EE 459/559: Control and Applications of Power Electronics

Topic 3: (State Space) Control of Power Electronics

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Outline

- **Why do we need control?**
- State feedback control for stabilization $u = Kx$ $K =$ feedback matrix of gains.
- Integral control for reference tracking $e_I = \int (v^{ref} - v_c)\, dt \longleftrightarrow \dot{e}_I = v^{ref} - v_c$ (one more state)
- Inner/outer loop control for power electronics

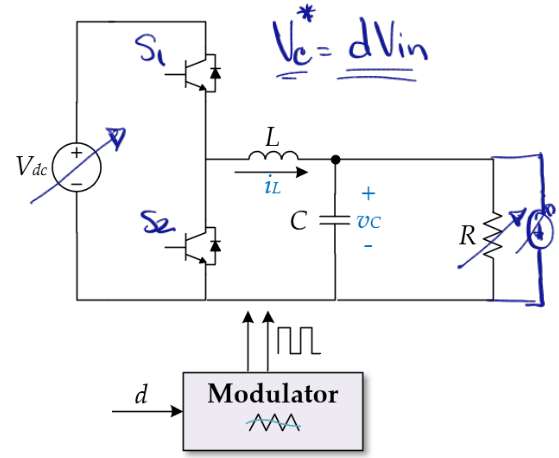
(affect the time scales → fast states → slower states Singular perturbation)

State Space Average Model of a Buck Converter

- The state space average model of a buck converter

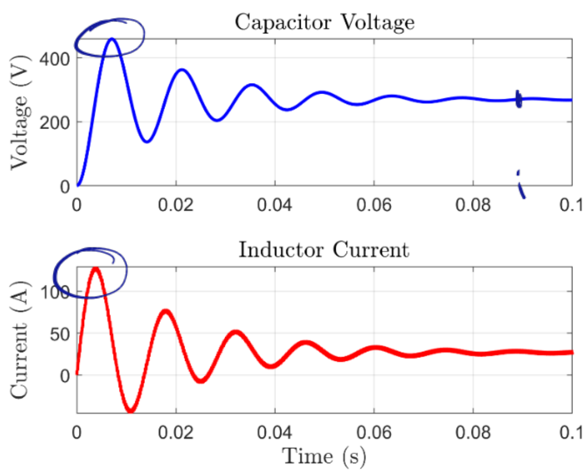
$$\frac{d}{dt}\begin{pmatrix} i_L \\ v_C \end{pmatrix} = \underbrace{\begin{pmatrix} 0 & -\frac{1}{L} \\ \frac{1}{C} & -\frac{1}{RC} \end{pmatrix}}_{A \text{ (constant)}} \begin{pmatrix} i_L \\ v_C \end{pmatrix} + \begin{pmatrix} \frac{1}{L} \\ 0 \end{pmatrix} \underbrace{dV_{\mathrm{dc}}}_{u}$$

$u = dV_{dc}$



- Assume that we need to control the output side voltage to 270 V we can simply let $d = 0.5$ (if $V_{in} = 540$)
- The **open loop** transients are not desired:
  - What is the settling time? (time it takes for V_c to reach 270) ~90 ms
  - Are there any oscillations? Voltage/current overshoot?
- What can we do?
  - Feedback / closed loop control
  - $d = K(x, V_c^{ref}) = K(i_L, V_c, V_c^{ref})$
- Duty cycle will be decided automatically
  - improve our response
  - robust to disturbances (change in V_{in})

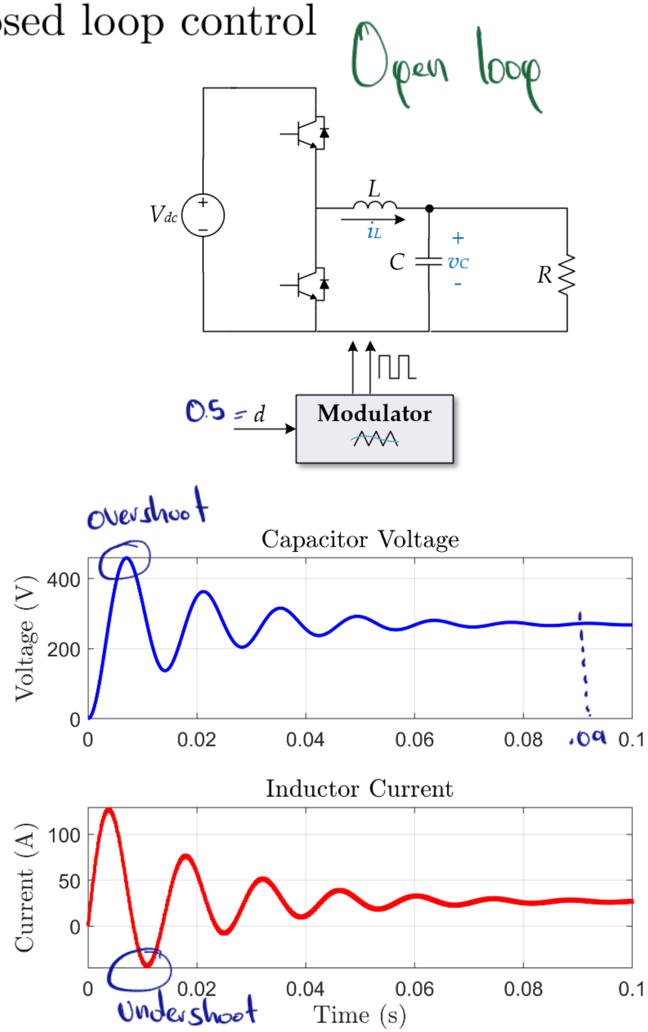
L	5 mH
C	1 mF
V_{in}	540 V
d	0.5
R	10Ω

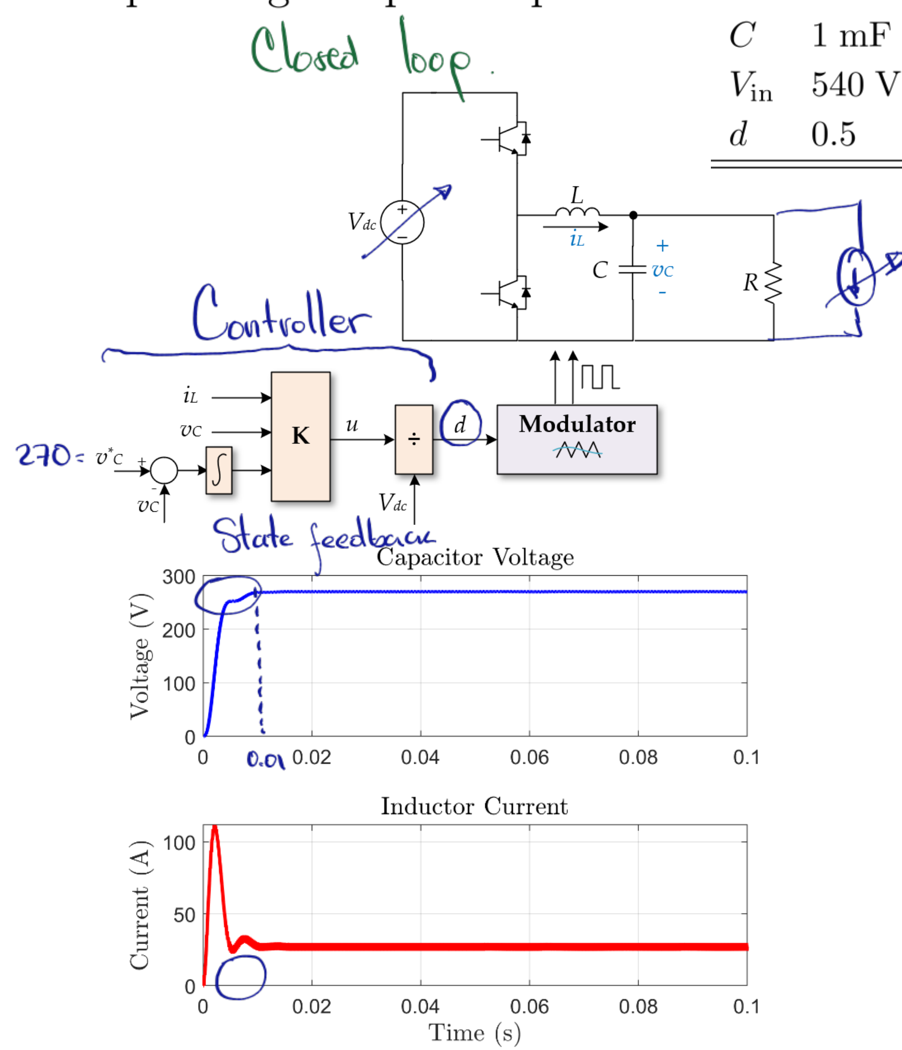


Open Loop vs Closed Loop Performance

- Compare the performance of the converter when operating as open loop and closed loop control

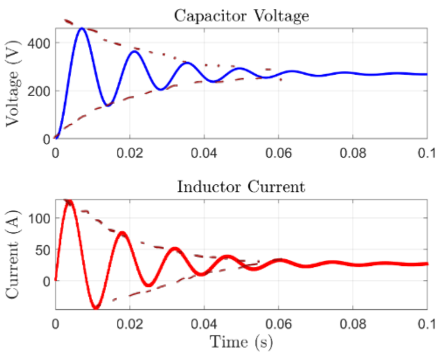
L	5 mH
C	1 mF
V_{in}	540 V
d	0.5





Analysis of Closed Loop Behavior

- What caused the behavior of the buck converter to improve in closed loop mode?



