EE 459/559: Control and Applications of Power Electronics

Dynamic Model

Topic 2: State Space Modeling of Power Converters

Luis Herrera
Dept. of Electrical Engineering
University at Buffalo

Spring 2023

University at Buffalo

Outline

- Overview of Modeling and Control

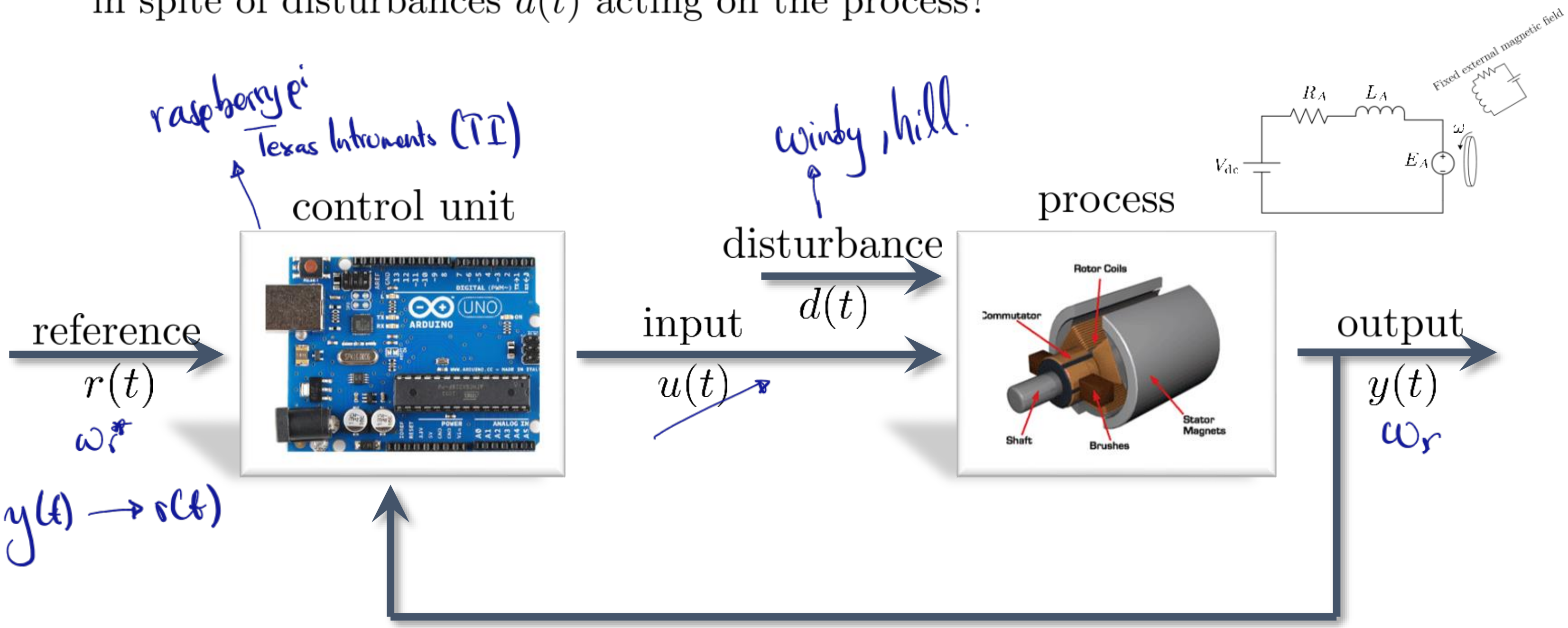
Modern Control

- State Space Modeling of Circuits
- Nonlinear Systems and Linearization
- Power Electronics State Space Averaging

Classical Control

- Transfer Function Analysis

- Control engineering is a discipline that applies automatic control theory to design systems with desired behaviors in control environment
  - How to control the inputs $u(t)$ to the process automatically to make the output $y(t)$ track the given reference $r(t)$?
  - How to exploit the measurements of $y(t)$ to track the reference $r(t)$ in spite of disturbances $d(t)$ acting on the process?

