EE 419/519: Industrial Control Systems

Lecture 2: Modeling of Dynamic Systems

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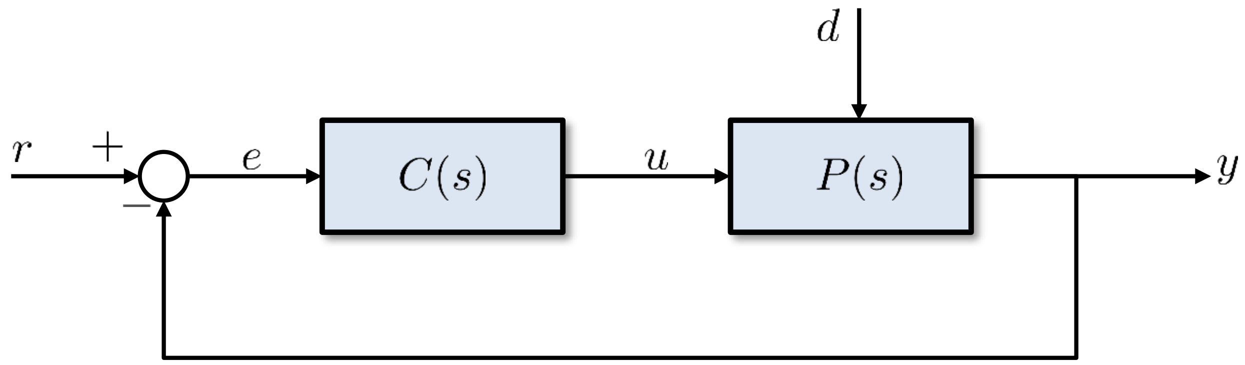
Topic Outline

- **Overview of Feedback Control Systems**
- **Dynamic Modeling Techniques**
- **State Space Methods**
- **Laplace/Frequency Domain Analysis**

Review the Components of Feedback Control Systems

- A feedback control system is composed of a **dynamic system/plant**, **controller**, **sensors/outputs**, **actuators/inputs**, **reference**, and **disturbance**



- **Main idea:** Having an accurate description of how the dynamic system (plant) naturally behaves, we would like to design a controller to modify this behavior in a desired way

- **First step:** we need to have a dynamic model

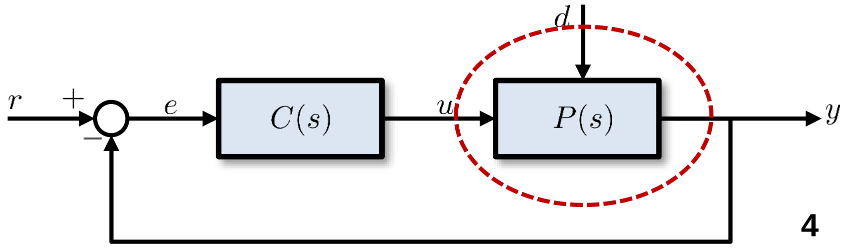
"Make everything as simple as possible, but not simpler." Albert Einstein

Types of Dynamic Models

- The first step is to obtain a **good** mathematical model of the system/plant
- How can we obtain a set of mathematic equations describing the behavior of the system?
- In this class, we will consider two types of mathematical models (commonly used for **linear systems**):

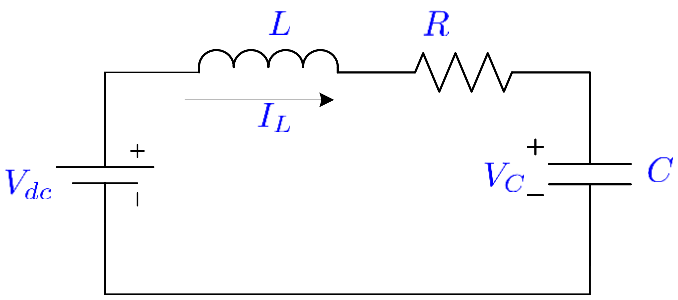
    1. Transfer-function/frequency domain approach