Open loop Response

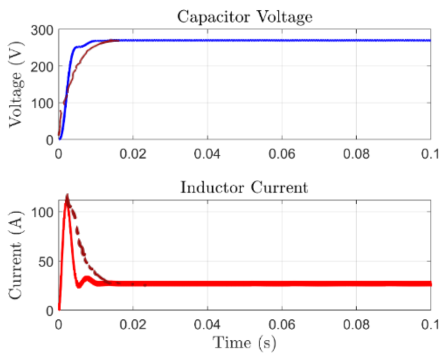
- $\dot{x} = Ax + Bu$ $\quad u = d \cdot V_{in}$ $\quad x = \begin{pmatrix} i_L \\ V_C \end{pmatrix}$
- $d = 0.5 \rightarrow \bar{u} = 0.5 \times V_{in}$ (Constant value)
- $\dot{x} = Ax + B\bar{u}$ ($\bar{u}$ is constant)
- $x(t)$ is decided by the eigenvalues of A

λ_1^o, λ_2^o = eigenvalues of A

$\lambda_{1,2}^o = \alpha \pm j\omega$

$\alpha^o < 0$

eg $\alpha^o = -2$



Closed loop Response

- $\dot{x} = Ax + Bu$ but now $u = Kx$ e.g. $u = \overbrace{[k_1 \ \ k_2]}^{\text{feedback gain } K} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = k_1 x_1 + k_2 x_2$

$\Rightarrow \dot{x} = Ax + BKx \rightarrow \dot{x} = \underbrace{(A + BK)}\, x \quad A_{CL} \triangleq A + BK$

$\Rightarrow \dot{x} = A_{CL}\, x$

- $x(t)$ is now decided by eigenvalues of $A_{CL} = A + BK$

$\lambda_1^c, \lambda_2^c = eig(A_{CL}) \quad \lambda_{1,2}^c = \alpha^c \pm j\omega^c, \ \alpha^c < 0$

- For a linear state space system, $\dot{x} = Ax + Bu$, how can we design the input u (e.g. duty cycle)?

eg. $\alpha^c = -20$

$\alpha^c < \alpha^o$

Outline

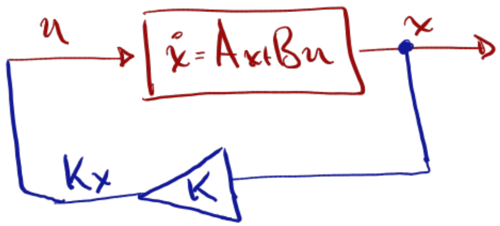
- Why do we need control?
- **State feedback control for stabilization** , $u = Kx$
- Integral control for reference tracking
- Inner/outer loop control for power electronics

State Feedback Problem Formulation

- Consider a LTI state space representation of a dynamical system:

$$\dot{x} = Ax + Bu$$

- **Problem statement:** Design a controller of the form $u = Kx$ (state feedback) such that the closed loop system is asymptotically stable



Openloop: • $u = 0$, or $u = c$ (0 or constant value)

• $\Rightarrow \dot{x} = Ax + \cancel{Bu} = Ax \rightarrow \dot{x} = Ax$

- Asymp. Stable: if $Re\{\lambda_i\} < 0 \ \forall i$
- (Marginally) Stable: if $\exists i \ Re\{\lambda_i\} = 0$ and all other eigenvalues $Re\{\lambda_j\} < 0$, $j \neq i$
- Unstable* if $\exists i$ s.t. $Re\{\lambda_i\} > 0$

Closed loop: $u = Kx$ (linear function of states)

• $\dot{x} = Ax + B\underbrace{Kx}_{u} = (A+BK)x \rightarrow \dot{x} = \overbrace{\boxed{(A+BK)}}^{A_{cl}}x$

• Stability of closed loop system is decided by eigenvalues of $A_{cl} = A + BK$

• How do we design K?

Controllability Definition

- Consider a LTI state space representation of a dynamical system:

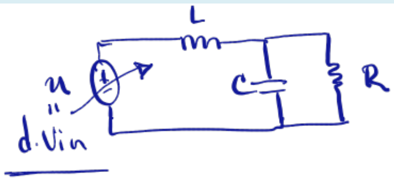
$$\text{LTI}\ \left(\dot{x} = Ax + Bu\right)$$

? if $u \in \mathbb{R}^1$ (1 input)

- Assume the input is given by a state feedback controller $u = \underline{K}x = [k_1\ k_2 \cdots k_n]\begin{bmatrix}x_1\\x_2\\\vdots\\x_n\end{bmatrix} = k_1x_1 + \dots + k_nx_n$

Controllability: An LTI system is controllable if for every $x(t) = x^*$, there exists an input $u(t),\ t \in [0,\ t]$, such that the system goes from $x(0) = 0$ to $x(t) = x^*$, $t < \infty$

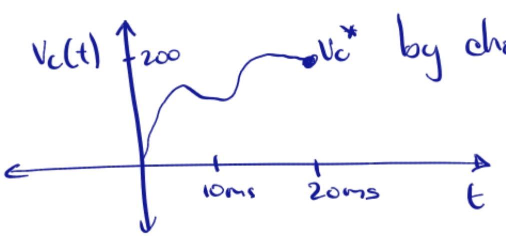
- Assume we have the following circuit



(Average model for buck Converter)

- 2^{nd} order, $x(t) = \begin{pmatrix} i_L \\ V_c \end{pmatrix}$
- e.g. $\underline{x^* = \begin{pmatrix} 5 \\ 200 \end{pmatrix} = \begin{pmatrix} i_L^* \\ V_c^* \end{pmatrix}}$ $t = 20ms$, can we find an input $u(t)$ $t \in [0, 20ms]$ s.t.

$x(0) = 0 \longrightarrow x(t) = x^*$ $t = 20ms$.



by changing $u(t)$

- There is a formula for finding $u(t)$
- We can find $u(t)$ if $\underline{W^c(t) \triangleq e^{At}BB^Te^{(At)^T}}$ is invertible for all t

Cayley Hamilton Theorem.

Controllability Matrix and Rank

- Consider a LTI state space representation of a dynamical system:

$$\dot{x} = Ax + Bu$$

$x \in \mathbb{R}^n \Rightarrow n$ states $\Rightarrow n^{th}$ order system.

$A \in \mathbb{R}^{n \times n}$

- Assume the input is given by a state feedback controller $u = Kx$

Controllability: An LTI system is controllable if and only if the rank of the matrix $\mathcal{M}_c = \begin{bmatrix} B & AB & \cdots & A^{n-1}B \end{bmatrix}$ is n

$W_c = \underbrace{e^{At} B B^T e^{(At)^T}}$ is invertible

# of linearly independent columns.

- If a system is controllable, then we can design an input $u = Kx$ such that we can choose the eigenvalues of $(A + BK)$

• If system ($\dot{x} = Ax + Bu$) is controllable, find K matrix s.t. $u = Kx \Rightarrow \dot{x} = (A+BK)x$
all eigenvalues of $(A+BK)$ are anywhere

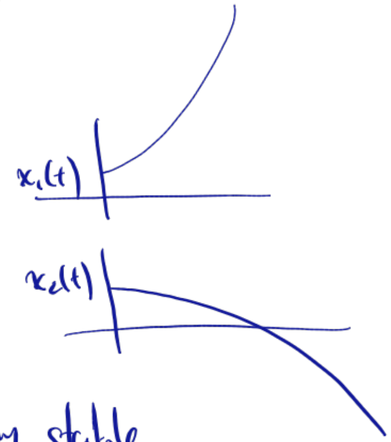
• What if $\text{rank}(M_c) < n$? $\dot{x} = Ax + Bu$ is not controllable, we cannot modify all eigenvalues of $(A+BK)$

Example 1: Controllability and State Feedback

$n=2$ (Second order system)

- Consider the following LTI system:

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \begin{pmatrix} 2 & 1 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} 2 \\ 1 \end{pmatrix} \underline{u}$$



- Is the open loop $(u=0)$ system asymptotically stable, stable, or unstable?
  - $u=0$ $\quad \dot{x} = Ax \quad \text{eig}(A) = \{2, -1\}$ Unstable

- Open loop system is unstable, ⇒ design a state feedback controller s.t. closed loop system is asymptotically stable.

(m= 1) one input. $\Rightarrow u = Kx = \begin{bmatrix} \overset{?}{k_1} & \overset{?}{k_2} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = k_1 x_1 + k_2 x_2$

$\Rightarrow \dot{x} = Ax + BKx = \underbrace{(A+BK)}\, x$ be asymp. stable.

⇒ we want all $\text{eig}(A+BK)$ to have neg. real part.

- Step 1: check the controllability of this system.

Example 1: Controllability and State Feedback

- Consider the following LTI system:

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \begin{pmatrix} 2 & 1 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} 2 \\ 1 \end{pmatrix} \underset{Kx}{\underset{\shortparallel}{u}} \Rightarrow \dot{x} = \underbrace{(A+BK)}x$$

- Design a state feedback matrix K such that the eig$(A + BK) = \{\underbrace{\overset{*}{-10}, \overset{*}{-20}}\}$

Step 1: is the system controllable? need to check rank of controllability matrix

$M_c = [B \;\; AB \cdots A^{n-1}B]$ • In our case, $\underline{n = \#\text{states}} = 2 \Rightarrow M_c = [B \;\; AB]$

$B = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$ $A \cdot B = \begin{bmatrix} 2 & 1 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 5 \\ -1 \end{bmatrix}$ $M_c = \begin{bmatrix} 2 & 5 \\ 1 & -1 \end{bmatrix}$ rank(M_c) = # of lin. independent columns = 2

- a set of vectors (columns), $v_1, v_2, \dots, v_n$, are linearly independent if $a_1v_1 + a_2v_2 + \dots + a_nv_n = 0$ is satisfied only by $a_1 = a_2 = \dots = a_n = 0$

- $a_1\begin{bmatrix} 2 \\ 1 \end{bmatrix} + a_2\begin{bmatrix} 5 \\ -1 \end{bmatrix} = 0 \Rightarrow \begin{bmatrix} 2 & 5 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ $a_1 = a_2 = 0$

$\Rightarrow$ rank $(M_c) = 2 \Rightarrow$ System is controllable!