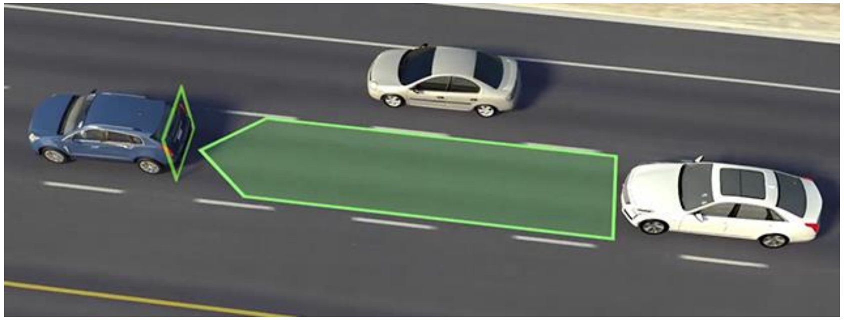


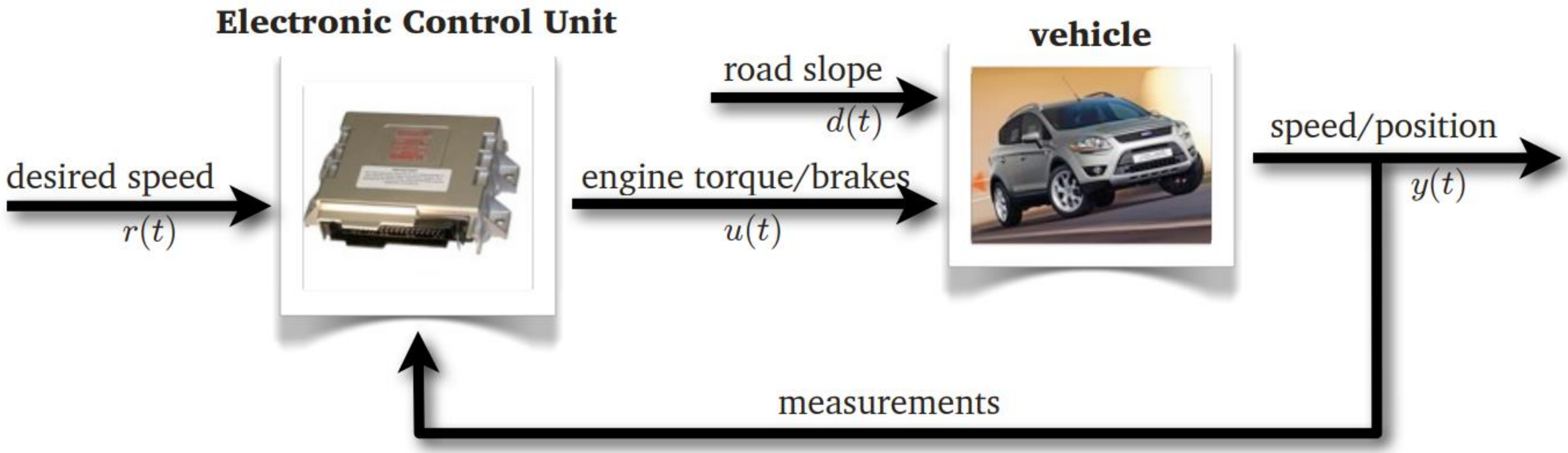
A. Bemporad, Lectures on automatic control, 2011

What is Control?

Example:

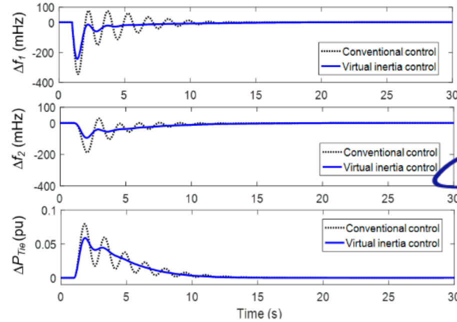
- Automatic Cruise Control (ACC) is a cruise control system that **automatically adjust the speed of the vehicle** to maintain a safe distance from cars ahead