    2. State-space time domain approach



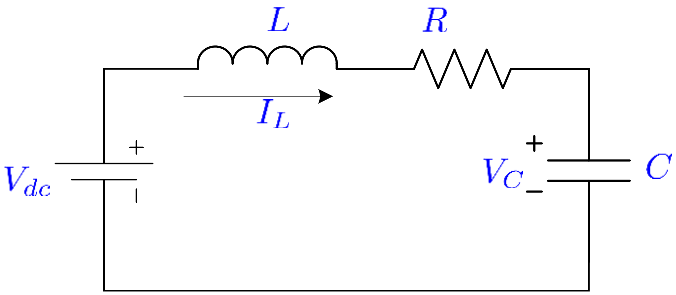
Dynamic Model Example 1 (Time domain)

- How can we obtain a set of mathematic equations describing the behavior of the system?
- Depends on the application, let's consider a few



Dynamic Model Example 1 (Frequency domain)

- How can we obtain a set of mathematic equations describing the behavior of the system?
- Depends on the application, let's consider a few



Dynamic Model Example 2

- How can we obtain a set of mathematic equations describing the behavior of the system?
- Depends on the application, let's consider a few

Dynamic Systems Review

- In this class, we will consider two types of mathematical models (commonly used for **linear systems**):

  1. Transfer-function/frequency domain approach
  2. State-space time domain approach

- The path we will follow is:

State space modeling $\rightarrow$

Laplace domain modeling and controller design $\rightarrow$

State space analysis and controller design

Tentative Topics

- Overview of Feedback Control Systems
- Dynamic Modeling Techniques
- **State Space Methods**
- Laplace/Frequency Domain Analysis

Ordinary Differential Equations

- Many physical systems (plants), can be modeled by an n^{th} order differential equation:

$$F\left(t,\ x,\ \dot{x},\ \ddot{x},\ \cdots,\ x^{(n)}\right) = 0$$

- The **order** of the differential equation is decided by the highest derivative
- The ODE is **time varying** if the equation explicitly depends on time
- Otherwise is known as **time invariant**

LTV and LTI Differential Equations

- Two classes of ordinary differential equations (ODE) that we will use in this class are:

1. **Linear Time Varying (LTV):**

$$a_n(t)x^{(n)} + \cdots + a_2(t)\ddot{x} + a_1(t)\dot{x} + a_0(t)x = 0$$

2. **Linear Time Invariant (LTI):**

$$a_n x^{(n)} + \cdots + a_2\ddot{x} + a_1\dot{x} + a_0 x = 0$$

Solving an ODE

- Consider a second order LTI ODE: $a_2 \frac{d^2x}{dt^2} + a_1 \frac{dx}{dt} + a_0 x = 0$
- If we wanted to solve this equation for $x(t)$, what else is needed?

- While we will look at the general solution of a system of LTI ODE, this will **not** be the main focus of this course
- **We will try to change the behavior of the system without obtaining a solution**

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System of First Order ODE – State Space Form

- Any n^{th} order LTI/LTV ODE can be transformed into a **system** of first order equations (**HW problem**)

- **Example:** convert this second order ODE into a system of first order ODEs
$a_2\frac{d^2x}{dt^2} + a_1\frac{dx}{dt} + a_0 x = 0$

State Space Representation of ODEs

- Any n^{th} order LTI/LTV ODE can be transformed into a **system** of **n** first order equations
- This will be known as the **state space system**

$$
\begin{aligned}
\dot{x}_1 &= \frac{\mathrm{d}x_1}{\mathrm{d}t} = f_1(x_1,\ x_2,\ \ldots,\ x_n, u_1,\ u_2,\ \ldots,\ u_m,\ t) & x_1(0) &= x_{10} \\
\dot{x}_2 &= \frac{\mathrm{d}x_2}{\mathrm{d}t} = f_2(x_1,\ x_2,\ \ldots,\ x_n, u_1,\ u_2,\ \ldots,\ u_m,\ t) & x_2(0) &= x_{20} \\
&\ \vdots \qquad\qquad\qquad \vdots & &\ \vdots \\
\dot{x}_n &= \frac{\mathrm{d}x_n}{\mathrm{d}t} = f_n(x_1,\ x_2,\ \ldots,\ x_n, u_1,\ u_2,\ \ldots,\ u_m,\ t) & x_n(0) &= x_{n0}
\end{aligned}
$$