Step 2: If the system is controllable, find K s.t. $u = Kx$ and $\lambda(A+BK)$ are as chosen

Matlab $K = -\text{place}(A, B, [-10 \;\; -20])$

$K = [\underset{k_1}{-37.71} \;\; \underset{k_2}{44.43}] \longrightarrow u = \underline{Kx} = [k_1 \;\; k_2]\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = -37.71x_1 + 44.43x_2$

```
%% Define the system (A, B)
A = [2 0;0 -1];
B = [2 1]';

%% Find the matrix k using Matlab's place
eig_des = [-10 -20];                    % Desired eigenvalues

%(careful u = -Kx using place)
Kpl = -1*place(A, B, eig_des);          % -1 to make it u = Kx;

% Check the closed loop
Acl = A+B*Kpl
eig(Acl)
```

Pole placement

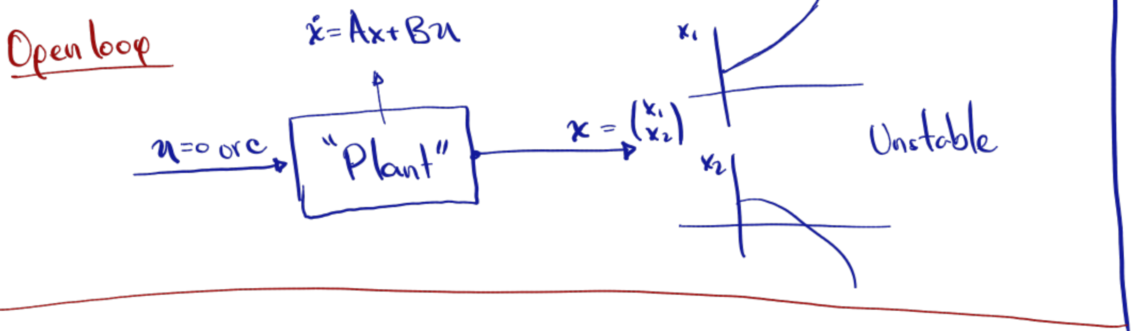
Example 1: Controllability and State Feedback

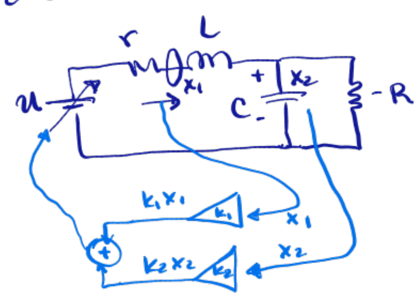
- Consider the following LTI system:

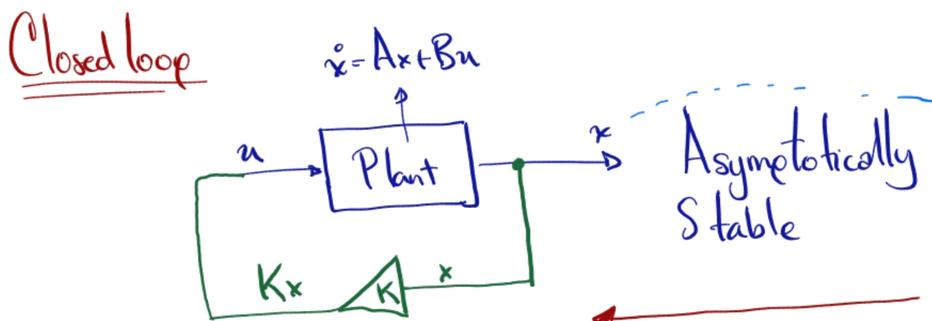
$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \begin{pmatrix} 2 & 1 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} 2 \\ 1 \end{pmatrix} u$$

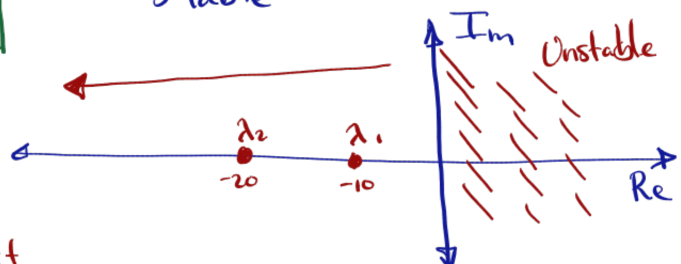
"Suppose" $\dot{x} = Ax + Bu$ are the equations for a circuit.

- The overall controller is then as follows:





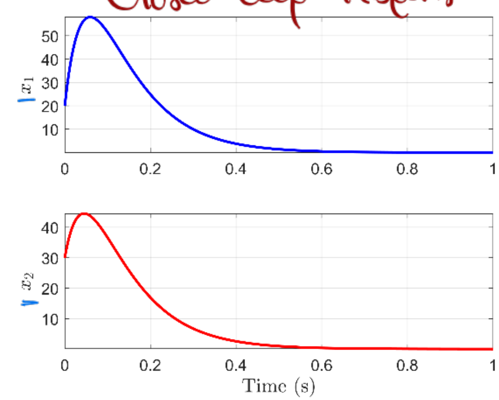




$\lambda(A+BK) = \{\lambda_1, \lambda_2\}$

$x(t) = \sum_{i=1}^{2} v_i e^{\lambda_i t} x_i(0)$

Closed Loop Response



$\lambda_1 = -10 \quad \lambda_2 = -20$

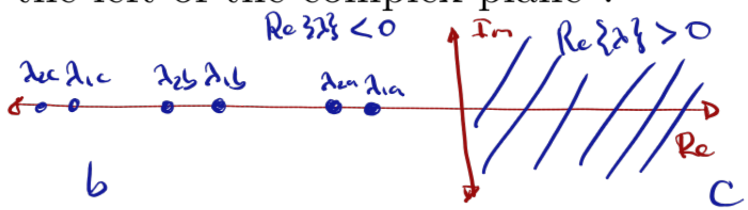
Example 1: Controllability and State Feedback

- Consider the following LTI system:

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \begin{pmatrix} 2 & 1 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} 2 \\ 1 \end{pmatrix} u \longrightarrow \underline{u = Kx}$$

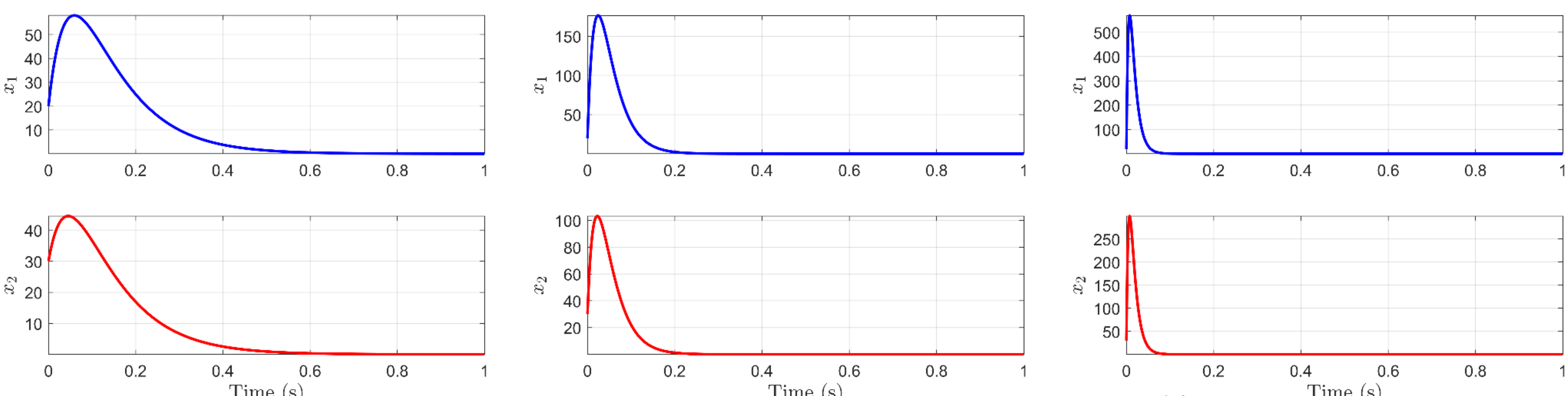
- Designed a state feedback matrix K such that the $\mathrm{eig}(A + BK) = \{-10,\ -20\}$

- What if we placed the poles further on the left of the complex plane ?
  $\mathrm{eig}(A + BK) = \{-30,\ -50\}$
  $\mathrm{eig}(A + BK) = \{-80,\ -200\}$



a $\qquad$ b $\qquad$ c

$\mathrm{eig}(A + BK) = \{-10,\ -20\}$ $\qquad$ $\mathrm{eig}(A + BK) = \{-30,\ -50\}$ $\qquad$ $\mathrm{eig}(A + BK) = \{-80,\ -200\}$



Other techniques for obtaining K: Linear Quadratic Regulator (LQR), Linear Matrix Inequalities (LMI)

Outline

- Why do we need control?
- State feedback control for stabilization
- **Integral control for reference tracking**
- Inner/outer loop control for power electronics

State Feedback and Reference Tracking

- If we consider an LTI model of the form:

$$\dot{x} = Ax + Bu$$
$$\underbrace{y = Cx}$$

y = outputs or measurements.

and design a state feedback controller $\underline{u = Kx}$ such that

$$\dot{x} = (\underbrace{A + BK})x$$

→ $\lambda(A+BK)$ have neg. real part.

is asymptotically stable, this implies:

$$x(t),\ y(t) \to 0 \ \text{ as } \ t \to \infty$$

- What if we don't want $\underline{x(t) \to 0}$? or
- What if we would like the output $y = Cx$ to track a reference?