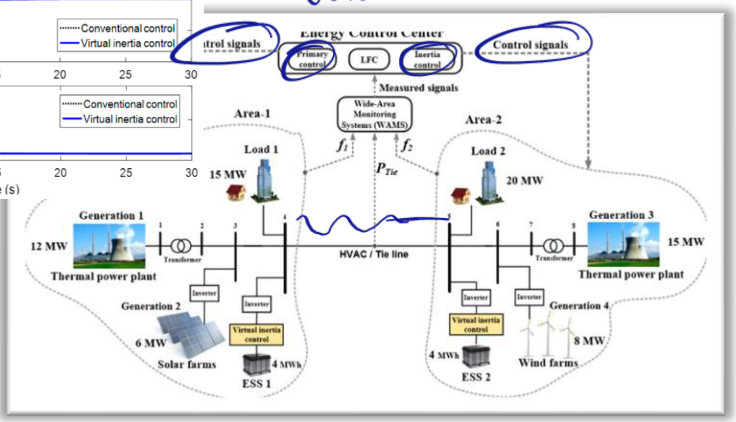


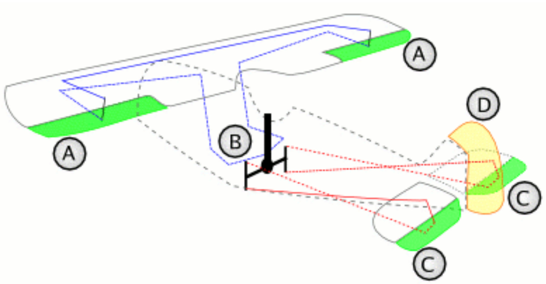
Applications of Control Theory

- Control systems can be used in many applications, ranging from:
  - Automotive
  - Robotics
  - Aeronautics & aerospace
  - Power systems
  - Power electronics
  - …



Wide Area Control



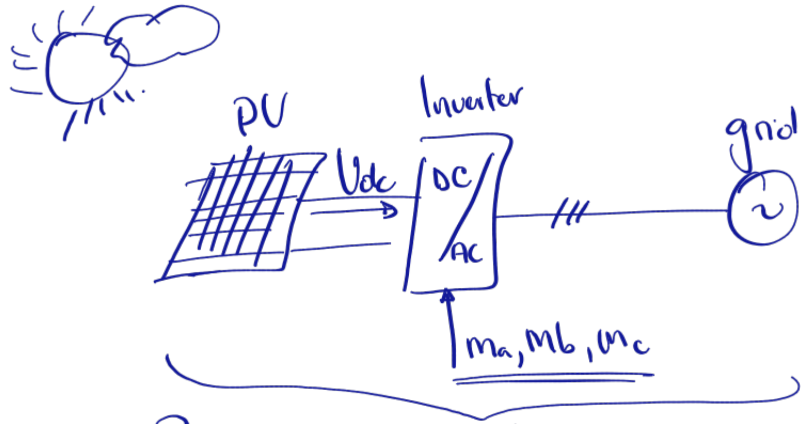


Example: Power Electronics

- Consider a dc/ac inverter connecting a PV plant to the grid
- What are the "inputs"?

modulation signals to the inverter

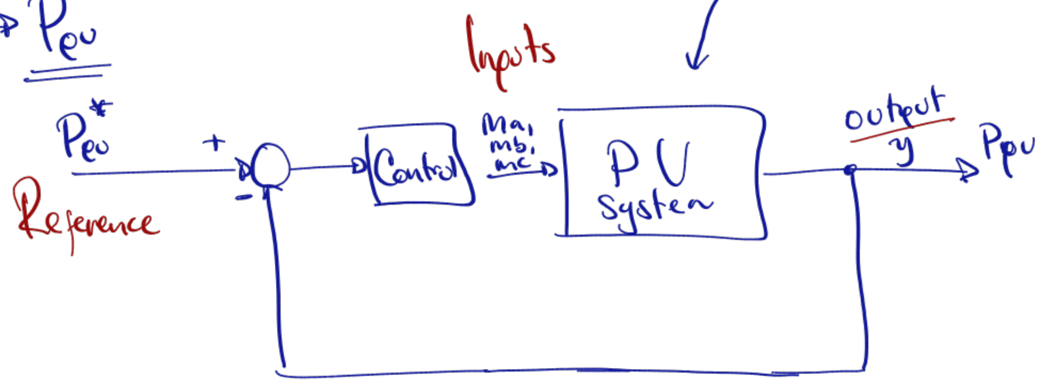
m_a, m_b, m_c



- What are the outputs?