- It will be typically written in vector form as follows:

$$
\begin{aligned}
\dot{x} &= f(x,\ u,\ t), \qquad \text{where } x \in \mathbb{R}^n,\ u \in \mathbb{R}^m,\ t \in \mathbb{R} \\
x(0) &= x_0
\end{aligned}
$$

Linear Time Invariant (LTI) State Space System

- Any n^{th} order LTI/LTV ODE can be transformed into a **system** of **n** first order equations

$$\dot{x} = f(x,\ u,\ t), \qquad \text{where } x \in \mathbb{R}^n,\ u \in \mathbb{R}^m,\ t \in \mathbb{R}$$
$$x(0) = x_0$$

- If the system of equations is **Linear Time Invariant**, the state space system is as follows:

$$\dot{x} = Ax + Bu, \quad x(0) = x_0 \qquad \text{where } x \in \mathbb{R}^n,\ u \in \mathbb{R}^m,\ A \in \mathbb{R}^{n\times n},\ B \in \mathbb{R}^{n\times m}$$

LTI State Space System with Outputs

- An LTI State Space model with outputs is of the form:

$$\dot{x} = Ax + Bu, \quad x(0) = x_0$$
$$y = Cx + Du$$

- The vector $x \in \mathbb{R}^n$ are the **states**
- The vector $u \in \mathbb{R}^m$ are the **inputs**
- The vector $y \in \mathbb{R}^p$ are the **outputs**

- When $m = 1$ and $p = 1$, this is known as a ________________________ system

- When $m > 1$ and $p > 1$, this is known as a ________________________ system

LTI State Space System with Outputs

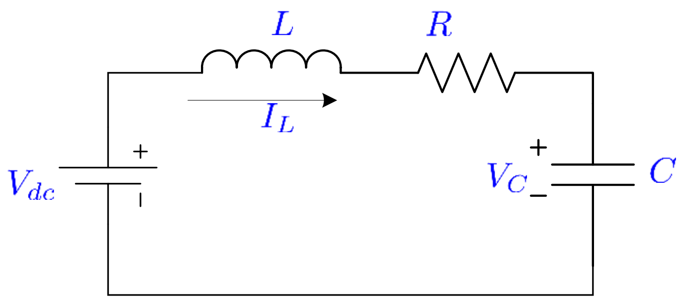
- An LTI State Space model with outputs is of the form:

$$\dot{x} = Ax + Bu, \quad x(0) = x_0$$
$$y = Cx$$

- Let's look at this system in our control diagram:

Dynamic Model Example 1 (Time domain)

- Place an RLC circuit in state space form:



Dynamic Model Example 2

- Place the spring and damper system in state space form

LTI Systems to Laplace/Frequency Domain

- An LTI State Space model with outputs is of the form:

$$\dot{x} = Ax + Bu, \quad x(0) = x_0$$
$$y = Cx$$

- The main motivation for using state space systems is the possibility of using **Linear Algebra** for the analysis

- Before we do this, we will first analyze **SISO** systems in frequency domain

Topic Outline

- Overview of Feedback Control Systems
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- State Space Methods
- **Laplace/Frequency Domain Analysis**

Laplace Transform (Review)

- The **Laplace Transform** $\mathcal{L}[f(t)]$ of $f(t)$ is the function defined as follows:

$$\mathcal{L}[f(t)] = F(s) = \int_0^\infty e^{-st} f(t)dt, \quad \forall s \in \mathbb{C} \text{ for which the integral exists}$$

- Find the Laplace Transform for the following functions
  - $f(t) = e^{at}$
  - $f(t) = \cos(\omega t)$
  - $f(t) = \sin(\omega t)$
  - $f(t) = e^{-at}\cos(\omega t)$
  - $f(t) = e^{-at}\sin(\omega t)$