- $y \longrightarrow r^*$ as $t \to \infty$
- We can define error $e(t) = y(t) - r^*$

$\Rightarrow e(t) \longrightarrow 0$ as $t \to \infty$

$\Rightarrow y(t) \to r^*$

Problem Definition

- Consider an LTI model of the form:

$$\dot{x} = Ax + Bu$$
$$y = Cx \qquad y \in \mathbb{R}^1$$

- **Problem definition:** Given a reference signal $r(t)$, design a controller $u(t)$ such that $y(t) \to r(t)$ and make sure $\lambda(A+BK)$ have neg. real part, i.e. system asymp. stable.
- We will assume the reference signal is a step function or constant value

i.e. $r(t) = r^* H(t) \qquad H(t) = \begin{cases} 1 & \text{for } t \geq 0 \\ 0 & t < 0 \end{cases}$

- Define the error as $e_c(t) = r^* - y(t) = r^* - Cx$
- Create a new state (integral of error) $\longrightarrow$ $e_I(t) = \int_0^t r - y \, dt$
- Now we have $\Rightarrow \dot{e}_I(t) = r - Cx = r - Cx$ (1 more state)

$$\left\{ \begin{array}{l} \dot{x} = Ax + Bu \\ \dot{e} = r^* - Cx \end{array} \right. \longrightarrow \begin{pmatrix} \dot{x} \\ \dot{e} \end{pmatrix} = \begin{pmatrix} A & 0 \\ -C & 0 \end{pmatrix} \begin{pmatrix} x \\ e \end{pmatrix} + \begin{pmatrix} B \\ 0 \end{pmatrix} u + \begin{pmatrix} 0 \\ 1 \end{pmatrix} r^*$$

Increasing the order of system by 1

Augmented State Space Model, and now we can use State feedback

Integral Control

- Consider an LTI model of the form:

$$\dot{x} = Ax + Bu$$
$$y = Cx$$

- **Problem definition:** Given a reference signal $r(t)$, design a controller $u(t)$ such that $y(t) \to r(t)$, **define the error as** $\mathbf{e(t) = r(t) - y(t)}$

- Integral control begins by integrating the error, i.e. $\int e(t)dt = \int r(t) - y(t)dt$

  $r(t) = r^* H(t)$

- **How does this affect the state space model?**

- $\dot{e}_I = r - Cx$ adds one more state $\rightarrow$ $\underbrace{\begin{pmatrix} \dot{x} \\ \dot{e} \end{pmatrix}}_{\dot{x}_a} = \underbrace{\begin{pmatrix} A & 0 \\ -C & 0 \end{pmatrix}}_{A_a} \underbrace{\begin{pmatrix} x \\ e \end{pmatrix}}_{x_a} + \underbrace{\begin{pmatrix} B \\ 0 \end{pmatrix}}_{B_a} u + \underbrace{\begin{pmatrix} 0 \\ 1 \end{pmatrix}}_{P_a} r^*$

- Design a state feedback controller for this augmented system $\dot{x}_a = A_a x_a + B_a u + P_a r^*$

  $x_a \in \mathbb{R}^{\tilde{n}}$

Step 1: Check controllability of Aug. system: (A_a, B_a)

$M_c = [B_a \;\; A_a B_a \;\; \cdots \;\; A_a^{\tilde{n}-1} B_a]$ $\quad$ rank$(M_c) = \tilde{n}$ (controllable!)

Step 2: Obtain K_a s.t. $\lambda(A_a + B_a K_a)$ have neg. real part. (Asym. Stable)

$u = K_a x_a = [K \;\; k_I] \begin{bmatrix} x \\ e_I \end{bmatrix} = Kx + k_I e_I$ $\quad$ (State feedback + Integral control)

Integrator + State Feedback

- The new system becomes:

$$\underbrace{\begin{pmatrix} \dot{x} \\ \dot{e}_I \end{pmatrix}}_{\dot{x}_a} = \underbrace{\begin{pmatrix} A & 0 \\ -C & 0 \end{pmatrix}}_{A_a} \underbrace{\begin{pmatrix} x \\ e_I \end{pmatrix}}_{x_a} + \underbrace{\begin{pmatrix} B \\ 0 \end{pmatrix}}_{B_a} u + \underbrace{\begin{pmatrix} 0 \\ 1 \end{pmatrix}}_{P} r$$

- Design an input $u = K_a x_a$ such that $(A_a + B_a K_a)$ has eigenvalues with negative real part (asymptotically stable)

- Augmented System: $\dot{x}_a = A_a x_a + B_a u + P r$ $\quad u = K_a x_a$

$\Rightarrow \dot{x}_a = A_a x_a + B_a \underbrace{K_a x_a}_{u} + P r = \underline{(A_a + B_a K_a)} x_a + P r$

↳ Eigenvalues have neg. real part.

- In our original system

$u = K_a x_a = [K \quad K_I] \begin{bmatrix} x \\ e_I \end{bmatrix} \longrightarrow \begin{pmatrix} \dot{x} \\ \dot{e}_I \end{pmatrix} = \begin{pmatrix} A+BK & BK_I \\ -C & 0 \end{pmatrix} \begin{pmatrix} x \\ e_I \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \end{pmatrix} r$

$= Kx + K_I e_I$

- Now the augmented system is Asymptotically Stable, but will $y(t) = Cx(t) \longrightarrow r^*$?

Integrator + State Feedback: Steady State

- With a feedback matrix $K_a = \begin{pmatrix} K_x & K_I \end{pmatrix}$, the closed loop system becomes:

$e_I = \int r - y(t)$

$$\begin{pmatrix} \dot{x} \\ \dot{e}_I \end{pmatrix} = \underbrace{\begin{pmatrix} A + BK_x & BK_I \\ -C & 0 \end{pmatrix}}_{A_{cl}} \begin{pmatrix} x \\ e_I \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \end{pmatrix} r$$

- Can we show that $y(t) \to r(t)$? or in steady state $y^* = r^* = Cx^*$

(Assuming $r(t) = r^* H(t)$)

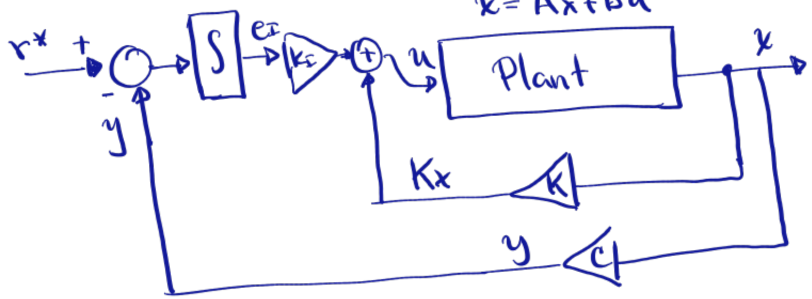
- find the steady state or equilibrium points of the system $r(t) = r^* \neq 0$

$$\Rightarrow \begin{pmatrix} 0 \\ 0 \end{pmatrix} = \begin{pmatrix} A+BK & BK_I \\ -C & 0 \end{pmatrix} \begin{pmatrix} x^* \\ e_I^* \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \end{pmatrix} r$$

- The second row/equation: $0 = -Cx^* + r^* \Rightarrow \boxed{Cx^* = y^* = r^*}$ $\Rightarrow y(t) \to r^*$ assuming aug. system is asymp. stable.

or $e^* = 0$

$\dot{x} = Ax + Bu$



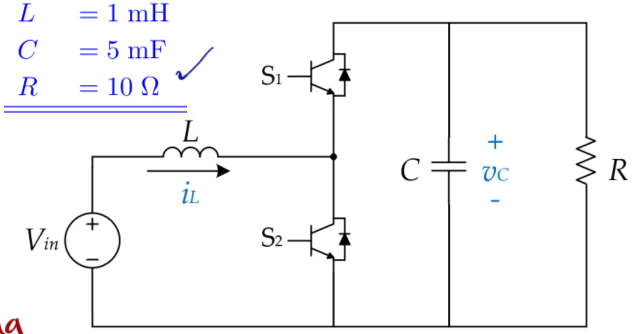
Boost Converter Example: Voltage Tracking

- From topic 2, the state space average model of a boost converter is as follows: $u=d$ $\quad \dot{x} = A(u)x + BV_{in}$

$$\begin{pmatrix} \dot{i}_L \\ \dot{v}_C \end{pmatrix} = \begin{pmatrix} 0 & \frac{-(1-d)}{L} \\ \frac{1-d}{C} & \frac{-1}{RC} \end{pmatrix} \begin{pmatrix} i_L \\ v_C \end{pmatrix} + \begin{pmatrix} \frac{1}{L} \\ 0 \end{pmatrix} V_{\text{in}}$$

* Cannot use for $u = Kx$

V_{in}	$= 100$ V
L	$= 1$ mH
C	$= 5$ mF
R	$= 10\ \Omega$



- Our controllable input is d ($u=d$). Therefore, we cannot directly use state feedback techniques. What can we do? Linearize ⇔ obtaining a linear approx. around a certain operating point

$V_{in}^* = 100V$ e.g. $D^* = 0.5$

- The linearized model is as follows:

$$\begin{pmatrix} \dot{\tilde{i}}_L \\ \dot{\tilde{v}}_C \end{pmatrix} = \begin{pmatrix} 0 & \frac{-(1-D^*)}{L} \\ \frac{1-D^*}{C} & \frac{-1}{RC} \end{pmatrix} \begin{pmatrix} \tilde{i}_L \\ \tilde{v}_C \end{pmatrix} + \begin{pmatrix} \frac{V_C^*}{L} \\ \frac{-I_L^*}{C} \end{pmatrix} \tilde{d} \qquad \Rightarrow \quad \dot{\tilde{x}} = A\tilde{x} + B_d\tilde{d}$$

- What is $V_C^* = ?$ and $I_L^* = ?$ $\longrightarrow$ Equilibrium points $\Rightarrow$ $V_C^* = \frac{1}{1-D^*} V_{in}^* = \frac{1}{0.5}(100) = 200V$

$I_L^* = \frac{1}{R(1-D^*)^2} V_{in} = \frac{1}{10(0.5)^2} \cdot 100V$

- Goal: Design input or $\tilde{d} = K\tilde{x} + K_I e_I$ s.t. - closed loop system is asymptotically stable (locally)

