EE 419/519: Industrial Control Systems

Lecture 3: Control in Frequency Domain

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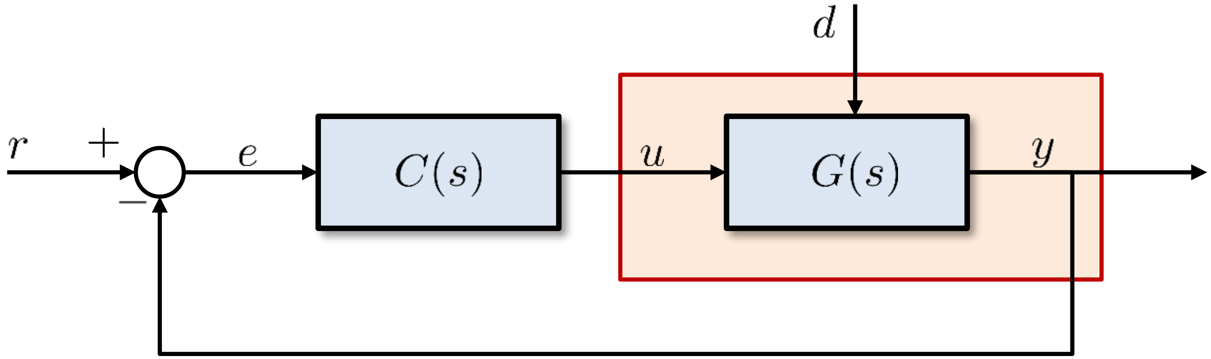
Topic Outline

- **Stability Definitions**
- **Block Diagram Algebra**
- **Proportional Control and Root Locus**
- **PI Control**
- **Bandwidth, Gain Margin, Phase Margin**

Review the Components of Feedback Control Systems

- A feedback control system is composed of a **dynamic system/plant**, **controller**, **sensors/outputs**, **actuators/inputs**, **reference**, and **disturbance**



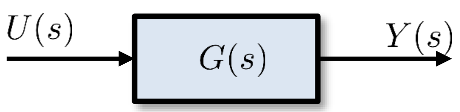
- **Main idea:** Having an accurate description of how the dynamic system (plant) naturally behaves, we would like to design a controller to modify this behavior in a desired way

- **First step:** we need to have a dynamic model

Types of Dynamic Models

- In this class, we will consider two types of mathematical models (commonly used for **linear systems**):

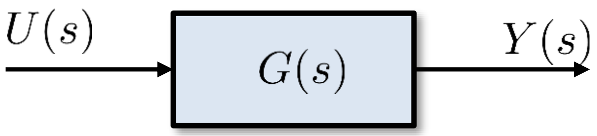
  1. Transfer-function/frequency domain approach



  2. State-space time domain approach

Motivation for Stability Analysis

- Consider the following **open loop system**

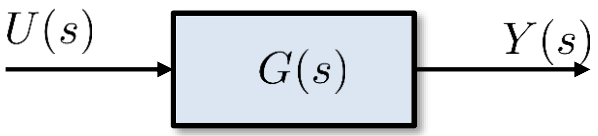


1. If an input is applied (e.g. constant), what will happen to the output?

2. (Assume $u = 0$) If the system is not at rest at $t = 0$, what will happen to the output? will it reach a steady state value?

Stability Definitions

- Consider the following **open loop system**



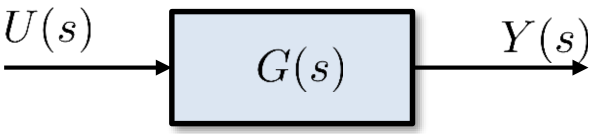
- In general, there are two categories of stability that we will be interesed in:

  1. **Stability around an equilibrium**: If the system is moved away from the equilibrium (e.g. $x(0) \neq 0$), will the system return to the same equilibrium ($x_e = 0$)?

  2. **Bounded Input Bounded Output**: If a bounded input ($|u(t)| \leq \bar{u}$) is applied to the system, will the output also be bounded?

Bounded Input Bounded Output – Time Domain

- Consider the following **open loop system**



- We can analyze the **Bounded Input Bounded Output** (BIBO) question
- First, review the open loop transfer function model

$$\frac{Y(s)}{U(s)} = G(s) \quad \Leftrightarrow \quad Y(s) = G(s)\,U(s)$$

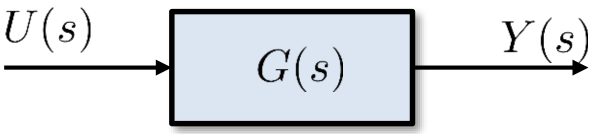
- We can use inverse Laplace transform to obtain $y(t)$:

$$Y(s) = G(s)\,U(s)$$

- What is $g(t)$?

Bounded Input Bounded Output – Derivation

- Consider the following **open loop system**



- We can analyze the **Bounded Input Bounded Output** (BIBO) question

$$Y(s) = G(s)\, U(s) \quad \xrightarrow{\mathcal{L}^{-1}} \quad y(t) = \int_0^t g(t-\tau)u(\tau)d\tau$$

- If the input applied is bounded, $|u(t)| \leq \bar{u} \;\; \forall t$,
under what conditions will the output be bounded for all t?

$$|y(t)| = \left| \int_0^t g(t-\tau)u(\tau)d\tau \right| \leq \int_0^t |g(t-\tau)|\; |u(\tau)|d\tau$$

Bounded Input Bounded Output

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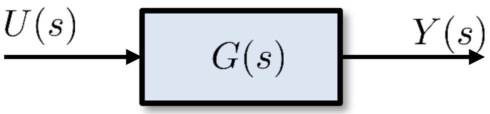

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$$|y(t)| = \left| \int_0^t g(t-\tau)u(\tau)d\tau \right| \leq \int_0^t |g(t-\tau)|\,|u(\tau)|d\tau$$

$$\Rightarrow \quad |y(t)| \leq \bar{u} \int_0^t |g(t-\tau)|d\tau \qquad \text{where } g(t) = a_1 e^{p_1 t} + \cdots + a_n e^{p_n t}$$

- We can say the output is bounded $|y(t)| \leq \bar{y}$, $\forall t$, including $t \to \infty$, if the integral term, $\int_0^t |g(t-\tau)|d\tau$ is bounded as $t \to \infty$

- This occurs if the exponential terms decay to 0 as $t \to \infty$

Bounded Input Bounded Output

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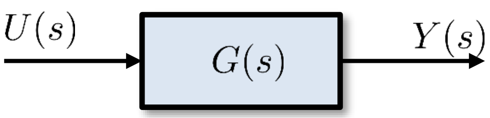

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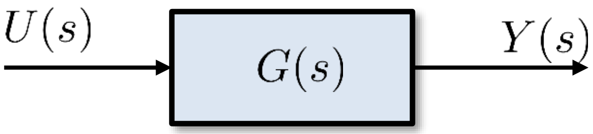
- This occurs if the exponential terms decay to 0 as $t \to \infty$

- Essentially, we need $\text{Re}\{p_i\} < 0 \ \forall i$

Asymptotic Stability -> Bounded Input Bounded Output

- Consider the following **open loop system**



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- If the input applied is bounded, $|u(t)| \leq \bar{u} \;\; \forall t$,
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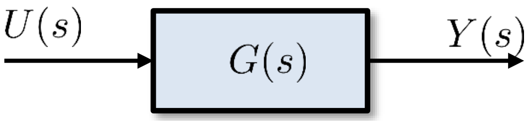
$$|y(t)| = \left| \int_0^t g(t-\tau)u(\tau)d\tau \right| \leq \int_0^t |g(t-\tau)|\;|u(\tau)|d\tau \leq \bar{u}\int_0^t |g(t-\tau)|d\tau$$

- A system is **asymmptotically stable** if the real part of all of the poles are negative, less than 0, $\text{Re}\{p_i\} < 0 \; \forall i$

- If an LTI system is **asymptotically stable**, then it is also **Bounded Input Bounded Output (BIBO)**

Examples

- Decide whether the following transfer functions are asymptotically stable



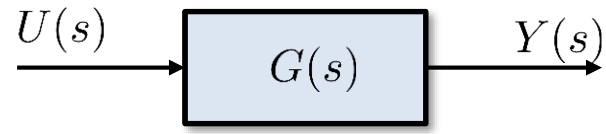
- $G(s) = \dfrac{3s+1}{s^2+6s+25}$

- $G(s) = \dfrac{10}{(s-3)(s+12)(s+4)}$

- $G(s) = \dfrac{5s+2}{(s)(s+2)(s+3)}$

General Impulse Response and Stability Conditions

- Consider the following **open loop system**



- For LTI systems, the stability of the system is defined by the poles of $G(s)$

$$\frac{Y(s)}{U(s)} = G(s) \qquad G(s) = K\frac{\prod_{i=1}^{m}(s-z_i)}{\prod_{i=1}^{n}(s-p_i)}, \qquad \text{where } m < n$$

- Assume we have $k \leq n$ **distinct poles**: $\{p_1,\ p_2,\ \ldots,\ p_k\}$

- Using partial fraction decomposition:

$$G(s) = G_1(s) + G_2(s) + \cdots + G_k(s),$$

$$\text{where } G_i(s) = \frac{a_{i1}}{(s-p_i)^1} + \frac{a_{i2}}{(s-p_i)^2} + \cdots + \frac{a_{iv}}{(s-p_i)^v}$$

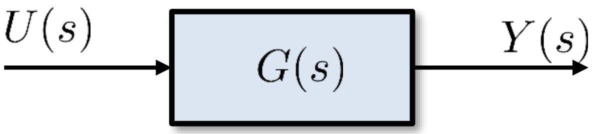
- The impulse response of the system is then as follows:

$$g(t) = \mathcal{L}^{-1}\left\{G(s)\right\} = g_1(t) + \cdots + g_k(t)$$

$$\text{where } g_i(t) = \left[a_{i1} + a_{2i}t + \cdots + a_{vi}t^{v_i-1}/(v_i-1)!\right]e^{p_i t}$$

General Impulse Response and Stability Conditions

- Consider the following **open loop system**



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- Assume we have $k \leq n$ **distinct poles**: $\{p_1,\ p_2,\ \ldots,\ p_k\}$

$$g(t) = \mathcal{L}^{-1}\left\{G(s)\right\} = g_1(t) + \cdots + g_k(t)$$

$$\text{where } g_i(t) = \left[a_{i1} + a_{2i}t + \cdots + a_{vi}t^{v_i-1}/(v_i-1)!\right]e^{p_i t}, \qquad p_i = \sigma_i + j\omega_i$$