$f(t)$	$F(s) = \mathcal{L}[f(t)]$	
$f(t) = 1$	$F(s) = \frac{1}{s}$	$s > 0$
$f(t) = e^{at}$	$F(s) = \frac{1}{(s-a)}$	$s > a$
$f(t) = t^n$	$F(s) = \frac{n!}{s^{(n+1)}}$	$s > 0$
$f(t) = \sin(at)$	$F(s) = \frac{a}{s^2 + a^2}$	$s > 0$
$f(t) = \cos(at)$	$F(s) = \frac{s}{s^2 + a^2}$	$s > 0$
$f(t) = \sinh(at)$	$F(s) = \frac{a}{s^2 - a^2}$	$s > \lvert a \rvert$
$f(t) = \cosh(at)$	$F(s) = \frac{s}{s^2 - a^2}$	$s > \lvert a \rvert$
$f(t) = t^n e^{at}$	$F(s) = \frac{n!}{(s-a)^{(n+1)}}$	$s > a$
$f(t) = e^{at}\sin(bt)$	$F(s) = \frac{b}{(s-a)^2 + b^2}$	$s > a$
$f(t) = e^{at}\cos(bt)$	$F(s) = \frac{(s-a)}{(s-a)^2 + b^2}$	$s > a$
$f(t) = e^{at}\sinh(bt)$	$F(s) = \frac{b}{(s-a)^2 - b^2}$	$s - a > \lvert b \rvert$
$f(t) = e^{at}\cosh(bt)$	$F(s) = \frac{(s-a)}{(s-a)^2 - b^2}$	$s - a > \lvert b \rvert$

Inverse Laplace Transform (Review)

- The **inverse Laplace Transform** $\mathcal{L}^{-1}[F(s)]$ of $F(s)$ is defined as follows:

$$f(t) = \mathcal{L}^{-1}[F(s)](t) = \frac{1}{2\pi j}\int_{\gamma - j\infty}^{\gamma + j\infty} F(s)e^{st}ds$$

- We typically use tables for both Laplace and inverse Laplace transforms:

- $F(s) = 10\frac{1}{s+3}$

- $F(s) = 4\frac{4}{s^2+16}$

- $F(s) = \frac{10}{s^2+7s+12}$

$f(t)$	$F(s) = \mathcal{L}[f(t)]$	
$f(t) = 1$	$F(s) = \frac{1}{s}$	$s > 0$
$f(t) = e^{at}$	$F(s) = \frac{1}{(s-a)}$	$s > a$
$f(t) = t^n$	$F(s) = \frac{n!}{s^{(n+1)}}$	$s > 0$
$f(t) = \sin(at)$	$F(s) = \frac{a}{s^2+a^2}$	$s > 0$
$f(t) = \cos(at)$	$F(s) = \frac{s}{s^2+a^2}$	$s > 0$
$f(t) = \sinh(at)$	$F(s) = \frac{a}{s^2-a^2}$	$s > \lvert a \rvert$
$f(t) = \cosh(at)$	$F(s) = \frac{s}{s^2-a^2}$	$s > \lvert a \rvert$
$f(t) = t^n e^{at}$	$F(s) = \frac{n!}{(s-a)^{(n+1)}}$	$s > a$
$f(t) = e^{at}\sin(bt)$	$F(s) = \frac{b}{(s-a)^2+b^2}$	$s > a$
$f(t) = e^{at}\cos(bt)$	$F(s) = \frac{(s-a)}{(s-a)^2+b^2}$	$s > a$
$f(t) = e^{at}\sinh(bt)$	$F(s) = \frac{b}{(s-a)^2-b^2}$	$s - a > \lvert b \rvert$
$f(t) = e^{at}\cosh(bt)$	$F(s) = \frac{(s-a)}{(s-a)^2-b^2}$	$s - a > \lvert b \rvert$

Properties of the Laplace Transform

- The **convolution** $h(t) = f(t) * g(t)$ of two signals $f(t)$ and $g(t)$ is given by:

$$h(t) = \int_0^t f(\tau)g(t-\tau)d\tau$$

- The Laplace Transform of $\mathcal{L}\left[h(t)\right]$ as defined above has an easier representation:

$$\mathcal{L}\left[h(t)\right] = \mathcal{L}\left[f(t) * g(t)\right] = \mathcal{L}\left[f(t)\right]\mathcal{L}\left[g(t)\right] = F(s)G(s)$$