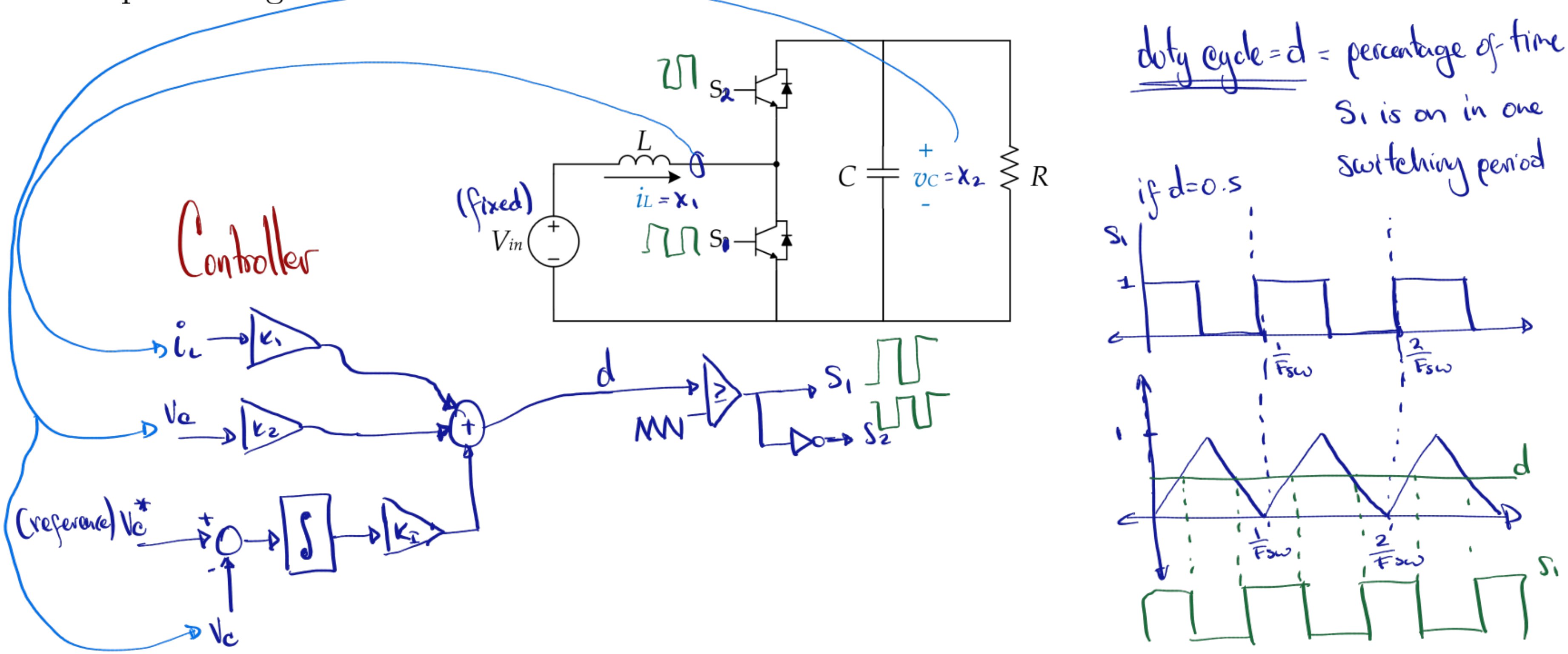
- $V_c \longrightarrow r^* = V_c^*$

Boost Converter Example: Voltage Tracking Diagram

- The linearized model is as follows:

$$\begin{pmatrix} \dot{\tilde{i}}_L \\ \dot{\tilde{v}}_C \end{pmatrix} = \begin{pmatrix} 0 & \frac{-(1-D^*)}{L} \\ \frac{1-D^*}{C} & \frac{-1}{RC} \end{pmatrix} \begin{pmatrix} \tilde{i}_L \\ \tilde{v}_C \end{pmatrix} + \begin{pmatrix} \frac{V_C^*}{L} \\ \frac{-I_L^*}{C} \end{pmatrix} \tilde{d} \qquad \Rightarrow \quad \dot{\tilde{x}} = A\tilde{x} + B_d\tilde{d}$$

- Use the linearized model to design an integral+state feedback controller to set the output voltage to a reference



Boost Converter Example: Voltage Tracking Design

- The linearized model is as follows:

$$\begin{pmatrix} \dot{\tilde{i}}_L \\ \dot{\tilde{v}}_C \end{pmatrix} = \begin{pmatrix} 0 & \frac{-(1-D^*)}{L} \\ \frac{1-D^*}{C} & \frac{-1}{RC} \end{pmatrix} \begin{pmatrix} \tilde{i}_L \\ \tilde{v}_C \end{pmatrix} + \begin{pmatrix} \frac{V_C^*}{L} \\ \frac{-I_L^*}{C} \end{pmatrix} \tilde{d} \qquad \Rightarrow \quad \dot{\tilde{x}} = A\tilde{x} + B_d\tilde{d}$$

- Use the linearized model to design an integral+state feedback controller to set the output voltage to a reference

- Goal: 1) (Locally) Asymp. Stable
2) $V_c \to V_c^*$ if $e = V_c^* - V_c \Rightarrow e \to 0$

- $u = \tilde{d} = K\tilde{x} + k_I e_I$ $\qquad e_I \triangleq \int V_c^* - V_c \, dt.$ or $\dot{e}_I = \tilde{V}_c^* - \tilde{V}_c$

If we augment $x_a = \begin{pmatrix} \tilde{x} \\ e_I \end{pmatrix} \Rightarrow u = \tilde{d} = K_a x_a$

1) Obtain augmented state space model

$$\underbrace{\begin{bmatrix} \dot{\tilde{i}}_L \\ \dot{\tilde{V}}_c \\ \dot{e}_I \end{bmatrix}}_{\dot{x}_a} = \underbrace{\begin{bmatrix} 0 & \frac{-(1-D^*)}{L} & 0 \\ \frac{1-D^*}{C} & \frac{-1}{RC} & 0 \\ 0 & -1 & 0 \end{bmatrix}}_{A_a} \underbrace{\begin{bmatrix} \tilde{i}_L \\ \tilde{V}_c \\ e_I \end{bmatrix}}_{x_a} + \underbrace{\begin{bmatrix} \frac{V_c^*}{L} \\ \frac{-I_L^*}{C} \\ 0 \end{bmatrix}}_{B_a} \underbrace{\tilde{d}}_{u} + \underbrace{\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}}_{P} V_c^r$$

$u = \tilde{d} = K_a x_a$

$\Downarrow$

$\dot{x}_a = (A_a + B_a K_a)x + P v^r$

2) Use (A_a, B_a) to place the eigenvalues of $\lambda(A_a + B_a K_a)$ by finding $K_a = [K \;\; k_I]$

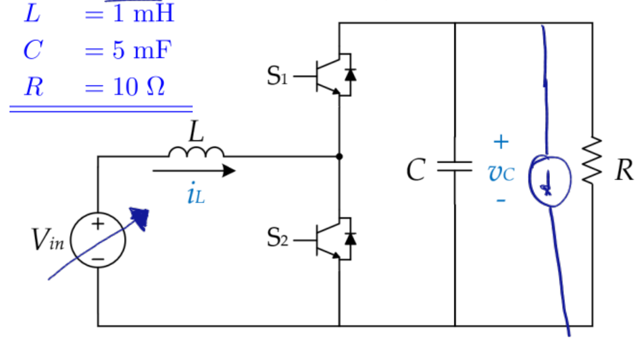
Boost Converter Example: Voltage Tracking Results

- The linearized model is as follows:

$$\begin{pmatrix} \dot{\tilde{i}}_L \\ \dot{\tilde{v}}_C \\ \dot{e}_V \end{pmatrix} = \begin{pmatrix} 0 & \frac{-(1-D^*)}{L} & \\ \frac{1-D^*}{C} & \frac{-1}{RC} & \\ & & \end{pmatrix} \begin{pmatrix} \tilde{i}_L \\ \tilde{v}_C \\ e_V \end{pmatrix} + \begin{pmatrix} \frac{V_C^*}{L} \\ \frac{-I_L^*}{C} \\ 0 \end{pmatrix} \tilde{d}$$

V_{in}	$= 100$ V
L	$= 1$ mH
C	$= 5$ mF
R	$= 10\ \Omega$

$V_o = \frac{1}{1-d} V_{in}$



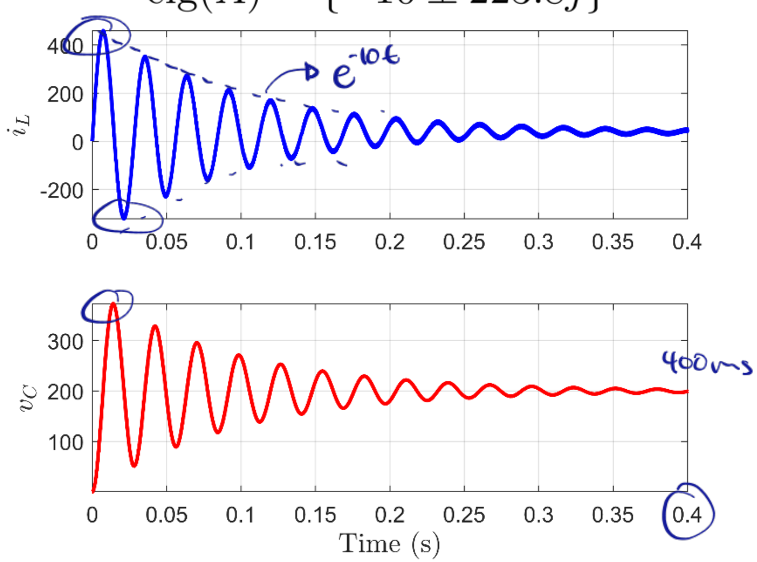
- Place the poles of the augmented sytem at