General Impulse Response and Stability Conditions

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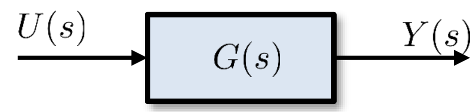
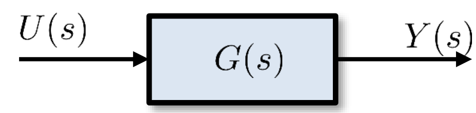
- We can then state the following stability conditions:

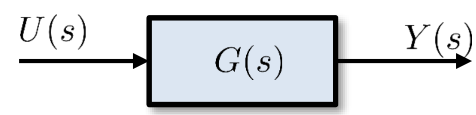
a) **Asymptotically stable**: $\mathrm{Re}\{p_i\} < 0,\ \forall i = 1,\ \cdots,\ k$



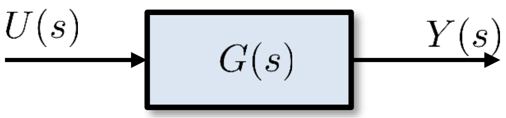
a) **Marginally stable**: If for all $\mathrm{Re}\{p_i\} = 0,\ v_i = 1,$
(all other poles must have negative real part)



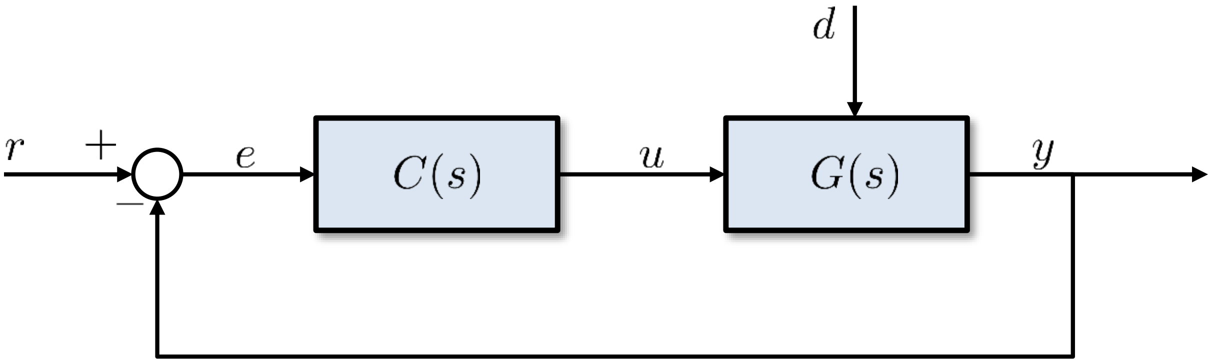
a) **Unstable**: If there exists an i s.t. $\mathrm{Re}\{p_i\} > 0$ **Or**
If there exists an i s.t. $\mathrm{Re}\{p_i\} = 0,\ \ v_i > 1$



Control Motivation



- We have now discussed some properties of $G(s)$ (Asymptotic Stability, BIBO)
- What if $G(s)$ is not asymptotically stable (or BIBO)?
- Can we design a controller, so that the closed loop system is BIBO or asymptotically stable?
- Can we design a controller to track a reference?



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- Stability Definitions
- **Block Diagram Algebra**
- Proportional Control and Root Locus
- PI Control
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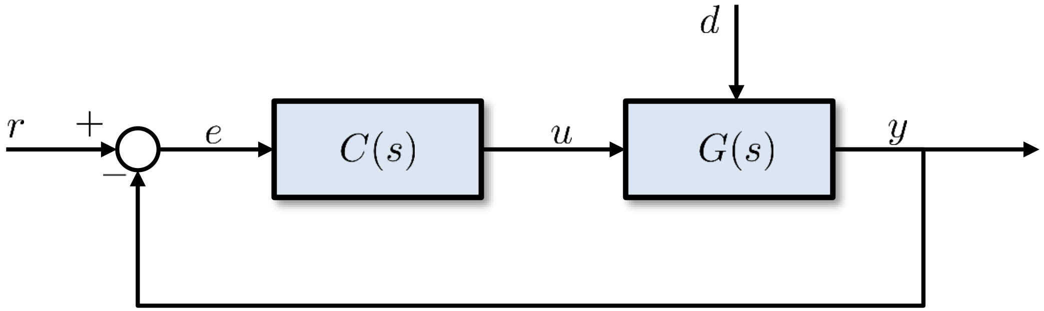
Motivation – Closed Loop System

- Before we analyze the design of $C(s)$, we need to know how how to write a mathematical model for the **closed loop system**

- For example, we need to find a mathematical model/description for:

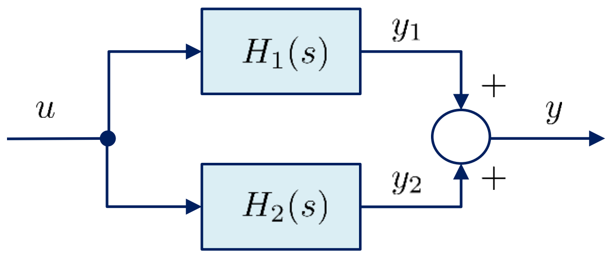
  - $\dfrac{Y(s)}{R(s)} = ?$

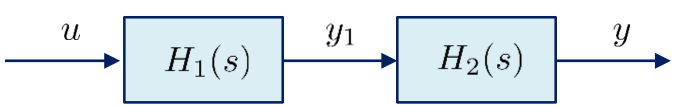
  - $\dfrac{E(s)}{R(s)} = ?$



Block Diagram Algebra

- In order to find transfer functions for the closed loop system, we can first analyze simpler interconnections:





Block Diagram Algebra (cont'd)

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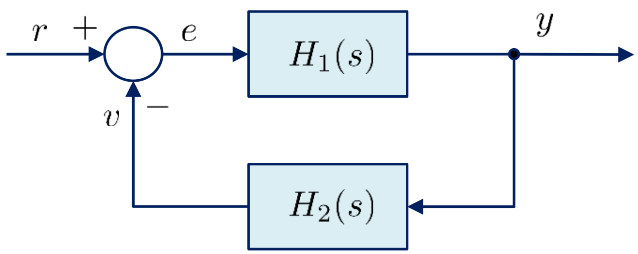
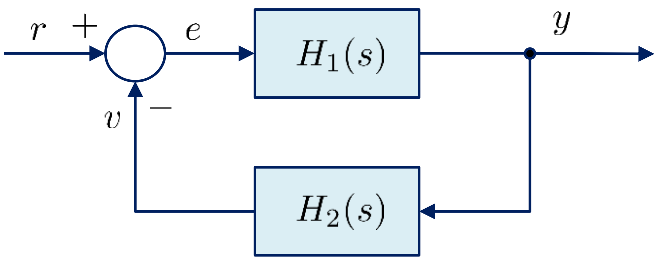
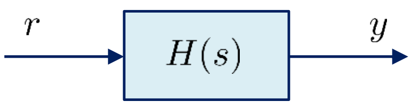


Transfer Function for a Simple Feedback Loop

- A simple feedback loop can be written as follows:



$\Longleftrightarrow$



$$y(s) = H(s)r(s), \qquad H(s) = \frac{H_1(s)}{1 + H_1(s)H_2(s)}$$

- The denominator of $H(s)$, which is $F(s) = 1 + H_1(s)H_2(s)$ will be known as the **return difference** or for a SISO as the **characteristic equation**

- The roots of $F(s) = 1 + H_1(s)H_2(s)$ are the **closed loop poles** of the system

Example: DC Motor

- **Goal:** we want to control the speed of the motor
- **Problem:** assume we use proportional feedback, write the closed loop reference/output transfer function

Example: DC Motor (cont'd)

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- **Problem:** assume we use proportional feedback, write the closed loop reference/output transfer function

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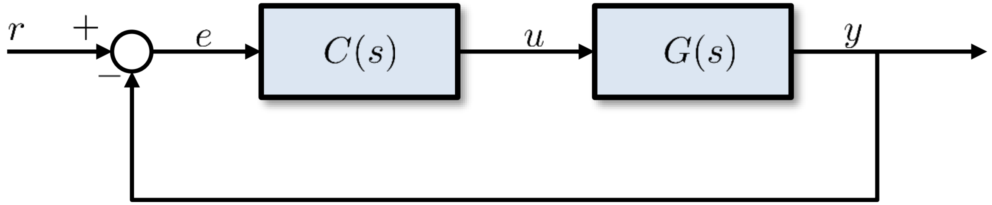
Proportional Control

- Some objectives of the controller will be:

  a) Tracking: attempt to track a reference, $y(t) \rightarrow r(t)$

  b) Stability: achieve asymptotic stability



- As a first step, let's consider proportional control

$$C(s) = K, \qquad G(s) = \frac{N(s)}{D(s)}$$

Closed Loop System Derivation

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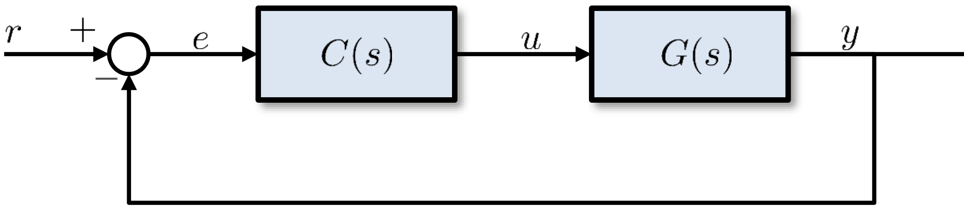

- Derive the transfer function $\frac{Y(s)}{R(s)} \triangleq G_{\text{cl}}(s)$

Closed Loop System Derivation (cont'd)

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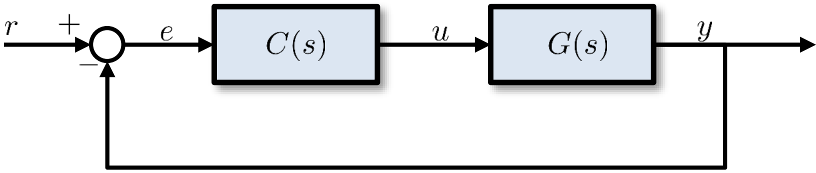
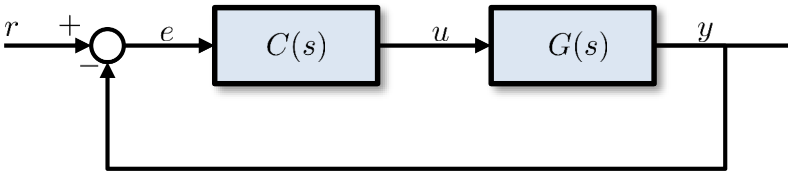


$$C(s) = K, \qquad G(s) = \frac{N(s)}{D(s)}$$

- The reference/output transfer function is $\dfrac{Y(s)}{R(s)} = G_{\text{cl}} = \dfrac{KG(s)}{1 + KG(s)}$
- If $G(s) = \dfrac{N(s)}{D(s)}$, simplify $G_{\text{cl}}(s)$:

Analyzing the Feedback Gain K



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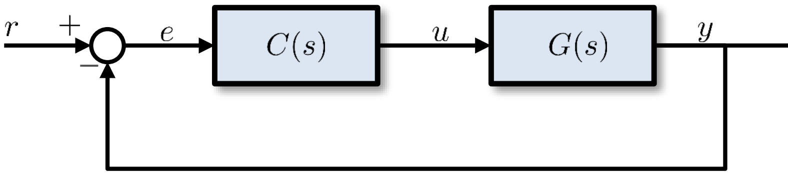
$$C(s) = K, \qquad G(s) = \frac{N(s)}{D(s)}$$

- The reference/output transfer function is $\dfrac{Y(s)}{R(s)} = G_{\text{cl}}(s) = \dfrac{KG(s)}{1+KG(s)} = \dfrac{KN(s)}{D(s)+KN(s)}$

- What happens to the poles of $G_{\text{cl}}(s)$ as $K \to 0$? as $K \to \infty$?