- Convolution in time domain is multiplication in Laplace form

- The Laplace Transform of a **time derivative** is $\mathcal{L}\left[\frac{df(t)}{dt}\right] = sF(s) - f(0)$

- The Laplace Transform of an **integral** is $\mathcal{L}\left[\int_0^t f(\tau)d\tau\right] = \frac{F(s)}{s}$

From State Space to Laplace Transform

- An LTI State Space model is of the form:

$$\dot{x} = Ax + Bu, \quad x(0) = x_0$$
$$y = Cx$$

- Assume $m = p = 1$ (SISO), apply Laplace transform to this system

From State Space to Laplace Transform

- An LTI State Space model with outputs is of the form:

$$\dot{x} = Ax + Bu, \quad x(0) = x_0$$
$$y = Cx$$

- Assume $m = p = 1$ (SISO), apply Laplace transform to this system

- The output of the system in frequency domain is:

$$Y(s) = \underbrace{C(sI - A)^{-1}x_0}_{\text{natural response}} + \underbrace{C(sI - A)^{-1}BU(s)}_{\text{forced response}}$$

- If $x_0 = 0$, we can obtain the input/output transfer function

$$\frac{Y(s)}{U(s)} = C(sI - A)^{-1}B$$

Input/Output Transfer Function

$$\dot{x} = Ax + Bu, \quad x(0) = x_0$$
$$y = Cx$$

- If $x_0 = 0$, we can obtain the input/output transfer function

$$\frac{Y(s)}{U(s)} = C(sI - A)^{-1}B$$

- How can we compute $(sI - A)^{-1}$?

$$(sI - A)^{-1} = \frac{\operatorname{adj}(sI - A)}{\det(sI - A)} = \frac{\operatorname{adj}(sI - A)}{s^n + a_{n-1}s^{n-1} + \cdots + a_1 s + a_0}$$

Input/Output Transfer Function and Solution

$$\dot{x} = Ax + Bu, \quad x(0) = x_0$$
$$y = Cx$$

- If $x_0 = 0$, we can obtain the input/output transfer function

$$\frac{Y(s)}{U(s)} = C(sI - A)^{-1}B = C\frac{\text{adj}\,(sI - A)}{\det\,(sI - A)}B = \frac{N(s)}{D(s)}$$

Proper and Strictly Proper Transfer Functions

- If $x_0 = 0$, we can obtain the input/output transfer function

$$\frac{Y(s)}{U(s)} = C(sI - A)^{-1}B = C\frac{\text{adj}\,(sI - A)}{\det\,(sI - A)}B = \frac{N(s)}{D(s)}$$

- In the most general form, we can write this as follows:

$$\frac{Y(s)}{U(s)} = \frac{N(s)}{D(s)} = \frac{b_{n-1}s^{n-1} + b_{n-2}s^{n-2} + \cdots + b_1 s + b_0}{s^n + a_{n-1}s^{n-1} + \cdots + a_1 s + a_0}$$

- A **proper transfer function** satisfies the following: $\deg\,(N(s)) \leq \deg\,(D(s))$

- A **strictly proper transfer function** satisfies the following: $\deg\,(N(s)) < \deg\,(D(s))$

Poles and Zeros of the System/Transfer Function

- If $x_0 = 0$, we can obtain the input/output transfer function

$$\frac{Y(s)}{U(s)} = C(sI - A)^{-1}B = C\frac{\text{adj}\,(sI - A)}{\det\,(sI - A)}B = \frac{N(s)}{D(s)}$$

- In the most general form, we can write this as follows:

$$\frac{Y(s)}{U(s)} = \frac{N(s)}{D(s)} = \frac{b_{n-1}s^{n-1} + b_{n-2}s^{n-2} + \cdots + b_1 s + b_0}{s^n + a_{n-1}s^{n-1} + \cdots + a_1 s + a_0}$$

- The roots of the denominator, $D(s)$, are called the **poles** of the system

- The roots of the numerator, $N(s)$, are called the **zeros** of the system

Partial Fraction Decomposition (no repeated poles)