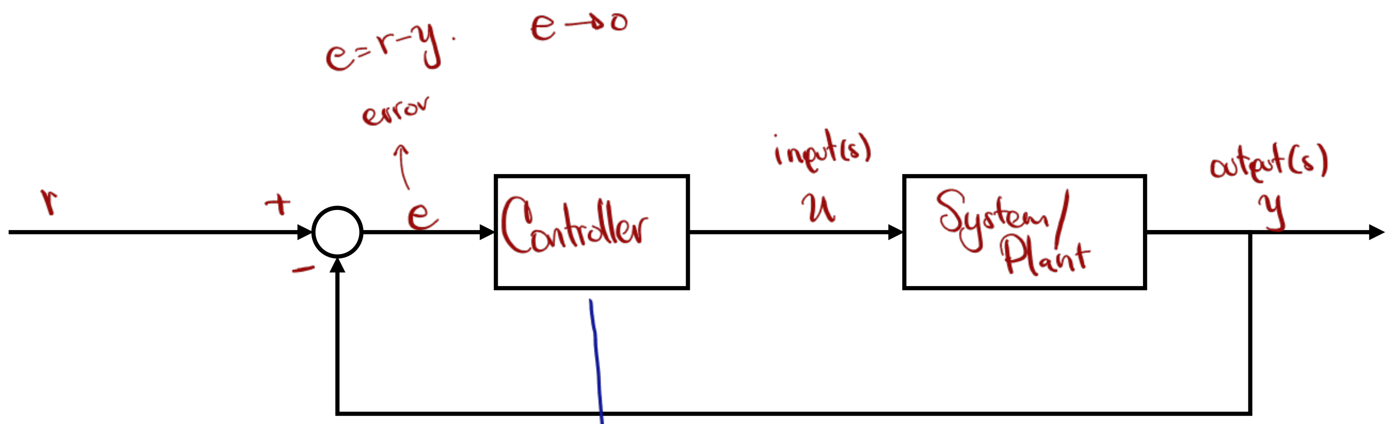
What are we trying to control?

$V_{dc}, I_{pv} \rightarrow P_{pv}$



Control System Components

- A typical feedback control system can be summarized as follows:



Model Based control

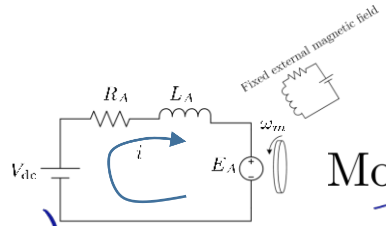
- Dynamic model of the system/plant
- How can we obtain this model? Physics, electric circuits.

Simple: Newton's 2^{nd} law — ODE

Complex: Maxwell's equations — PDE

Classical vs Modern Control Systems

- Frequency (Laplace) domain analysis of feedback systems falls under **classical control**
- Time domain, in particular state space techniques, are generally known as **modern control systems**



Classical control (1900s-1950s)

Laplace Transform (Frequency Domain)

$$H_1(s) = \frac{I(s)}{V_{dc}(s)} = \frac{Js + K_f}{(JL_a)s^2 + (JR_a + K_f L_a)s + (K^2 + K_f R_a)}$$

$$H_2(s) = \frac{\omega(s)}{V_{dc}(s)} = \frac{K}{(JL_a)s^2 + (JR_a + K_f L_a)s + (K^2 + K_f R_a)}$$

Single Input Single Output.

Transfer functions, SISO, frequency response methods, tracking and regulation, etc.

Modern control (1960s-present)

Time Domain

$$\frac{di}{dt} = \frac{-R_A}{L_A} i - \frac{K}{L_A}\omega_m + \frac{1}{L_A} V_{dc}$$

$$\frac{d\omega_m}{dt} = \frac{K}{J} i - \frac{K_f}{J}\omega_m - \frac{1}{J}\tau_{load}$$

$$\dot{x} = Ax + Bu + P\tau_{load}$$

$$y = Cx \qquad y \in \mathbb{R}^p$$

Multiple Input Multiple Output.

State space models, MIMO, Linear Algebra, etc.