Analyzing the Feedback Gain K (cont'd)



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Root Locus Definition

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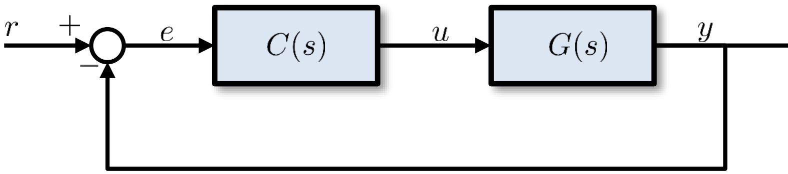

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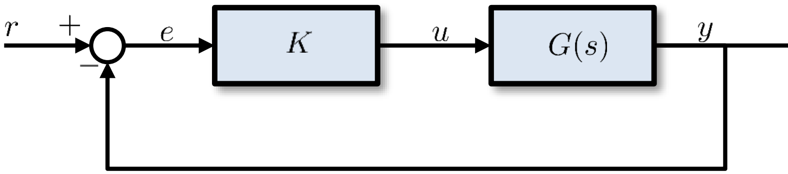
- The reference/output transfer function is $\dfrac{Y(s)}{R(s)} = G_{\text{cl}}(s) = \dfrac{KG(s)}{1+KG(s)} = \dfrac{KN(s)}{D(s)+KN(s)}$

- What happens to the poles of $G_{\text{cl}}(s)$ as $K \to 0$? as $K \to \infty$?

- The **Root Locus Method** is a plot in the complex plane of how the poles of $G_{\text{cl}}(s)$ vary as K is varied (from 0 to ∞)

$$\frac{Y(s)}{R(s)} = G_{\text{cl}}(s) = \frac{KG(s)}{1+KG(s)} = \frac{KN(s)}{D(s)+KN(s)}$$

Root Locus Matlab



- The reference/output transfer function is

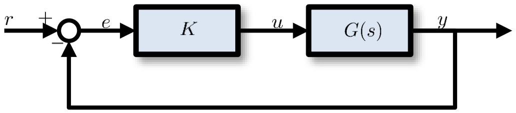
$$\frac{Y(s)}{R(s)} = G_{\text{cl}}(s) = \frac{KG(s)}{1+KG(s)} = \frac{KN(s)}{D(s)+KN(s)}$$

- The root locus plots the poles and zeros of $G_{\text{cl}}(s)$ as $K \to \infty$

- **Matlab**: Given $G(s)$ (open loop transfer function), the root locus can be obtained using the command **rlocus(sys)**

```
clc; clear all;close all;

% Define the system and plot the root locus
Ns = [1 -2];                  % Numerator:   s-2
Ds = [1 6 25];                % Denominator: s^2+6s+25
sys1 = tf(Ns, Ds)             % Creates the transfer function Gs = Ns/Ds

rlocus(sys1);
```

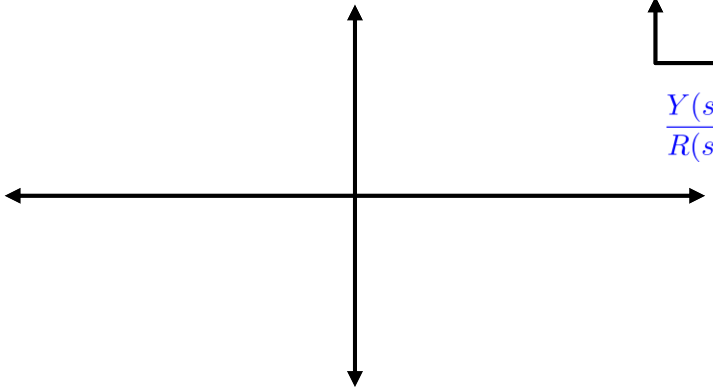
Root Locus Examples



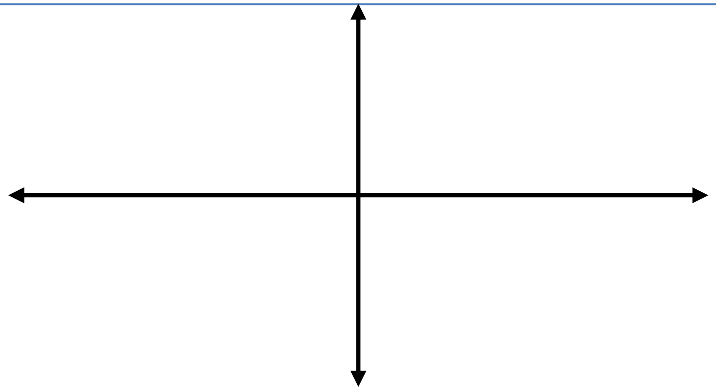
$$\frac{Y(s)}{R(s)} = G_{\text{cl}}(s) = \frac{KG(s)}{1+KG(s)}$$

- Let's look at a few cases:

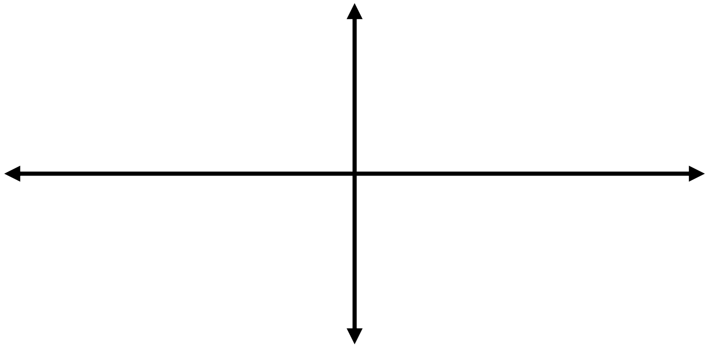
a) $G(s) = \dfrac{s-2}{s^2+6s+25}$



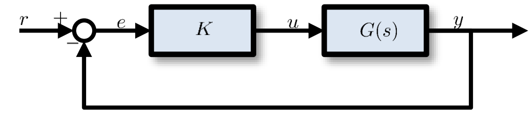
b) $G(s) = \dfrac{s+1}{s^2\,(s+4)}$



c) $G(s) = \dfrac{10}{s^2+6s+25}$



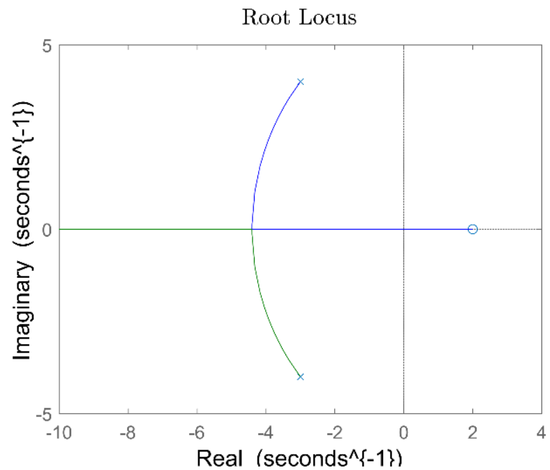
Root Locus Examples



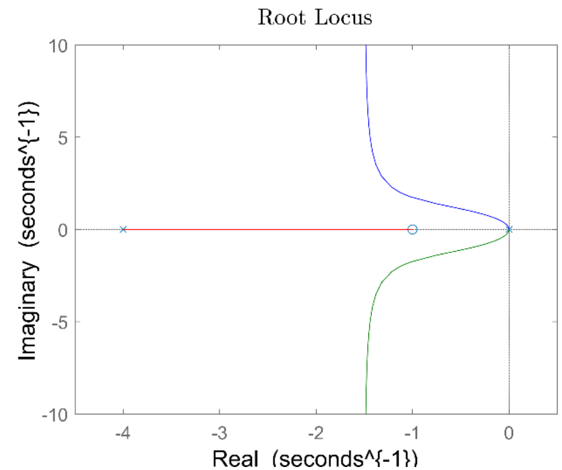
$$\frac{Y(s)}{R(s)} = G_{\text{cl}}(s) = \frac{KG(s)}{1+KG(s)}$$

- Let's look at a few cases:

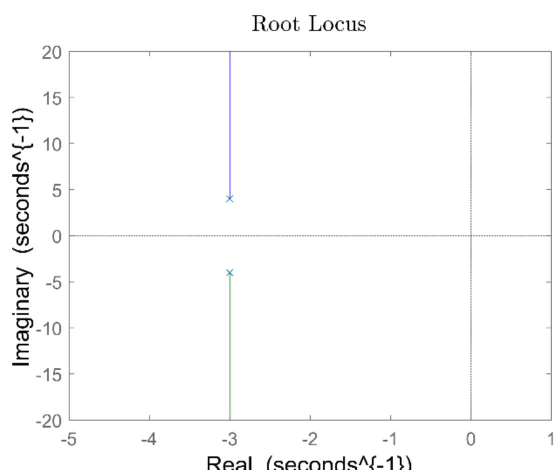
a) $G(s) = \dfrac{s-2}{s^2+6s+25}$



b) $G(s) = \dfrac{s+1}{s^2\,(s+4)}$

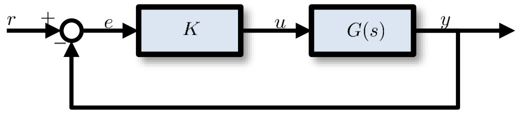


c) $G(s) = \dfrac{10}{s^2+6s+25}$



Root Locus Examples (Impact of Zeros)