- Consider the following **strictly proper** Input/Output transfer function

$$\frac{Y(s)}{U(s)} = \frac{N(s)}{D(s)} = \frac{b_{n-1}s^{n-1} + b_{n-2}s^{n-2} + \cdots + b_1 s + b_0}{s^n + a_{n-1}s^{n-1} + \cdots + a_1 s + a_0} = G(s)$$

- This transfer function can be placed in the following form:

$$\frac{Y(s)}{U(s)} = K\frac{(s-z_1)(s-z_2)\cdots(s-z_m)}{(s-p_1)(s-p_2)\cdots(s-p_n)}, \qquad m < n$$

- This transfer function can be expanded using partial fraction decomposition:

$$\frac{Y(s)}{U(s)} = \frac{a_1}{s-p_1} + \cdots + \frac{a_n}{s-p_n}, \qquad \text{where } a_i = \lim_{s \to p_i}(s-p_i)G(s)$$

Partial Fraction Decomposition (repeated poles)

- This transfer function can be placed in the following form:

$$\frac{Y(s)}{U(s)} = K\frac{(s-z_1)(s-z_2)\cdots(s-z_m)}{(s-p_1)(s-p_2)\cdots(s-p_n)}, \qquad m < n$$

- If a pole is repeated (multiplicity of $k > 1$), its partial fraction is as follows:

Impulse Response of the System

- The poles and partial fraction decomposition allows us to obtain the impulse response of the system
- If $u(t) = \delta(t)$, what happens to the output $y(t)$?

Impulse Response of the System Solution

- If $u(t) = \delta(t) \quad \Rightarrow \quad U(s) = 1$ (Laplace transform!)

- The output then becomes:

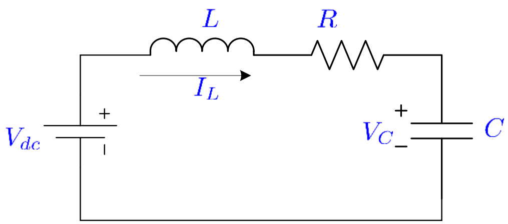
$$\frac{Y(s)}{U(s)} = Y(s) = K\frac{(s-z_1)(s-z_2)\cdots(s-z_m)}{(s-p_1)(s-p_2)\cdots(s-p_n)} = \frac{a_1}{s-p_1} + \cdots + \frac{a_n}{s-p_n}$$

- The output in time domain, $y(t)$, can be obtained by the inverse Laplace transform (no repeated poles):

Example 1 – RLC circuit Input/Output Transfer Function

- The state space equations for an RLC circuit are as follows:

$$\begin{pmatrix} \dot{i}_L \\ \dot{v}_C \end{pmatrix} = \begin{pmatrix} \frac{-R}{L} & \frac{-1}{L} \\ \frac{1}{C} & 0 \end{pmatrix} \begin{pmatrix} i_L \\ v_C \end{pmatrix} + \begin{pmatrix} \frac{1}{L} \\ 0 \end{pmatrix} V_{\mathrm{dc}} \qquad y = i_L = \begin{pmatrix} 1 & 0 \end{pmatrix} \begin{pmatrix} i_L \\ v_C \end{pmatrix}$$



- Find the input/output transfer function, assuming $x(0) = 0$

Example 1 – RLC circuit Input/Output Transfer Function

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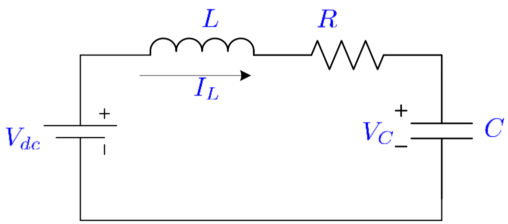
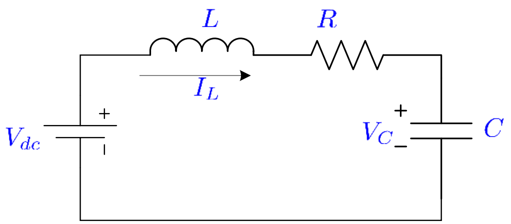