Outline

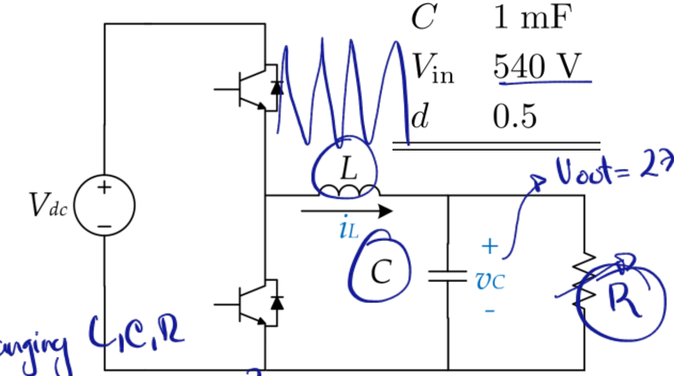
- Overview of Modeling and Control
- State Space Modeling of Circuits
- Nonlinear Systems and Linearization
- Power Electronics State Space Averaging
- Transfer Function Analysis

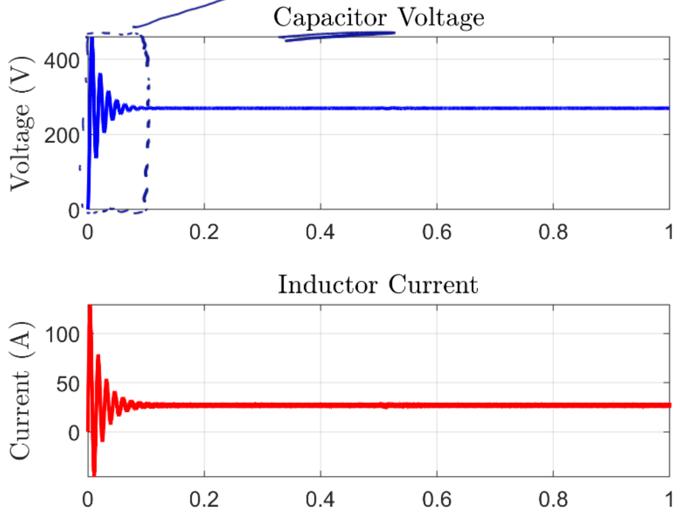
- The behavior of electrical circuits can be modeled by **physical laws**
- Assume the buck converter turns on at $t = 0$ with duty cycle $d = 0.5$ $(i_L(0) = 0,\ V_C(0) = 0)$
- It takes about 100 ms for the current/voltage to settle → reach steady state
  - Is this acceptable? No
  - How can we improve the transient response? → Changing L, C, R → What if L, C, R fixed?