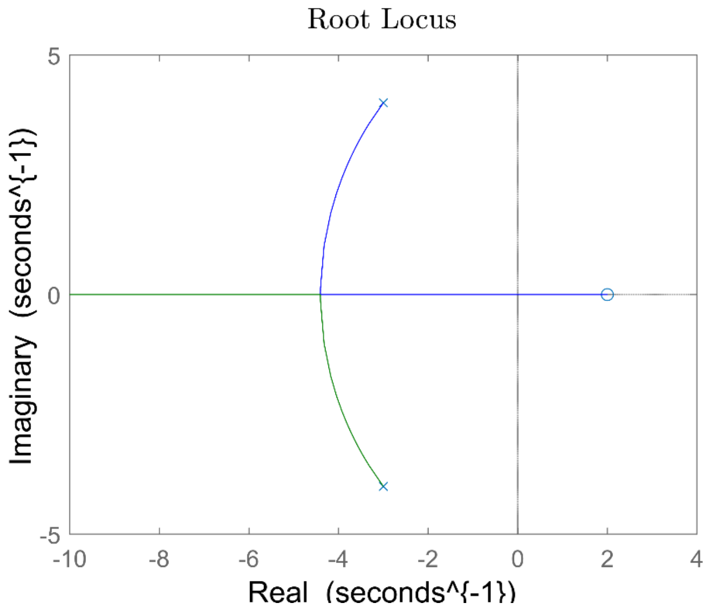
- Let's look at a few cases:

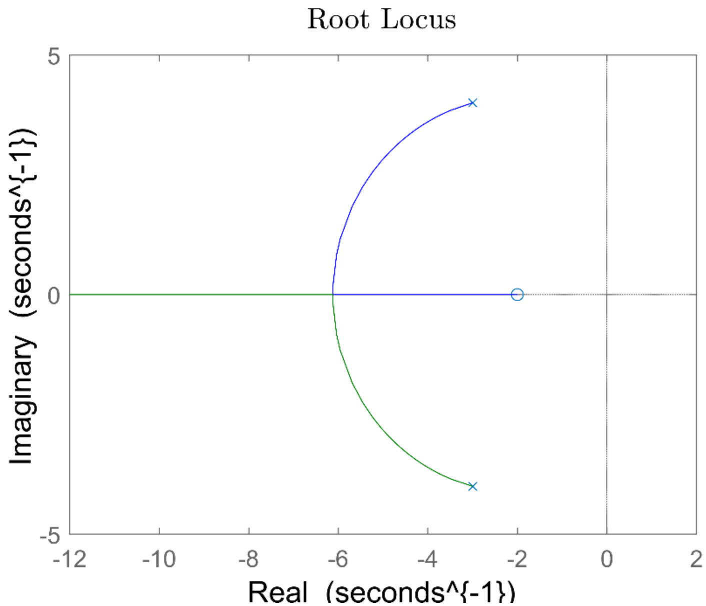


$$\frac{Y(s)}{R(s)} = G_{\text{cl}}(s) = \frac{KG(s)}{1+KG(s)}$$

a) $G(s) = \dfrac{s-2}{s^2+6s+25}$

b) $G(s) = \dfrac{s+2}{s^2+6s+25}$





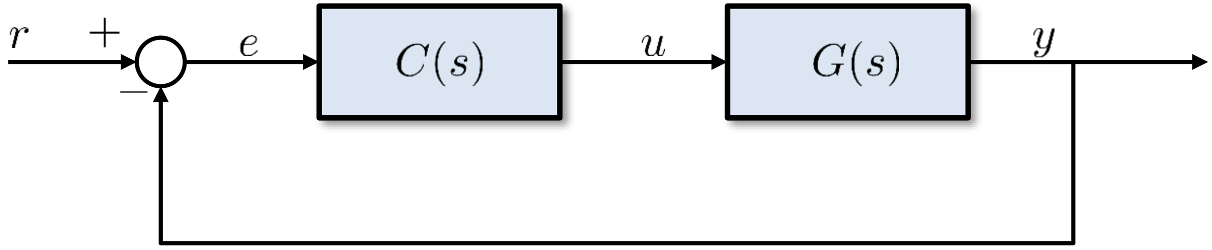
Topic Outline

- Stability Definitions
- Block Diagram Algebra
- Proportional Control and Root Locus
- **PI Control**
- Bandwidth, Gain Margin, Phase Margin

Reference Tracking

- We have looked at the case where $C(s) = K$ **proportional control**, and how K changes the closed loop poles of the system
- Assume K is chosen such that the closed loop poles have negative real part

- Will the output track the reference?, i.e. $y(t) \to r(t)$ or $e(t) \to 0$?
- What if the reference is a step response? a ramp? a polynomial? a sinusoid?

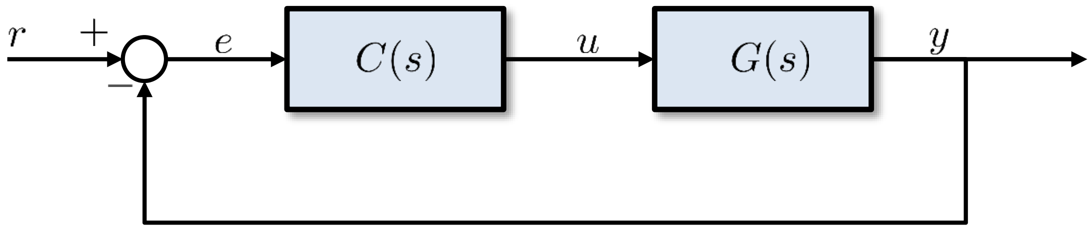


Error Modeling

- To prove the output will converge to the reference ($y(t) \to r(t)$), we need to show:

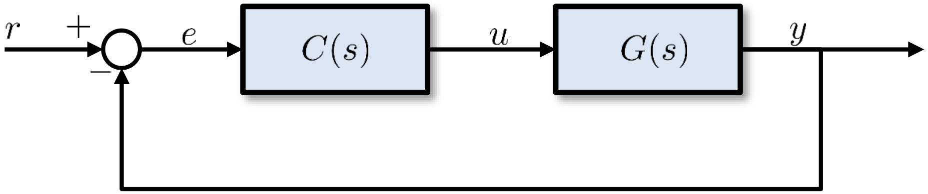
$$\lim_{t\to\infty} e(t) = 0$$

- What can we use to show this?
- We need a model for $E(s) = \mathcal{L}\{e(t)\}$



Error Modeling Derivation

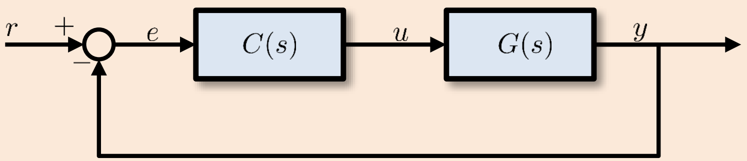
- Derive the transfer function $\dfrac{E(s)}{R(s)}$



Error Transfer Function and Laplace Final Value Theorem

- Therefore, the error in frequency domain is:

$$E(s) = \frac{1}{1 + G(s)C(s)} R(s)$$



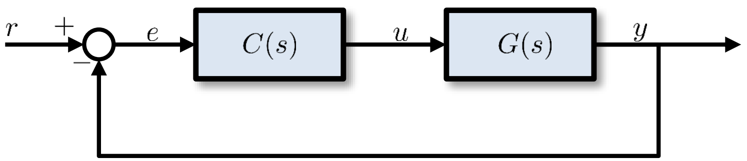
- Given a "type" of reference, $R(s)$, we can then show using Laplace Final Value Theorem:

$$\lim_{t \to \infty} e(t) = \lim_{s \to 0} sE(s) \stackrel{?}{=} 0$$

- Example of typical references to track:

Time Domain $r(t)$	Frequency Domain $R(s)$
$m\mathbf{1}(t)$	
mt	
$m \sin(\omega t + \phi)$	

Constant/Step Reference Tracking



- Therefore, the error in frequency domain is:

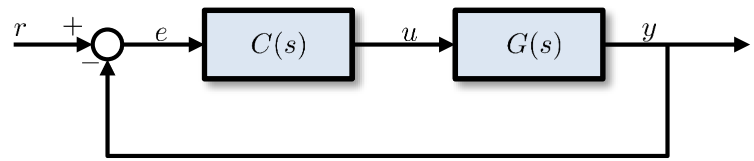
$$E(s) = \frac{1}{1+G(s)C(s)} R(s)$$

- Assume the reference is a step function $r(t) = r\mathbf{1}(t)$

- Use Laplace Final Value theorem to find $\lim_{t\to\infty} e(t) = \lim_{s\to 0} sE(s) \stackrel{?}{=} 0$

Constant/Step Reference Tracking

- Therefore, the error in frequency domain is:



$$E(s) = \frac{1}{1 + G(s)C(s)} R(s)$$

- Assume the reference is a step function $r(t) = r\mathbf{1}(t) \quad \Rightarrow \quad R(s) = \frac{r}{s}$

- Use Laplace Final Value theorem to find $\lim_{t\to\infty} e(t) = \lim_{s\to 0} sE(s) \stackrel{?}{=} 0$

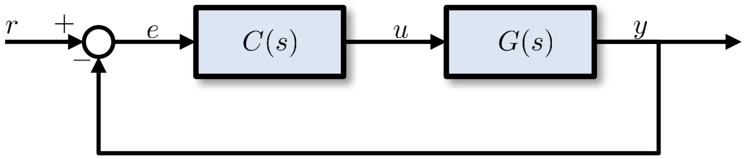
$$\lim_{t\to\infty} e(t) = \lim_{s\to 0} sE(s) = \lim_{s\to 0} \frac{r}{1 + G(s)C(s)}$$

- Let's consider two types of controllers $C(s)$

  a) Proportional control $C(s) = K_p$

  b) Integral control $C(s) = \frac{K_i}{s}$,

Constant/Step Reference Tracking – Proportional Control



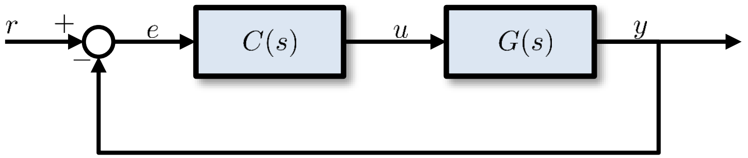
- Assume the reference is a step function
  $r(t) = r\mathbf{1}(t) \quad \Rightarrow \quad R(s) = \frac{r}{s}$

- **Assumptions**: $G(s)$ has only left hand poles, $G(0) = g_0 \neq 0$ is finite, $C(s) = K_p$

- **What is the steady state error under these conditions?**

$$\lim_{t\to\infty} e(t) = \lim_{s\to 0} sE(s) = \lim_{s\to 0} \frac{r}{1 + G(s)C(s)}$$

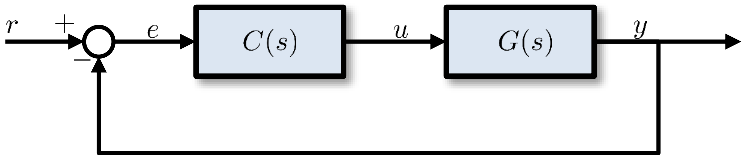
Constant/Step Reference Tracking – Integral Control