- Find the input/output transfer function, assuming $x(0) = 0$

Example 1 – RLC circuit Impulse Response of the System

- The state space equations for an RLC circuit are as follows:

$$\begin{pmatrix} \dot{i}_L \\ \dot{v}_C \end{pmatrix} = \begin{pmatrix} \frac{-R}{L} & \frac{-1}{L} \\ \frac{1}{C} & 0 \end{pmatrix} \begin{pmatrix} i_L \\ v_C \end{pmatrix} + \begin{pmatrix} \frac{1}{L} \\ 0 \end{pmatrix} V_{\mathrm{dc}} \qquad y = i_L = \begin{pmatrix} 1 & 0 \end{pmatrix} \begin{pmatrix} i_L \\ v_C \end{pmatrix}$$



- Find the impulse respone of the system, assuming $R = 1\ \Omega,\ L = 1/15\ \mathrm{H},\ C = 1/10\ F$

$$G(s) = \frac{I(s)}{V_{\mathrm{dc}}(s)} = \frac{\frac{1}{L}s}{s^2 + \frac{R}{L}s + \frac{1}{LC}}$$

Steady State Solution

- Let's again consider the transfer function for a SISO system

$$\frac{Y(s)}{U(s)} = C(sI - A)^{-1}B \quad \Leftrightarrow \quad Y(s) = \underbrace{C(sI - A)^{-1}B}_{G(s)} \; U(s)$$

- If the input is now **constant**, $u(t) = u_c$, what will be the steady state value of the output?

- Essentially, we are interested in $\lim_{t \to \infty} y(t)$

- **Laplace Final Value Theorem**

$$\lim_{t \to \infty} f(t) = \lim_{s \to 0} sF(s)$$

Steady State Solution (cont'd)

- Let's again consider the transfer function for a SISO system

$$\frac{Y(s)}{U(s)} = C(sI - A)^{-1}B \quad \Leftrightarrow \quad Y(s) = \underbrace{C(sI - A)^{-1}B}_{G(s)} \; U(s)$$

- If the input is now **constant**, $u(t) = u_c$, what will be the steady state value of the output?
- Essentially, we are interested in $\lim_{t \to \infty} y(t)$ $\quad u(t) = \begin{cases} 0 & t < 0 \\ u_c & t \geq 0 \end{cases} \xrightarrow{\mathcal{L}} U(s) = \frac{u_c}{s}$

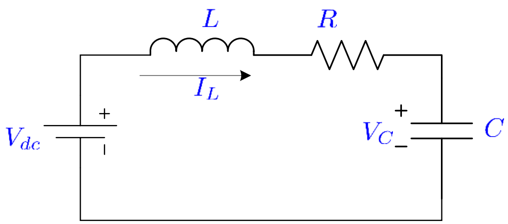
$$\Rightarrow \lim_{t \to \infty} y(t) = \lim_{s \to 0} sY(S) = \lim_{s \to 0} sG(s)U(s) = G(0)u_c = -CA^{-1}B \, u_c$$

- The term $-CA^{-1}B$ is known as the **dc gain** of the system

DC Gain of RLC Circuit

- The state space equations for an RLC circuit are as follows:

$$\begin{pmatrix} \dot{i}_L \\ \dot{v}_C \end{pmatrix} = \begin{pmatrix} \frac{-R}{L} & \frac{-1}{L} \\ \frac{1}{C} & 0 \end{pmatrix} \begin{pmatrix} i_L \\ v_C \end{pmatrix} + \begin{pmatrix} \frac{1}{L} \\ 0 \end{pmatrix} V_{\text{dc}} \qquad y = i_L = \begin{pmatrix} 1 & 0 \end{pmatrix} \begin{pmatrix} i_L \\ v_C \end{pmatrix}$$



- Assuming $R = 1\ \Omega,\ L = 1/15\ \text{H},\ C = 1/10\ F$, if we applied $V_{\text{dc}} = 10$, *what is the steady state output?*

$$G(s) = \frac{I(s)}{V_{\text{dc}}(s)} = \frac{\frac{1}{L}s}{s^2 + \frac{R}{L}s + \frac{1}{LC}}$$

Next: *Controller analysis in frequency domain*