$$\text{eig}(A_a + B_a K_a) = \{-200 + 342.13j,\ \ -200 - 342.13j,\ \ -200\}$$

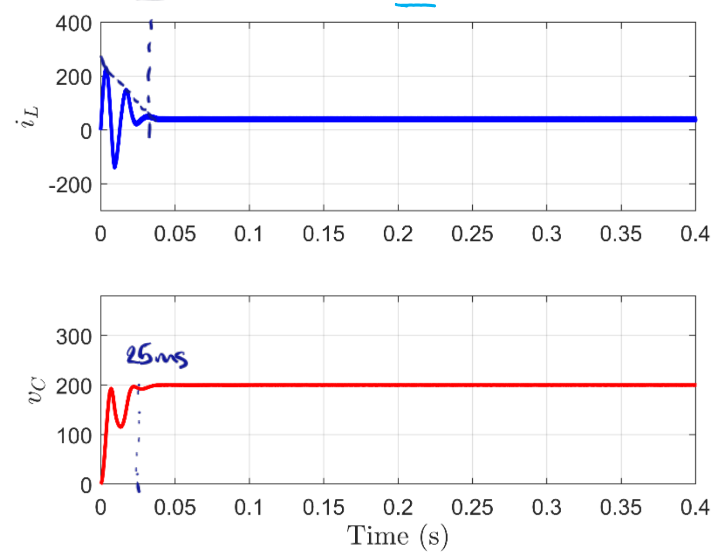
$d = Kx + K_I e_I$

Open loop, $\boxed{d = 0.5}$

$$\text{eig}(A) = \{-10 \pm 223.8j\}$$



Closed loop, $V_r^* = 200$ V



Outline

- Why do we need control?
- State feedback control for stabilization
- Integral control for reference tracking

and Topic 2

Midterm

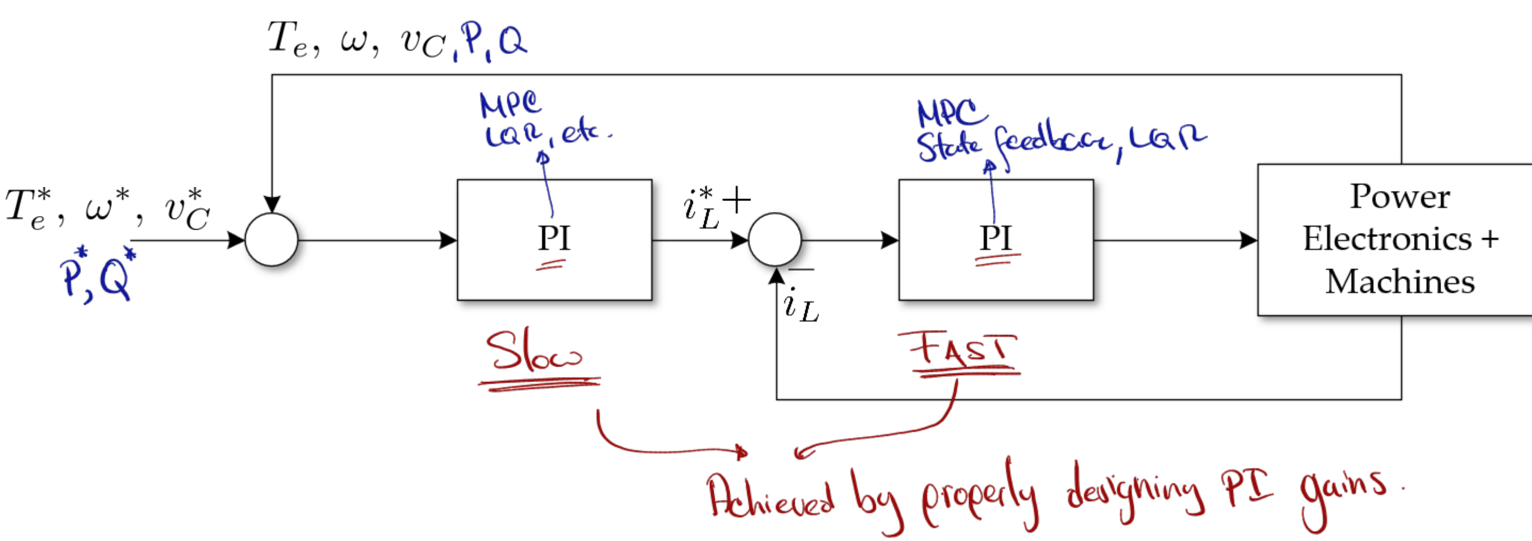
- **Inner/outer loop control for power electronics** (Not on Midterm)

Very common!

Power Electronics + Machines + Control

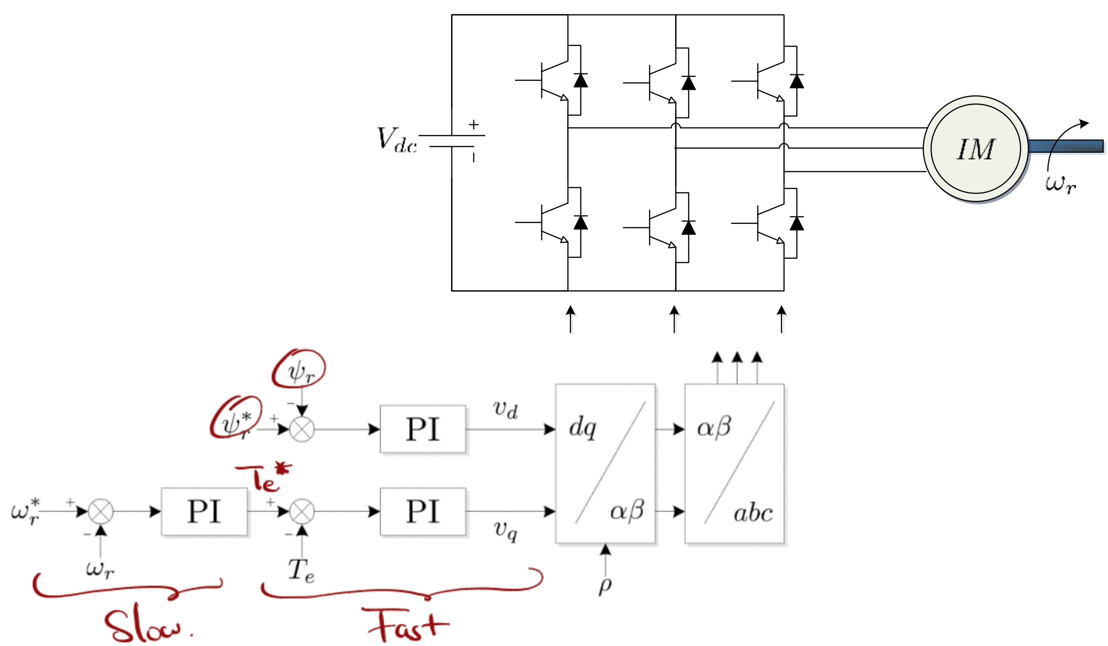
- Control techniques are able to command the power electronics and machines to regulate necessary variables such as:
  - Voltage, current
  - Active and reactive power
  - Torque and speed
  - …

- **In many applications,** the control structure to accomplish this is generally as follows



Motivation from Physical Systems

- One of the reasons this control structure has appeared is the ***time scales*** that exists in these systems:
  - **Current has a faster decay ratio (small inductance)** $\rightarrow V_L = L\frac{di}{dt} \Rightarrow \uparrow \frac{di}{dt} = \frac{V_L}{L \downarrow}$
  - **Voltage/speed is slower (larger capacitance/inertia)**
- Some examples can be seen in electric machines and dc/dc converters



https://www.tesla.com/models?redirect=no

- **The inner loop quickly regulates the current while the outer loop controls the speed/torque**

Assumptions for Inner/Outer Loop Control

- Control designs for these systems generally falls under singular perturbation techniques (Actual System)

input disturbance

$$\begin{pmatrix} x \\ z \end{pmatrix} \qquad \dot{x} = f(x,\ z,\ u,\ d) \qquad x = \text{Slow states (outer loop)}$$

$$\epsilon\dot{z} = g(x,\ z,\ u,\ d) \qquad z = \text{Fast states (Inner loop)}$$

where $0 < \epsilon \ll 1$ → Singular perturbation variable.

as $\epsilon \to 0 \to z$ (fast states) become instantaneous.
if $\epsilon = 0 \to 0 = g(x, z, u, d)$

- **Assumption 1:** The slow modes are relatively **constant** when seen through the fast dynamics

(When designing fast inner loop)

Constant

$$\epsilon\dot{z} = g(\bar{x},\ z,\ u,\ d) = \tilde{g}(z,\ u,\ d)$$

- **Assumption 2:** The fast modes are **instantaneous** when analyzing the slow states:

(for Outer loop design)

$$\dot{x} = f(x,\ z,\ u,\ d)$$

$$0 = g(x,\ z,\ u,\ d) \longrightarrow z = h(x,u,d)$$

$$\dot{x} = f(x, h(x,u,d), u, d) = \tilde{f}(x,u,d)$$

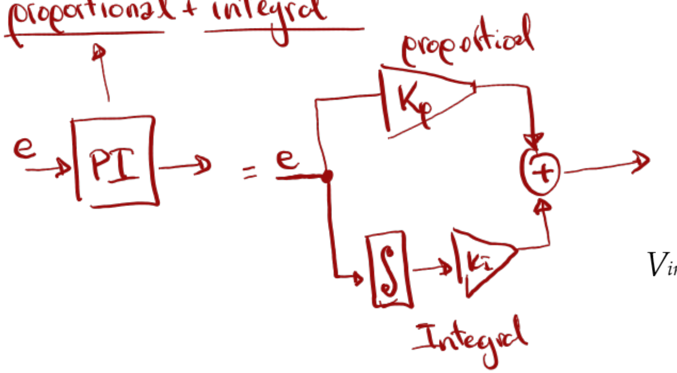
2 references → "Nonlinear Systems" H. Khalil
→ work P. Kokotovic