Overshoot → damage the load
~450V

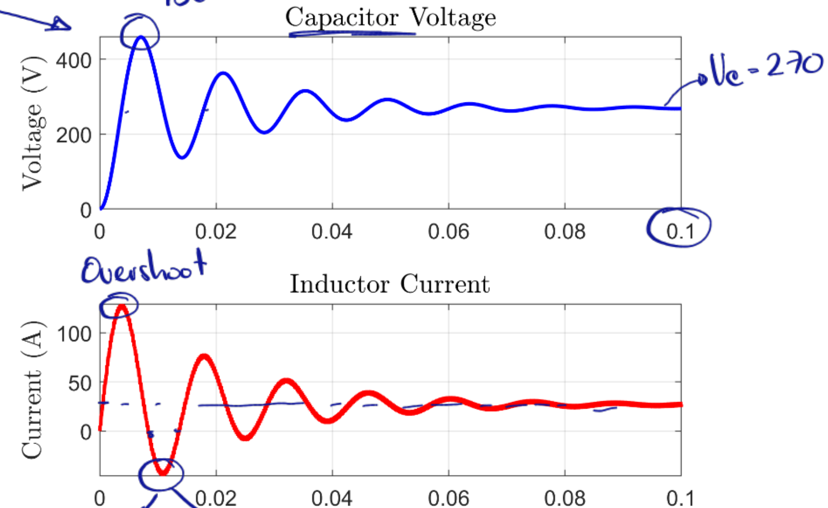
$V_{out} = d\,V_{in}$

L	5 mH
C	1 mF
V_{in}	540 V
d	0.5





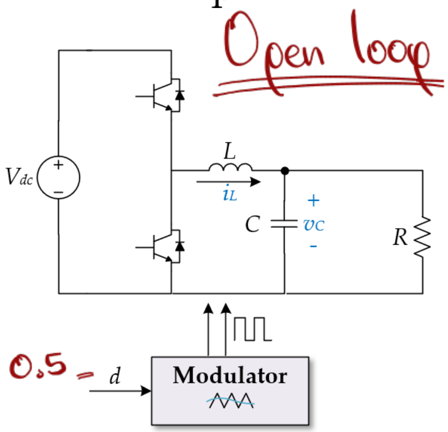
Natural Response

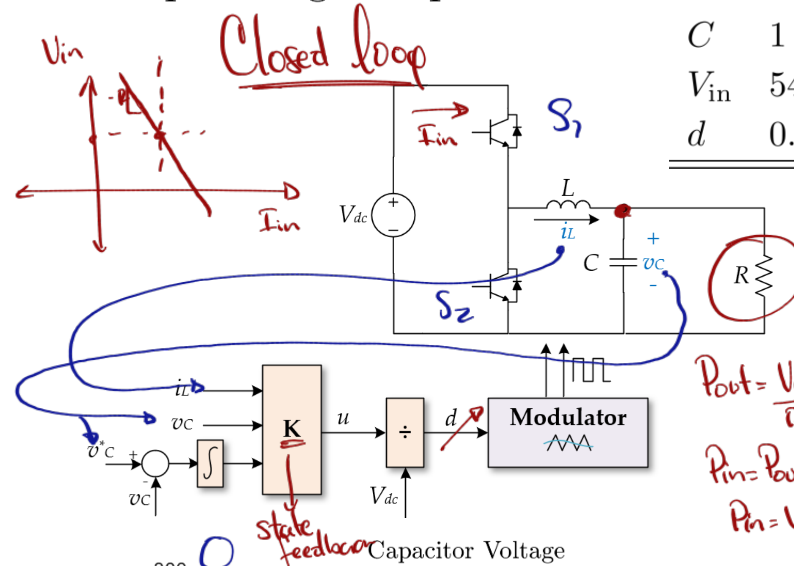


Open Loop vs Closed Loop Performance

- Compare the performance of the converter when operating as open loop and closed loop control

L	5 mH
C	1 mF
V_{in}	540 V
d	0.5

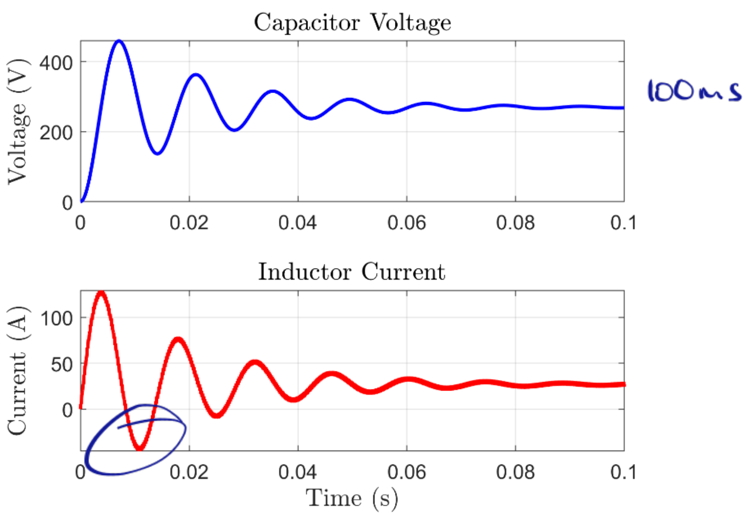


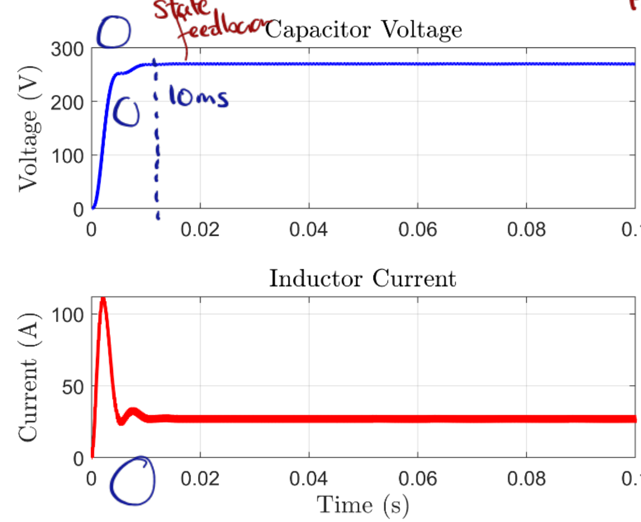


$P_{out} = \frac{V_c^2}{R}$ = fixed!