- Assume the reference is a step function
  $r(t) = r\mathbf{1}(t) \quad \Rightarrow \quad R(s) = \frac{r}{s}$
- **Assumptions**: $G(s)$ has only left hand poles, $G(0) = g_0 \neq 0$ is finite, $C(s) = \frac{K_i}{s}$
- **What is the steady state error under these conditions?**

$$\lim_{t\to\infty} e(t) = \lim_{s\to 0} sE(s) = \lim_{s\to 0} \frac{r}{1 + G(s)C(s)}$$

Constant/Step Reference Tracking – Integral Control



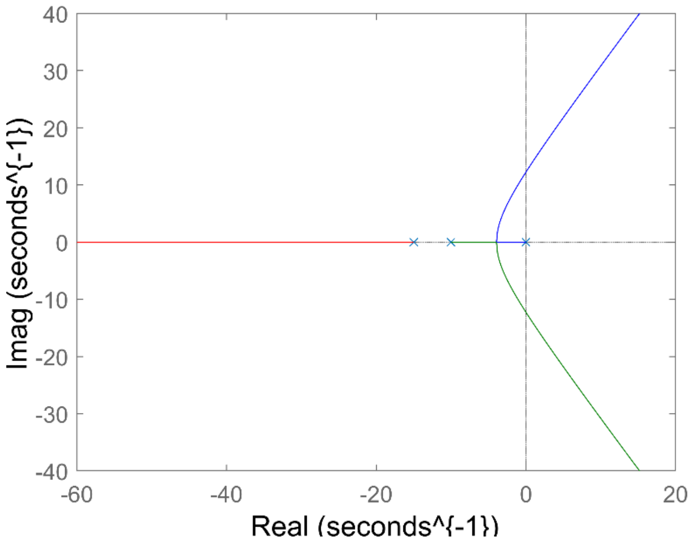
- Assume the reference is a step function
  $r(t) = r\mathbf{1}(t) \quad \Rightarrow \quad R(s) = \frac{r}{s}$
- **Assumptions**: $G(s)$ has only left hand poles, $G(0) = g_0 \neq 0$ is finite
- Integral control is sufficient to ensure perfect tracking of a step function
- However, having only integral gain typically makes the closed loop system less damped (large oscillation, slow convergence)

$$G(s) = \frac{1}{(s+10)(s+15)} \qquad C(s) = \frac{K_i}{s}$$

$$\Rightarrow \quad E(s) = \frac{1}{1 + G(s)C(s)} R(s)$$

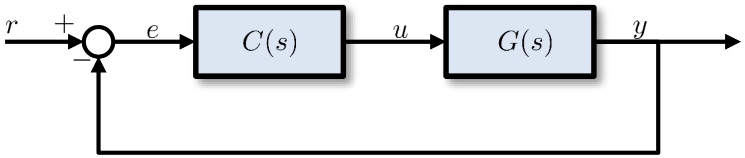
$$\Rightarrow \quad Y(s) = \frac{C(s)G(s)}{1 + C(s)G(s)} R(s)$$

$$\Rightarrow \quad Y(s) = \frac{C(s)G(s)}{1 + K_i \underbrace{\frac{1}{s}G(s)}_{\tilde{G}(s)}} R(s)$$



Constant/Step Reference Tracking – PI Control

- Assume the reference is a step function
  $r(t) = r\mathbf{1}(t) \quad \Rightarrow \quad R(s) = \frac{r}{s}$



- **Assumptions**: $G(s)$ has only left hand poles, $G(0) = g_0 \neq 0$ is finite
- To improve the performance, Integral control is typically combined with a proportional gain $\Rightarrow$ **Proportional Integral (PI) control**

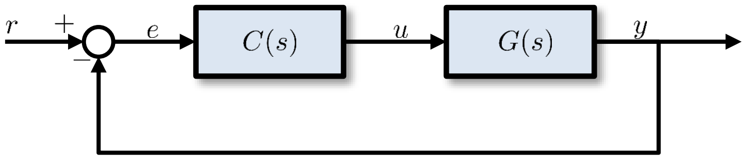
$$C(s) = K_p + \frac{K_i}{s} = \frac{K_p s + K_i}{s} \triangleq \frac{N_c(s)}{D_c(s)} \qquad G(s) = \frac{N_p(s)}{D_p(s)}$$

$$\Rightarrow \quad \frac{Y(s)}{R(s)} = \frac{C(s)G(s)}{1 + C(s)G(s)} = \frac{N_c(s)N_p(s)}{D_c(s)D_p(s) + N_c(s)N_p(s)}$$

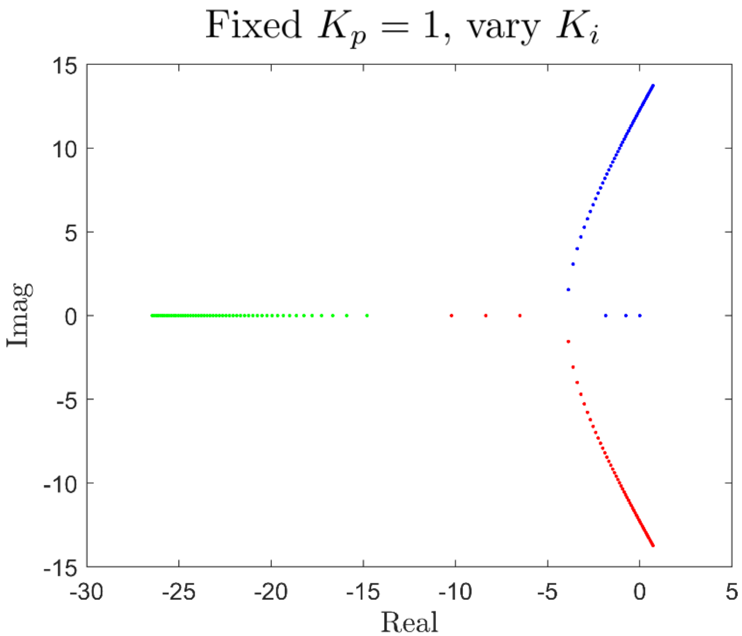
- Now we have two parameters to optimize/play with: K_p and K_i!

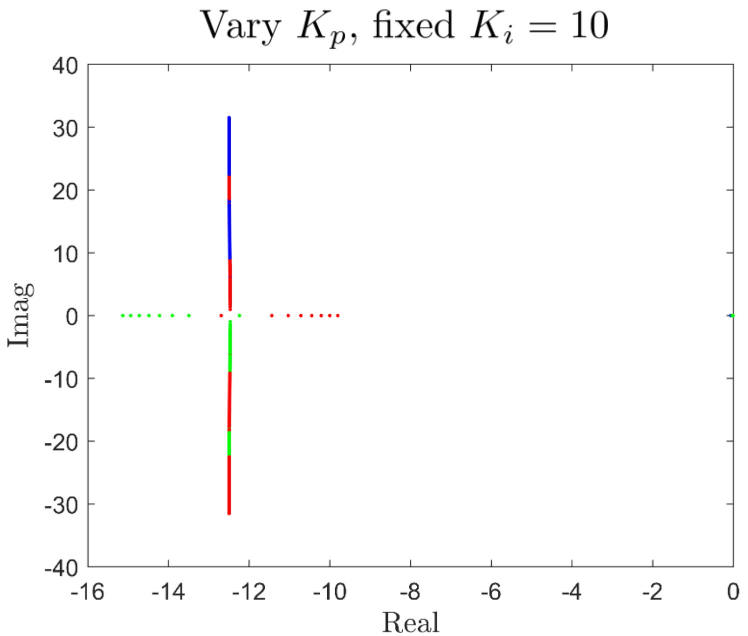
Constant/Step Reference Tracking – PI Control

- Assume the reference is a step function
  $r(t) = r\mathbf{1}(t) \quad \Rightarrow \quad R(s) = \frac{r}{s}$



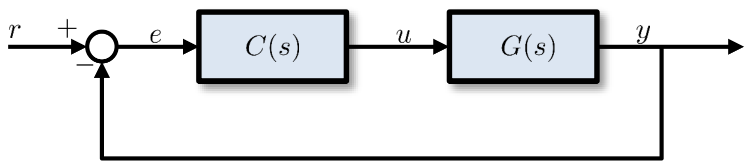
$$G(s) = \frac{1}{(s+10)(s+15)} \qquad C(s) = K_p + \frac{K_i}{s} = \frac{K_p s + K_i}{s} \qquad \Rightarrow \quad \frac{Y(s)}{R(s)} = \frac{C(s)G(s)}{1 + C(s)G(s)}$$





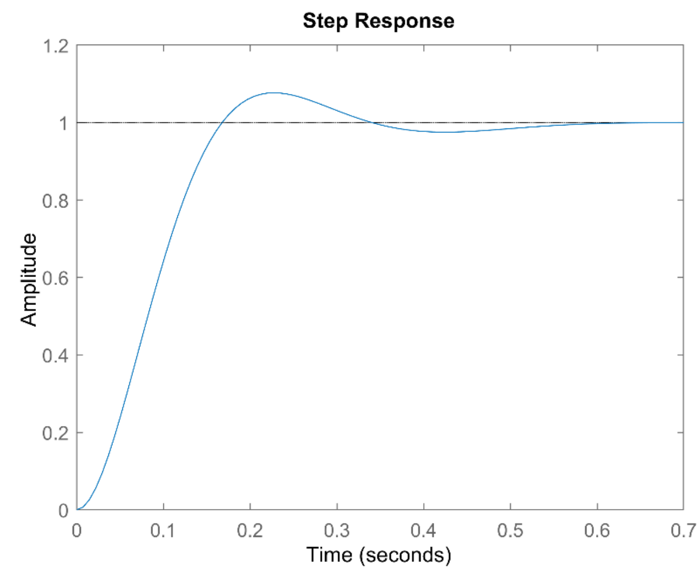
Matlab PI Tuner!