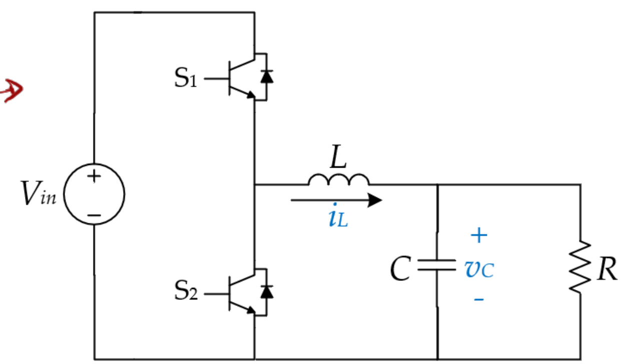
Example: Buck Converter

- The buck converter average state space model is:

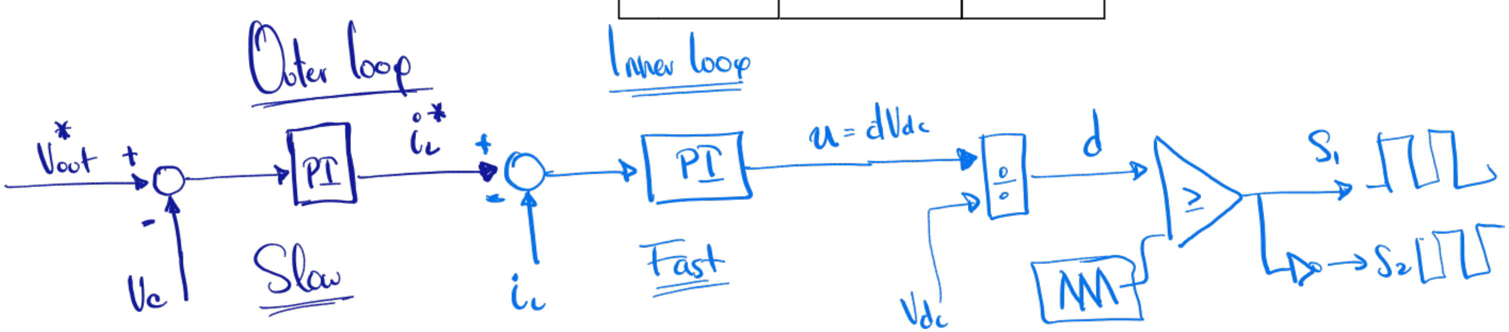
$$\frac{d}{dt}\begin{pmatrix} i_L \\ v_C \end{pmatrix} = \underbrace{\begin{pmatrix} 0 & -\frac{1}{L} \\ \frac{1}{C} & -\frac{1}{RC} \end{pmatrix}}_{A} \begin{pmatrix} i_L \\ v_C \end{pmatrix} + \underbrace{\begin{pmatrix} \frac{1}{L} \\ 0 \end{pmatrix}}_{B} \boxed{\underbrace{dV_{\mathrm{dc}}}_{u}} = Ax + Bu \quad \text{(linear System)}$$

- Design an inner/outer loop control to track the output voltage





2 integrators → The closed loop system will have 2 more states → 4th order system.

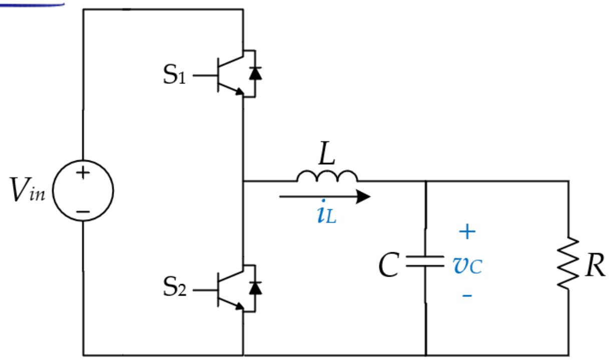


Example: Buck Converter - Simplification

Approximation

- The buck converter average state space model is:

$$\frac{d}{dt}\begin{pmatrix} i_L \\ v_C \end{pmatrix} = \begin{pmatrix} 0 & -\frac{1}{L} \\ \frac{1}{C} & -\frac{1}{RC} \end{pmatrix}\begin{pmatrix} i_L \\ v_C \end{pmatrix} + \begin{pmatrix} \frac{1}{L} \\ 0 \end{pmatrix} \underbrace{dV_{dc}}_{u}$$



- Design an inner/outer loop control to track the output voltage
- Apply the singular perturbation assumptions:

Slow state(s) $\dot{v}_c = -\frac{1}{Rc} v_c + \frac{1}{c} i_L$

fast state(s) $\dot{i}_L = -\frac{1}{L} v_c + \frac{1}{L} u$

- Assumption 1: Slow states are relatively constant when seen through fast state dynamics.

$\longrightarrow$ $\dot{i}_L \simeq -\frac{1}{L}\bar{v}_c + \frac{1}{L} u$ (first order system)

$\bar{v}_c$ → constant value

Assumption 2: The (closed loop) fast states are instantaneous when seen through slow state dyn.

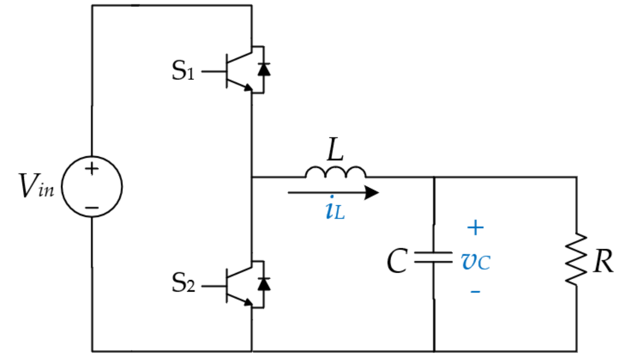
$\dot{v}_c = \frac{-1}{Rc} v_c + \frac{1}{c} i_L^*$, $\quad i_L = i_L^*$ (instantaneous)

(first order system)

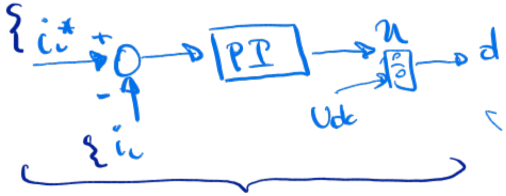
Example: Buck Converter – Inner Loop

- The buck converter average state space model is:

$$\frac{d}{dt}\begin{pmatrix} i_L \\ v_C \end{pmatrix} = \begin{pmatrix} 0 & -\frac{1}{L} \\ \frac{1}{C} & -\frac{1}{RC} \end{pmatrix}\begin{pmatrix} i_L \\ v_C \end{pmatrix} + \begin{pmatrix} \frac{1}{L} \\ 0 \end{pmatrix}\boxed{\underbrace{dV_{\text{dc}}}_{u}}$$



- Design an inner/outer loop control to track the output voltage
- Apply the singular perturbation assumptions: (1) Slow states are constants.



- Inner/fast dynamics are simplified to

2nd order system $\begin{cases} \dot{i}_L \approx \frac{1}{L}u - \frac{1}{L}\bar{V}_c \\ \dot{e}_I = i_L^* - i_L \end{cases}$ → constant value

$z = \begin{pmatrix} i_L \\ e_I \end{pmatrix}$ $\qquad u = K^I z$

- $\overbrace{\begin{pmatrix} \dot{i}_L \\ \dot{e}_I \end{pmatrix}}^{\dot{z}} = \underbrace{\begin{pmatrix} 0 & 0 \\ -1 & 0 \end{pmatrix}}_{A^I}\begin{pmatrix} i_L \\ e_I \end{pmatrix} + \underbrace{\begin{pmatrix} \frac{1}{L} \\ 0 \end{pmatrix}}_{B^I} u + \begin{pmatrix} -\frac{1}{L} \\ 0 \end{pmatrix}\bar{V}_c \Rightarrow \dot{z} = \underbrace{A^I} z + \underbrace{B^I} u + P^I \bar{V}_c = (A^I + B^I K^I) z + P^I \bar{V}_c$

$\Rightarrow \dot{z} = \underbrace{(A^I + B^I K^I)} z + P^I \bar{V}_c$

$u = \underbrace{K_p^i (i_L^* - i_L) + K_I^i e_I}_{PI} = \underbrace{\begin{bmatrix} -K_p & K_I \end{bmatrix}\begin{bmatrix} i_L \\ e_I \end{bmatrix}}_{K^I z} + K_p^i i_L^*$

- $K^I = -\text{place}(A^I, B^I, [\lambda_1^I \ \lambda_2^I])$

$K_p = -K^I(1)$

$K_I = K^I(2)$

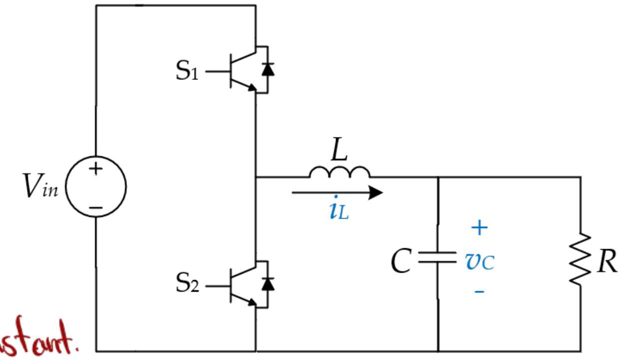
* desired location *

- FAST

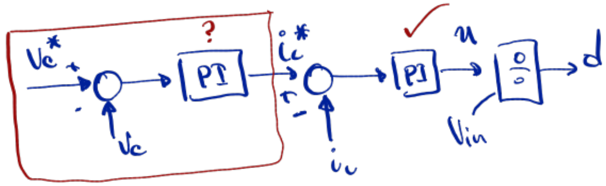
Example: Buck Converter – Outer Loop

- The buck converter average state space model is:

$$\frac{d}{dt}\begin{pmatrix} i_L \\ v_C \end{pmatrix} = \begin{pmatrix} 0 & -\frac{1}{L} \\ \frac{1}{C} & -\frac{1}{RC} \end{pmatrix}\begin{pmatrix} i_L \\ v_C \end{pmatrix} + \begin{pmatrix} \frac{1}{L} \\ 0 \end{pmatrix}\underbrace{dV_{\mathrm{dc}}}_{u}$$