$P_{in} = P_{out}$ = fixed

$P_{in} = V_{in} I_{in}$





Solving an ODE

- Consider a second order LTI ODE: $a_2 \frac{d^2x}{dt^2} + a_1 \frac{dx}{dt} + a_0 x = 0$

  Linear Time Invariant

  x: dependent variable
  t: indep. "
  a_0, a_1, a_2 are constants.

- If we wanted to solve this equation for $x(t)$, what else is needed?

  - Initial/Boundary conditions. 2^{nd} order System $\longrightarrow$ 2 initial conditions.

    $x(0)$, $\dot{x}(0)$ (N^{th} order ODE $\rightarrow$ N initial conds.)

  - $x(t) = f(t)$ is a solution to the ODE if it satisfies it. $\longrightarrow a_2 \ddot{x}(t) + a_1 \dot{x}(t) + a_0 x(t) = 0$

  - Laplace Transform $\mathcal{L}\{a_2\ddot{x} + a_1\dot{x} + a_0 x\} = 0 \longrightarrow a_2 \mathcal{L}\{\ddot{x}\} + a_1 \mathcal{L}\{\dot{x}\} + a_0 \mathcal{L}\{x\} = 0$

    $a_2(s^2X(s) - sx(0) - \dot{x}(0)) + a_1(sX(s) + x(0)) + a_2 X(s)$

    if $x(0) = \dot{x}(0) \neq 0$

    $X(s)\left[a_2 s^2 + a_1 s + a_0\right] = \left(h(x(0), \dot{x}(0))\right)$ → Partial fraction Decomposition → Inverse Laplace

- While we will look at the general solution of a system of LTI ODE, this will **not** be the main focus of this course

- **We will try to change the behavior of the system without obtaining a solution**

State Space Representation of ODEs

- Any n^{th} order LTI/LTV ODE can be transformed into a **system** of **n** first order equations

n^{th} order ODE $\Rightarrow$ $a_n x^{(n)}(t) + \cdots + a_1 \dot{x}(t) + a_0 x(t) = b_1 u_1 + \cdots + b_m u_m$ *

- This will be known as the **state space system**

$$
\begin{cases}
\dot{x}_1 = \dfrac{\mathrm{d}x_1}{\mathrm{d}t} = f_1(\underbrace{x_1,\ x_2,\ \ldots,\ x_n}_{\text{states}}, \underbrace{u_1,\ u_2,\ \ldots,\ u_m}_{\text{inputs}},\ t) & x_1(0) = x_{10} \\
\dot{x}_2 = \dfrac{\mathrm{d}x_2}{\mathrm{d}t} = f_2(x_1,\ x_2,\ \ldots,\ x_n, u_1,\ u_2,\ \ldots,\ u_m,\ t) & x_2(0) = x_{20} \\
\quad\vdots \qquad\qquad \vdots & \vdots \\
\dot{x}_n = \dfrac{\mathrm{d}x_n}{\mathrm{d}t} = f_n(x_1,\ x_2,\ \ldots,\ x_n, u_1,\ u_2,\ \ldots,\ u_m,\ t) & x_n(0) = x_{n0}
\end{cases}
$$

- It will be typically written in vector form as follows:

Typically $m < n$ (# inputs, # states).

$$\dot{x} = f(x,\ u,\ t), \qquad \text{where } x \in \mathbb{R}^n,\ u \in \mathbb{R}^m,\ t \in \mathbb{R}$$

$$x(0) = x_0$$

$$f(x,u,t) = \begin{bmatrix} f_1(x_1, \ldots x_n, u_1, \ldots u_m, t) \\ f_2(x, u, t) \\ \vdots \\ f_n(x, u, t) \end{bmatrix} \qquad x(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \\ \vdots \\ x_n(t) \end{bmatrix} \qquad u(t) = \begin{bmatrix} u_1(t) \\ u_2(t) \\ \vdots \\ u_m(t) \end{bmatrix}$$