- Matlab has a command that can help you optimally "tune" the K_p and K_i gains



$$\Rightarrow \frac{Y(s)}{R(s)} = \frac{C(s)G(s)}{1 + C(s)G(s)}$$

```
% Define the numerator and denominator of the plant
Gp = zpk([], [-10 -15], 1);                 % Zero, pole, gain form zpk(zeros, poles, gain)

% PI Tune
[C_pi,info] = pidtune(Gp,'PI');
Cs = tf([C_pi.Kp C_pi.Ki], [1 0]);          % Obtain the transfer function for the controller

G_cl2 = (Cs*Gp)/(1+Cs*Gp);
G_cl2 = minreal(G_cl2);                     % Cancel any zeros and poles that may occur

figure,
step(G_cl2)                                 % Plot the step response
```

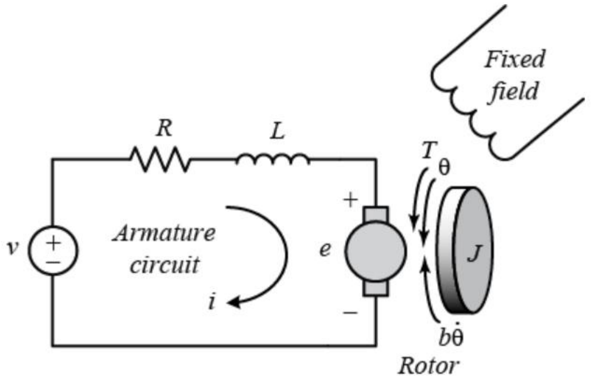


Example: DC Motor Speed Control

- The transfer function for the dc motor is:

$$\Rightarrow \frac{Y(s)}{U(s)} = \frac{W(s)}{V(s)} = \frac{\frac{\kappa}{JL}}{s^2 + \left(\frac{r}{L} + \frac{\beta}{J}\right)s + \left(\frac{\beta r + \kappa^2}{JL}\right)}$$

$$\Rightarrow \frac{Y(s)}{U(s)} = \frac{W(s)}{V(s)} = \frac{2}{s^2 + 12s + 20.02}$$



κ	0.01	Nm/A or Vs/rad
J	0.01	kgm^2
β	0.1	Nms
L	0.5	H
r	1	Ω

https://ctms.engin.umich.edu/CTMS/index.php?example=MotorSpeed&section=SystemModeling

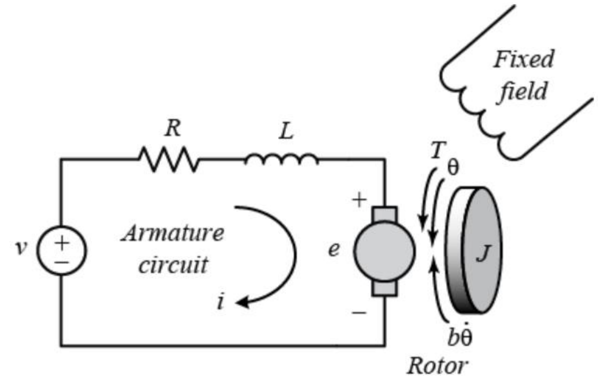
Example: DC Motor Speed Control (cont'd)

- The transfer function for the dc motor is:

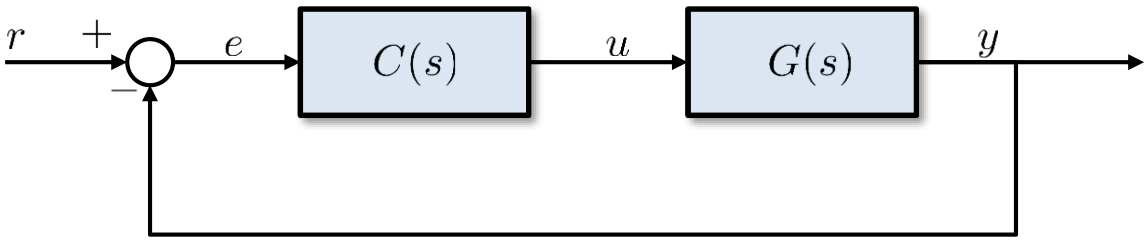
$$\Rightarrow \frac{Y(s)}{U(s)} = \frac{W(s)}{V(s)} = G(s) = \frac{2}{s^2 + 12s + 20.02}$$

$$C(s) = K_p + \frac{K_i}{s} = \frac{K_p s + K_i}{s}$$

$$\Rightarrow \frac{Y(s)}{R(s)} = \frac{C(s)G(s)}{1 + C(s)G(s)}$$

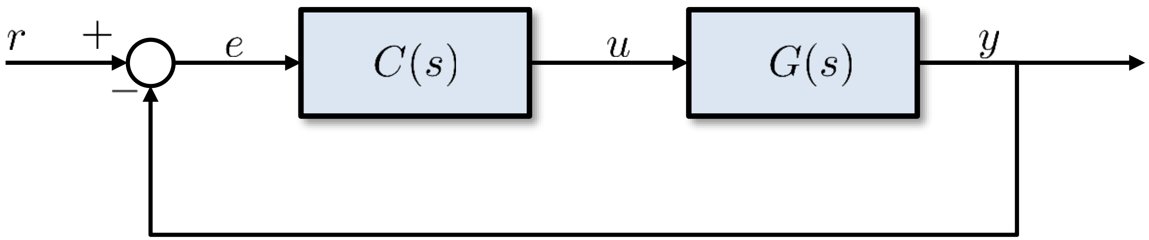


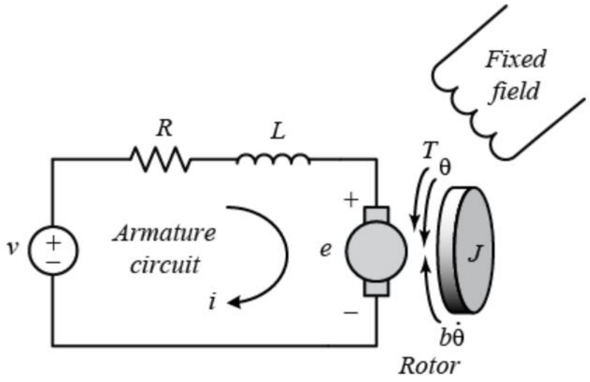
κ	0.01	Nm/A or Vs/rad
J	0.01	kgm^2
β	0.1	Nms
L	0.5	H
r	1	Ω

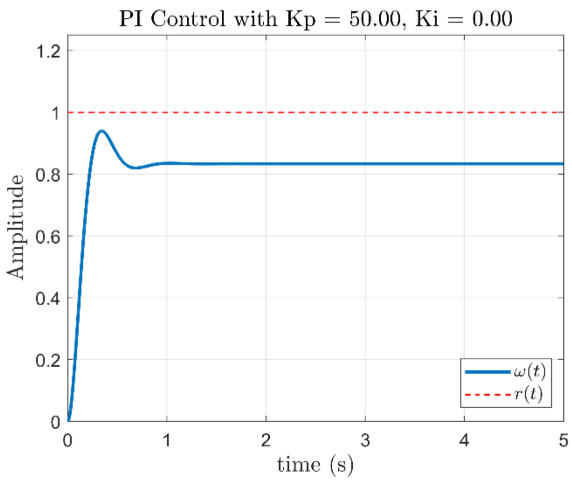


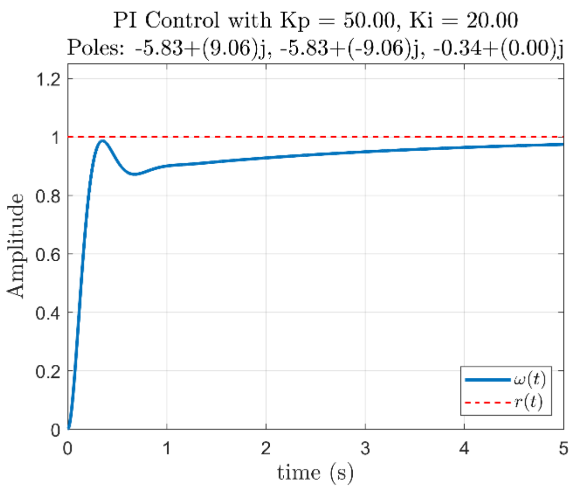
Example: DC Motor Speed Control (cont'd)

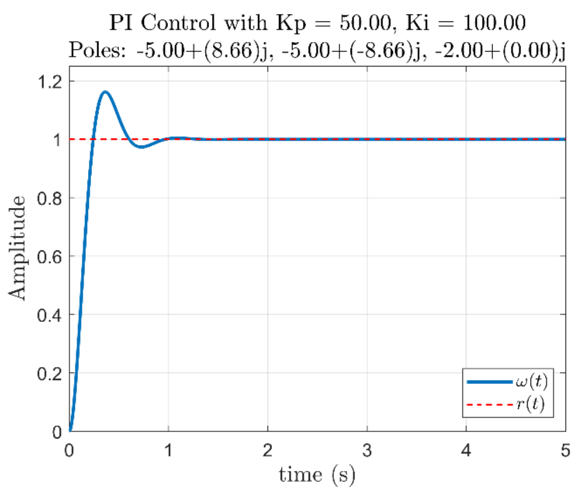
- The transfer function for the dc motor is:







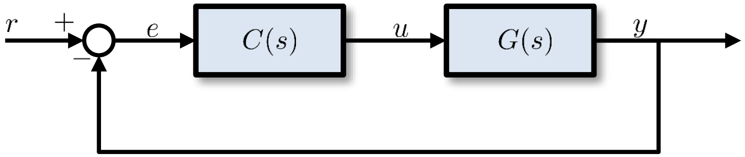




https://ctms.engin.umich.edu/CTMS/index.php?example=MotorSpeed&section=SystemModeling

Final Thoughts

- The error in frequency domain is:



$$E(s) = \frac{1}{1 + G(s)C(s)} R(s)$$

- Given a "type" of reference, $R(s)$, we can then show using Laplace Final Value Theorem:

$$\lim_{t \to \infty} e(t) = \lim_{s \to 0} sE(s) \overset{?}{=} 0$$

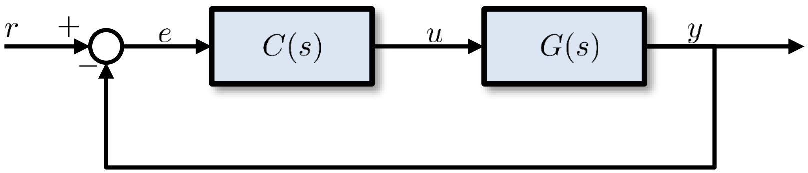
- We discussed that an integrator is needed to track a constant/step reference
- **What if we needed to track a ramp reference?**
- **What if we needed to track a polynomial signal of higher order?**

Topic Outline

- Stability Definitions
- Block Diagram Algebra
- Proportional Control and Root Locus
- PI Control
- **Bandwidth, Gain Margin, Phase Margin**

Transfer Functions and Complex Numbers

- Let's consider a feedback system as shown in the figure
  - Open loop system transfer function $\frac{Y(s)}{U(s)} = G(s)$
  - Controller transfer function $\frac{U(s)}{E(s)} = C(s)$
  - Closed loop transfer function $\frac{Y(s)}{R(s)} = \frac{G(s)C(s)}{1+G(s)C(s)} = G_{\text{cl}}(s)$
  - The return difference is defined as $T(s) \triangleq 1 + G(s)C(s)$



- For each of these transfer functions, we can consider $s = j\omega$, where ω is the frequency
- Therefore, each transfer function can be considered a complex value, i.e.

$$G(s) = G(j\omega) = |G(j\omega)|e^{j\phi}$$

  - **Magnitude** $|G(j\omega)|$
  - **Phase** ϕ

Decibels

- For each of these transfer functions, we can consider $s = j\omega$, where ω is the frequency
- Therefore, each transfer function can be considered a complex value, i.e.

$$G(s) = G(j\omega) = |G(j\omega)|e^{j\phi}$$

- However, rather than looking at the magnitude directly, it helps* if we instead consider the following:

$$D(\omega) = 20\log_{10}|G(j\omega)| \quad (\text{dB})$$

Zero, Pole, Gain Form and Amplitude

- For each of these transfer functions, we can consider $s = j\omega$, where ω is the frequency

$$G(j\omega) = G_0 \frac{\prod_{i=1}^{m}\left(\frac{j\omega}{z_i}+1\right)}{\prod_{i=1}^{n}\left(\frac{j\omega}{p_i}+1\right)}$$

- What is the amplitude in dB?

