v = \alpha_1 b_1 + \cdots + \alpha_n b_n$
- Similarly, let $C = \begin{bmatrix} c_1 & \cdots & c_n \end{bmatrix}$ be a matrix whose columns form a basis for $\mathbb{R}^n$, then $v = \gamma_1 c_1 + \cdots + \gamma_n c_n$
- Therefore, we can find the following transition matrices:

$$C v_{\mathscr{C}} = B v_{\mathscr{B}} \quad \Rightarrow \quad v_{\mathscr{C}} = \overbrace{C^{-1} B}\, v_{\mathscr{B}}$$

$$\Rightarrow \quad v_{\mathscr{B}} = \underbrace{B^{-1} C}\, v_{\mathscr{C}}$$

Linear Maps

- Let $\mathcal{V}$ be a vector space of dimension n
- and $\mathcal{W}$ be a vector space of dimension m over the field of $\mathbb{R}$
- **Definition:** A function $T : \mathcal{V} \to \mathcal{W}$ is called *linear* if

$$T(u+v) = T(u) + T(v) \qquad \forall \; u, \; v \in \mathcal{V}$$
$$T(\alpha v) = \alpha T(v) \qquad \forall \alpha \in \mathbb{R}, \;\; v \in \mathcal{V}$$

Linear Mapping Examples

- Show whether the transformation, $L : \mathbb{R}^2 \to \mathbb{R}^3$, is linear or not

$$L\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = \begin{bmatrix} 2x \\ x - 2y \\ 3x + 4y \end{bmatrix}$$

Linear Mapping Examples

- Show whether the transformation, $L : \mathbb{R}^2 \rightarrow \mathbb{R}^2$, is linear or not

$$L\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = \begin{bmatrix} 2x \\ xy \end{bmatrix}$$

Matrix as a Linear Map

- **Theorem:** Let $A \in \mathbb{R}^{m \times n}$. The mapping $L_A(x) : \mathbb{R}^n \to \mathbb{R}^m$ defined by:

$$L_A(x) = Ax, \quad \forall x \in \mathbb{R}^n$$

  is a linear transformation.

Matrix of a Linear Map

- **Theorem:** If L is a *linear* transformation from $\mathbb{R}^n$ to $\mathbb{R}^m$, then there is an $m \times n$ matrix A such that :

$$L(x) = Ax$$

for all $x \in \mathbb{R}^n$. The j^{th} column of A is given by:

$$a_j = L\left(e_j\right), \quad \text{for } j = 1, \cdots, n$$

That is $L = L_A$:

$$A = \begin{bmatrix} L(e_1) & L(e_2) & \cdots & L(e_n) \end{bmatrix}$$

The matrix A is called the **standard** matrix of a linear transformation L.

Linear Mapping – Matrix Examples

- Find the standard matrix of the following linear transformation:

$$L\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = \begin{bmatrix} 2x \\ x - 2y \\ 3x + 4y \end{bmatrix}$$

Linear Mapping – Different Basis

- We now know a linear transformation from two finite dimensional vector spaces, i.e. $L : \mathbb{R}^n \to \mathbb{R}^m$, has a **standard** matrix representation

- What if we were to use a different (not standard) basis for $\mathbb{R}^n$ and $\mathbb{R}^m$? What is the matrix representation of L w.r.t. these different basis?

Null Space of a Linear Map

- **Definition:** Let $A \in \mathbb{R}^{m \times n}$, i.e. $A : \mathbb{R}^n \to \mathbb{R}^m$, then the set

$$\mathcal{N}(A) = \{x \in \mathbb{R}^n \mid \; Ax = 0\}$$

is a subspace of R^n and is called the **null space** of A

Range/Column Space of a Linear Map

- **Definition:** Let $A \in \mathbb{R}^{m \times n}$, i.e. $A : \mathbb{R}^n \to \mathbb{R}^m$, then the set

$$\mathcal{R}(A) = \{y \in \mathbb{R}^m \mid \exists x \in \mathbb{R}^n, \;\; Ax = y\} = \{y = Ax \in \mathbb{R}^m \mid x \in \mathbb{R}^n\}$$

  is a subspace of R^m and is called the **range/column space** of A

- The range space for this matrix is also called the **column** space of A since:

$$\mathcal{R}(A) = \text{span}\,\{a_1,\; a_2,\; \cdots,\; a_n\}$$

  where a_i are the columns of A

Null/Range Space Example

- Find a **basis for** the null and column space of the following matrix:

$$A = \begin{bmatrix} 1 & 1 & 1 & 0 \\ 2 & 1 & 0 & 1 \end{bmatrix}$$

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Topic Outline

- Transfer Function vs State Space (Motivation)
- State Space Models and Examples
- From Nonlinear to Linear State Space Models
- Linear Algebra Review
- **Jordan Canonical Form (LN5 Slides)**