- Design an inner/outer loop control to track the output voltage
- Apply the singular perturbation assumptions: (2) Fast states are instant. $i_L = i_L^*$



• Slow dynamics $\dot{V}_C = \frac{1}{C} i_L - \frac{1}{RC} V_C \quad (i_L = i_L^*)$

$$\begin{cases} \dot{V}_C \approx -\frac{1}{RC} V_C + \frac{1}{C} i_L^* \\ \dot{e}_I^o = V_C^* - V_C \end{cases}$$

input $= i_L^* = K^o x$, $x = \begin{pmatrix} V_C \\ e_I^o \end{pmatrix}$

$$i_L^* = K^o x = \begin{bmatrix} k_1^o & k_2^o \end{bmatrix}\begin{bmatrix} V_C \\ e_I \end{bmatrix}$$

$$\dot{x} = A^o x + B^o i_L^* + P^o V_C^*, \quad i_L^* = K^o x \rightarrow \dot{x} = (A^o + B^o K^o) x + P V_C^*$$

$$K^o = -\mathrm{place}(A^o, B^o, [\lambda_1^o \ \lambda_2^o])$$

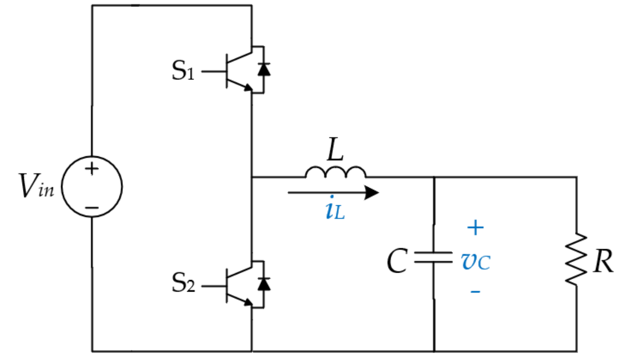
$$\underbrace{\begin{pmatrix} \dot{V}_C \\ \dot{e}_I^o \end{pmatrix}}_{\dot{x}} = \underbrace{\begin{pmatrix} -\frac{1}{RC} & 0 \\ -1 & 0 \end{pmatrix}}_{A^o}\underbrace{\begin{pmatrix} V_C \\ e_I^o \end{pmatrix}}_{x} + \underbrace{\begin{pmatrix} \frac{1}{C} \\ 0 \end{pmatrix}}_{B^o} i_L^* + \begin{pmatrix} 0 \\ 1 \end{pmatrix} V_C^*$$

$$K_I^o = K^o(2) \qquad K_P^o = -K^o(1)$$

Example: Buck Converter – Combined Controller

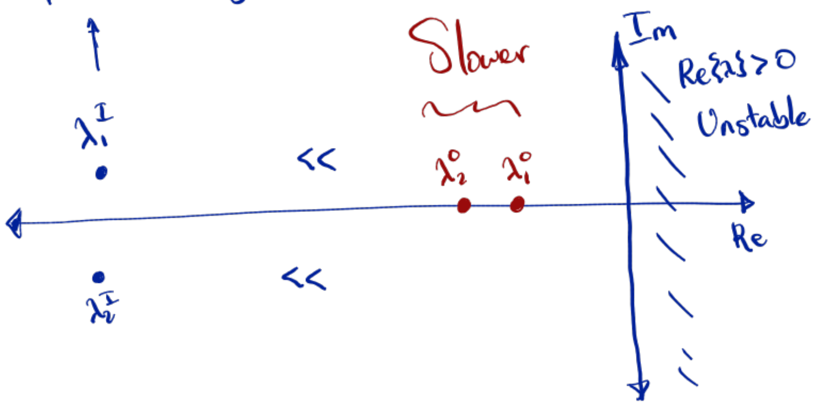
- The buck converter average state space model is:

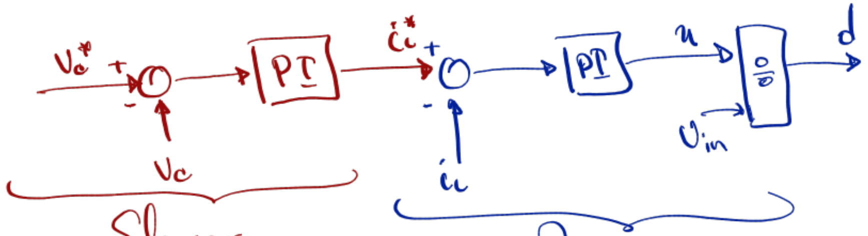
$$\frac{d}{dt}\begin{pmatrix} i_L \\ v_C \end{pmatrix} = \begin{pmatrix} 0 & -\frac{1}{L} \\ \frac{1}{C} & -\frac{1}{RC} \end{pmatrix}\begin{pmatrix} i_L \\ v_C \end{pmatrix} + \begin{pmatrix} \frac{1}{L} \\ 0 \end{pmatrix} \underbrace{dV_{\mathrm{dc}}}_{u}$$



- Design an inner/outer loop control to track the output voltage
- Apply the singular perturbation assumptions:

Inner/Fast loop eigenvalues.





$x = \begin{pmatrix} v_c \\ e_I^o \end{pmatrix}$

$\dot{x} = (A^o + B^o K^o)x + P^o v_c^*$

$\mathrm{eig}(A^o + B^o K^o) = \{\lambda_1^o, \lambda_2^o\}$

$z = \begin{pmatrix} i_L \\ e_I^I \end{pmatrix}$

$\dot{z} = (A^I + B^I K^I)z + P^I i_L^*$

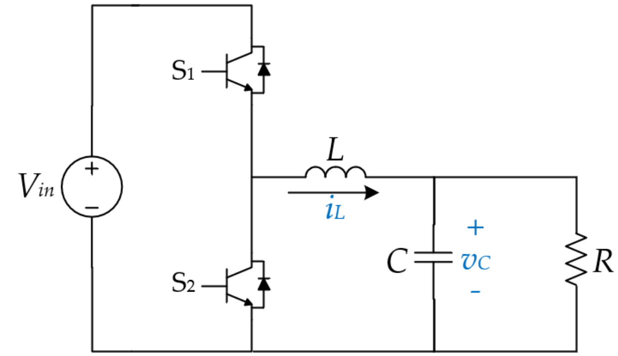
$\mathrm{eig}(A^I + B^I K^I) = \{\lambda_1^I, \lambda_2^I\}$.

- The location of these eigenvalues are obtained from 2 assumptions. or approx models
- May not be the actual location if we use actual model (next slide)

Example: Buck Converter – Actual Model

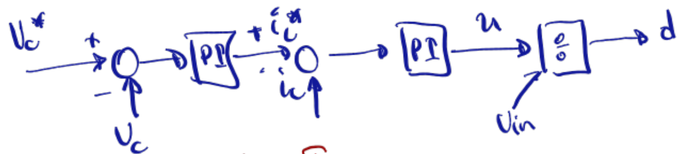
- The buck converter average state space model is:

$$\frac{d}{dt}\begin{pmatrix} i_L \\ v_C \end{pmatrix} = \begin{pmatrix} 0 & -\frac{1}{L} \\ \frac{1}{C} & -\frac{1}{RC} \end{pmatrix}\begin{pmatrix} i_L \\ v_C \end{pmatrix} + \begin{pmatrix} \frac{1}{L} \\ 0 \end{pmatrix}\underbrace{dV_{\text{dc}}}_{u}$$



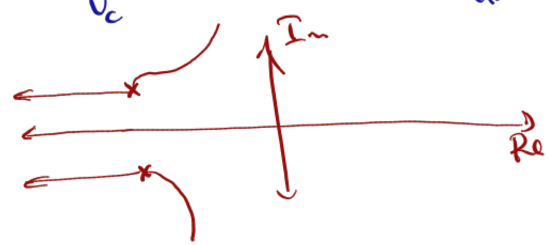
- Design an inner/outer loop control to track the output voltage
- ~~Apply the singular perturbation assumptions~~:

- We have 2 integrators (2 PI) ⇒ 2 more states.



- Write the additional states.

- Inner loop $\quad e_I = \int i_L^* - i_L \, dt \longrightarrow \dot{e}_I = i_L^* - i_L$



- Outer Loop $\quad e_o = \int V_c^* - V_c \, dt \longrightarrow \dot{e}_o = V_c^* - V_c$

Entire System

$$\begin{bmatrix} \dot{i}_L \\ \dot{V}_c \\ \dot{e}_I \\ \dot{e}_o \end{bmatrix} = \begin{bmatrix} 0 & -\frac{1}{L} & 0 & 0 \\ \frac{1}{C} & -\frac{1}{RC} & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{bmatrix}\begin{bmatrix} i_L \\ V_c \\ e_I \\ e_o \end{bmatrix} + \begin{bmatrix} \frac{1}{L} \\ 0 \\ 0 \\ 0 \end{bmatrix} u + \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} i_L^* + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} V_c^*$$

$u = K_p^i(i_L^* - i_L) + K_I^i e_I$

$i_L^* = K_p^o(V_c^* - V_c) + K_I^o e_o$

$K_p^i \quad K_I^i$

$K_p^o \quad K_I^o$

- Choosing Inner/outer loop PI gains is not trivial, we cannot use pole placement.
- Root locus: plot of the eigenvalues as a function of PI gains.

Next Topic

Three phase AC/DC and DC/AC converters

- abc to dq transformation
- Space vector PWM
- Phase Lock Loop (PLL)
- Controller Design Overview