Zero, Pole, Gain Form and Phase

- For each of these transfer functions, we can consider $s = j\omega$, where ω is the frequency

$$G(j\omega) = G_0 \frac{\prod_{i=1}^{m} \left(\frac{j\omega}{z_i} + 1\right)}{\prod_{i=1}^{n} \left(\frac{j\omega}{p_i} + 1\right)}$$

- What is the phase?

Summary

- For each of these transfer functions, we can consider $s = j\omega$, where ω is the frequency

$$G(j\omega) = G_0 \frac{\prod_{i=1}^{m}\left(\frac{j\omega}{z_i}+1\right)}{\prod_{i=1}^{n}\left(\frac{j\omega}{p_i}+1\right)}$$

- What is the amplitude and dB?

$$D(\omega) = 20\log|G_0| + 10\log\left[1+\left(\frac{\omega}{z_1}\right)^2\right] + \cdots + 10\log\left[1+\left(\frac{\omega}{z_m}\right)^2\right]$$
$$-10\log\left[1+\left(\frac{\omega}{p_1}\right)^2\right] - \cdots - 10\log\left[1+\left(\frac{\omega}{p_n}\right)^2\right]$$

- What is the phase?

$$\phi(\omega) = \tan^{-1}\left(\frac{\omega}{z_1}\right) + \cdots + \tan^{-1}\left(\frac{\omega}{z_m}\right) - \tan^{-1}\left(\frac{\omega}{p_1}\right) - \cdots - \tan^{-1}\left(\frac{\omega}{p_n}\right)$$

Impact of Zeros and Poles to D(w)

- What is the impact of poles and zeros to the magnitude $D(\omega)$?

  - For a **zero**, as frequency is increased it gains around 20 dB per decade

$$D(\omega) = 20\log|G_0| + 10\log\left[1+\left(\frac{\omega}{z_1}\right)^2\right] + \cdots + 10\log\left[1+\left(\frac{\omega}{z_m}\right)^2\right] - 10\log\left[1+\left(\frac{\omega}{p_1}\right)^2\right] - \cdots - 10\log\left[1+\left(\frac{\omega}{p_n}\right)^2\right]$$

  - For a **pole**, as frequency is increased it decreases around 20 dB per decade

Impact of Zeros and Poles to Phase

- What is the impact of poles and zeros to the magnitude $D(\omega)$?
  - For a **zero**, the phase shift at $\omega = z_i$ is 45^o

$$\phi(\omega) = \tan^{-1}\left(\frac{\omega}{z_1}\right) + \cdots + \tan^{-1}\left(\frac{\omega}{z_m}\right) - \tan^{-1}\left(\frac{\omega}{p_1}\right) - \cdots - \tan^{-1}\left(\frac{\omega}{p_n}\right)$$

  - For a **zero**, the phase shift at $\omega = p_i$ is -45^o

Bode Plot Example

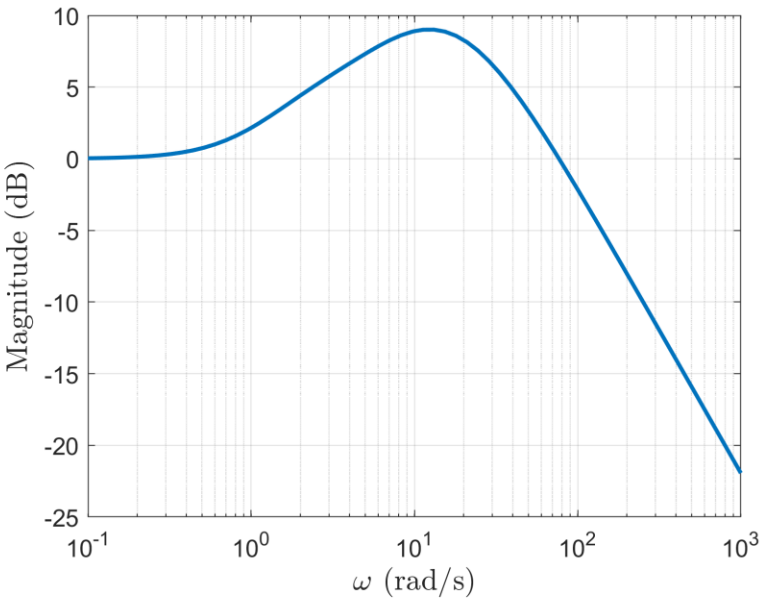
- Let's look at the bode plot of the following transfer function

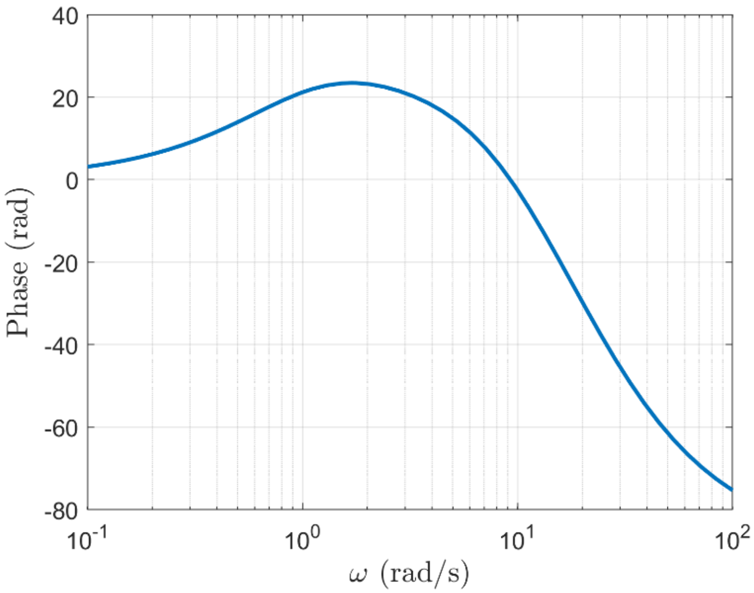
$$G(s) = \frac{(1+s)\left(1+\frac{s}{5}\right)}{\left(1+\frac{s}{2}\right)\left(1+\frac{s}{10}\right)\left(1+\frac{s}{20}\right)}$$

Bode Plot Example

- Let's look at the bode plot of the following transfer function

$$G(s) = \frac{(1+s)\left(1+\frac{s}{5}\right)}{\left(1+\frac{s}{2}\right)\left(1+\frac{s}{10}\right)\left(1+\frac{s}{20}\right)}$$

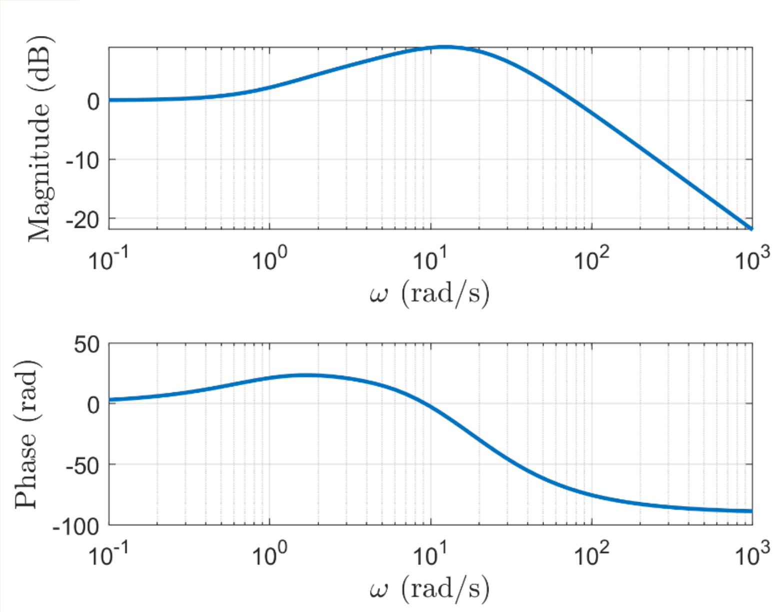




Bode Plot Example (Matlab Code)

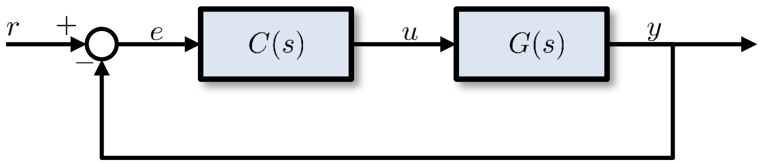
- The following is an example Matlab code to get the bode plot $G(s) = \dfrac{(1+s)\left(1+\frac{s}{5}\right)}{\left(1+\frac{s}{2}\right)\left(1+\frac{s}{10}\right)\left(1+\frac{s}{20}\right)}$

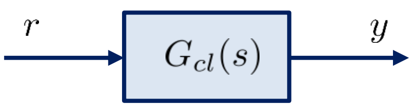
```
clc;clear all;close all;

%% Figure properties
fs = 15;                    % Font Size
gs = 12;                    % Axis size
lw = 2;                     % Line Width

%% Define the transfer function
s = tf('s');
sys = (s+1)*(s/5+1)/((s/2+1)*(s/10+1)*(s/20+1))

[mag, phase, wout] = bode(sys);
mag      = squeeze(mag);
phase    = squeeze(phase);

figure(1);
subplot(2,1,1);
semilogx(wout, 20*log(mag)./log(10),'LineWidth',lw); grid on;
set(gca,'FontSize',gs);
xlabel('$\omega$ (rad/s)','interpreter','latex','FontSize',fs);
ylabel('Magnitude (dB)','interpreter','latex','FontSize',fs);

subplot(2,1,2);
semilogx(wout, phase,'LineWidth',lw); grid on;
set(gca,'FontSize',gs);
xlabel('$\omega$ (rad/s)','interpreter','latex','FontSize',fs);
ylabel('Phase (rad)','interpreter','latex','FontSize',fs);
print(gcf,'bodesystem.png', '-dpng', '-r300');
```



Stability Review and Robustness

- Let's consider a feedback system as shown in the figure
  - Closed loop transfer function $\frac{Y(s)}{R(s)} = \frac{G(s)C(s)}{1+G(s)C(s)} = G_{\text{cl}}(s)$
  - The return difference is defined as $T(s) \triangleq 1 + G(s)C(s)$





- The **closed loop** system $G_{\text{cl}}(s)$ is asymptotically stable if the poles of the transfer function have all negative real parts

$$G_{cl}(s) = \frac{G(s)C(s)}{1 + G(s)C(s)}$$

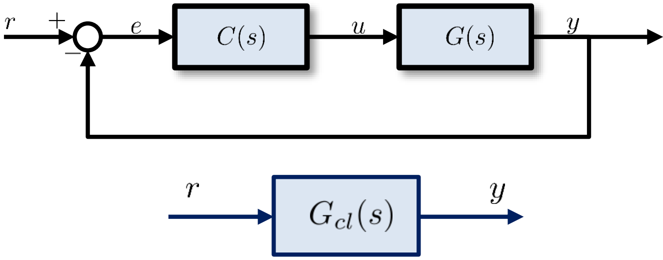
- How **robust** is the system to parameter changes?
- How **robust** is the system to **delays** in the control loop?

- We can answer some of these questions by analyzing the closed loop transfer function

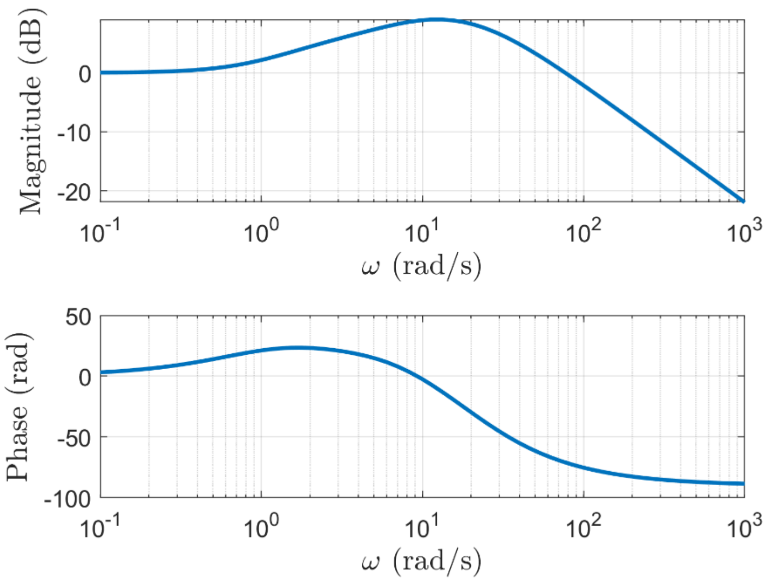
Closed Loop Frequency Domain Analysis

- Analyze the bode plot for the following closed loop system:

$$G_{cl}(s) = -4.1\frac{(s+15.47)(s-2)}{(s+15.22)(s^2+5.72s+8.25)}$$



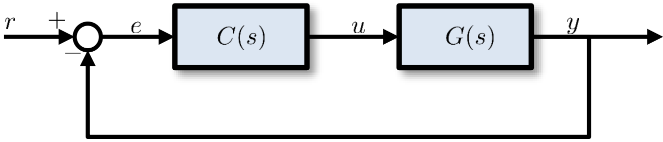
- Looking at the Bode plot, what does it mean to have a high gain at a certain frequency/frequencies?



- Looking at the Bode plot, what does it mean to have a low gain at a certain frequency/frequencies?

Closed Loop Frequency Domain Analysis

- Is it possible for the closed loop system at some frequency to have **infinite gain**? What would that imply?

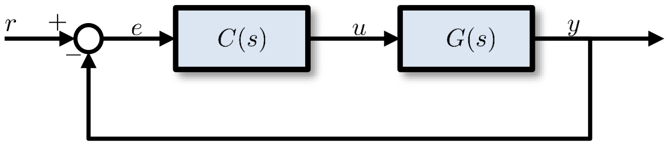


$$G_{cl}(s) = \frac{G(s)C(s)}{1 + G(s)C(s)}$$

Gain Margin

- Assume we have a closed loop system with proportional feedback

$$G_{cl}(s) = \frac{G(s)K}{1+G(s)K}$$

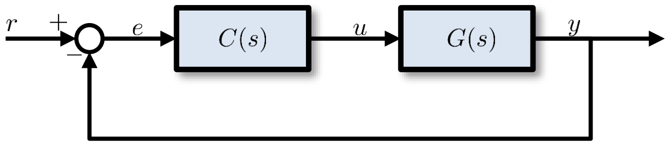


- **Gain Margin:** The amount that the loop gain (K) can be changed, at the frequency at which the phase shift is 180^o, without reducing the return difference to zero

Phase Margin

- Assume we have a closed loop system with proportional feedback

$$G_{cl}(s) = \frac{G(s)K}{1 + G(s)K}$$

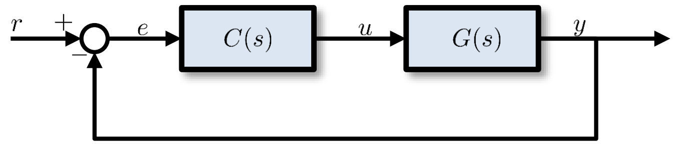


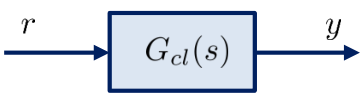
- **Phase Margin:** The amount of phase lag that can be added to the open loop transfer function, at the frequency at which its magnitude is unity, without making the return difference zero

Gain and Phase Margins – Matlab Code

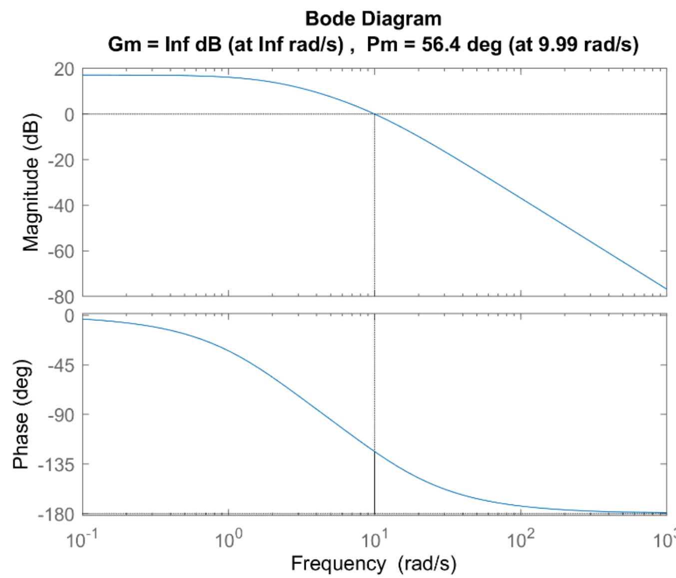
- Assume we have a closed loop system with proportional feedback

$$G_{cl}(s) = \frac{G(s)K}{1 + G(s)K}$$



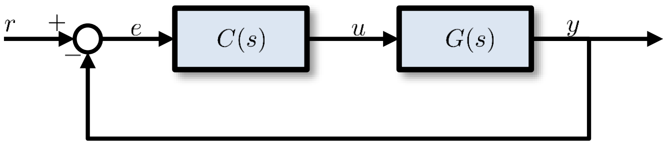


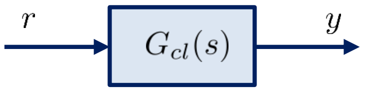

```matlab
%% DC Motor example
J = 0.01;
b = 0.1;
K = 0.01;
R = 1;
L = 0.5;
s = tf('s');
P_motor = K/((J*s+b)*(L*s+R)+K^2);

%% Assume proportional feedback
Cs = 72;                    % Kp

%% Plot the margins for the new system
margin(Cs*P_motor)
print(gcf,'margins.png', '-dpng', '-r300');
```

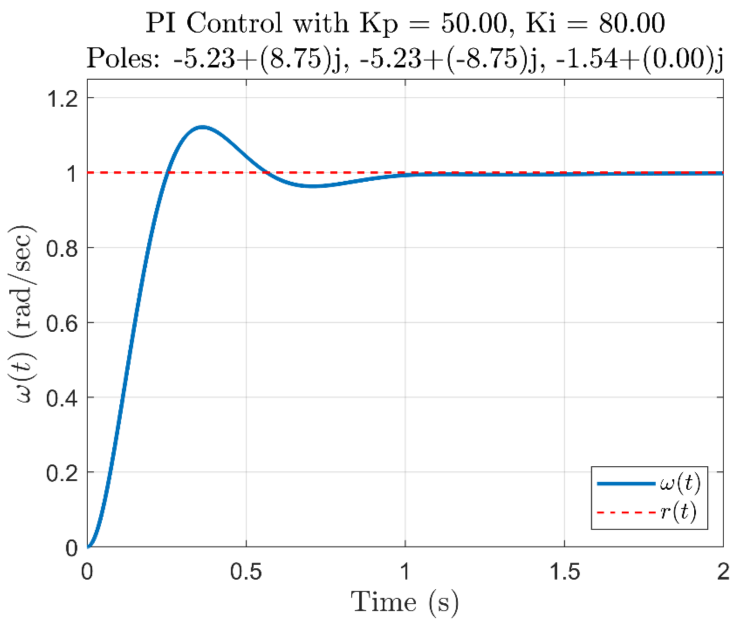


Types of Response: Overshoot, Rise Time

- Assume we have designed a closed loop system to track a certain step reference
- Let's look more closely at the step response

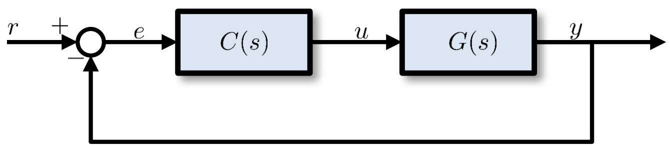


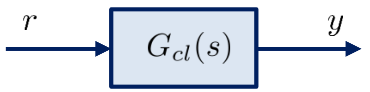




Rise Time and Bandwidth

- To track step references, we need a **fast** rise time
- A step reference is a signal that has high frequency components
- Therefore, we can characterize the frequencies a system can track by analyzing the bode plot





- **Bandwidth:** The first frequency where the gain drops below 70.79% (-3 dB) of its DC value

$$\textbf{bandwidth(Gcl)} = BW = 12.53 \text{ rad/s}$$