Linear Time Invariant (LTI) State Space System

- Any n^{th} order LTI/LTV ODE can be transformed into a **system** of **n** first order equations

$$\dot{x} = f(x,\ u,\ t), \qquad \text{where } x \in \mathbb{R}^n,\ u \in \mathbb{R}^m,\ t \in \mathbb{R}$$
$$x(0) = x_0$$

- If the system of equations is **Linear Time Invariant**, the state space system is as follows:

$\left[a_n x^{(n)}(t) + \cdots + a_1 \dot{x}(t) + a_0 x(t) = b_1 u_1 + \cdots + b_m u_m \right]$ n^{th} order ODE

$$\begin{cases} \dot{x}_1 = a_{11}x_1 + \cdots + a_{1n}x_n + b_{11}u_1 + \cdots + b_{1m}u_m \\ \quad \vdots \\ \dot{x}_n = a_{n1}x_1 + \cdots + a_{nn}x_n + b_{n1}u_1 + \cdots + b_{nm}u_m \end{cases}$$

System of Equations

$$\overbrace{\begin{pmatrix} \dot{x}_1 \\ \vdots \\ \dot{x}_n \end{pmatrix}}^{\dot{x}} = \overbrace{\underbrace{\begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{n1} & \cdots & a_{nn} \end{pmatrix}}_{\text{constant matrix}}}^{A} \overbrace{\begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}}^{x} + \overbrace{\underbrace{\begin{pmatrix} b_{11} & \cdots & b_{1m} \\ \vdots & \ddots & \vdots \\ b_{n1} & \cdots & b_{nm} \end{pmatrix}}_{\text{constant matrix}}}^{B} \overbrace{\begin{pmatrix} u_1 \\ \vdots \\ b_m \end{pmatrix}}^{u}$$

$$\dot{x} = Ax + Bu, \quad \underline{x(0) = x_0} \qquad \text{where } x \in \mathbb{R}^n,\ u \in \mathbb{R}^m,\ A \in \mathbb{R}^{n\times n},\ B \in \mathbb{R}^{n\times m}$$

System of First Order ODE – State Space Form

- **Example:** convert this second order ODE into a system of first order ODEs

$$\left(a_2\frac{d^2x}{dt^2} + a_1\frac{dx}{dt} + a_0x = 0 \right)$$

State Space form

- Define new variables 2^{nd} ODE $\longrightarrow$ 2 variables.

? $x_1 = x \longrightarrow \dot{x}_1 = \dot{x} = x_2 \longrightarrow \begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = \frac{-a_0}{a_2}x_1 - \frac{a_1}{a_2}x_2 \end{cases}$

$x_2 = \dot{x}$ $\dot{x}_2 = \ddot{x} = \frac{-a_0}{a_2}x_1 - \frac{a_1}{a_2}x_2 \Rightarrow$

$$a_2\ddot{x} + a_1\dot{x} + a_0x = 0$$

$$\Rightarrow \ddot{x} = \frac{-a_0}{a_2}\underset{\substack{\downarrow \\ x_1}}{x} - \frac{a_1}{a_2}\underset{\substack{\downarrow \\ x_2}}{\dot{x}}$$

- a_0, a_1, a_2 are fixed constants.

$$x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \longrightarrow \dot{x} = Ax \longleftrightarrow \begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \underbrace{\begin{pmatrix} 0 & 1 \\ \frac{-a_0}{a_2} & \frac{-a_1}{a_2} \end{pmatrix}}_{A}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \Rightarrow \begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = \frac{-a_0}{a_2}x_1 - \frac{a_1}{a_2}x_2 \end{cases}$$

eigenvalues of A will tell us the stability properties of the ODE / System

State Space Representation of Dynamic Systems

- **State space representation:** A mathematical model of a dynamical system as a set of input, output, and state variables related by **first order** differential equations
- Example of a rotating shaft (e.g. in a generator)

Not in state space form

Generator frequency equation

input — Electrical load Torque

$$J\frac{d^2\delta}{dt^2} = T_m - T_e$$

J: moment of inertia

$J\frac{d\omega}{dt} = T_m - T_e$

↑J (↓Δω (better))

$\delta(0) = \delta_0$

$\delta'(0) = \omega_0$

Frequency Regulation

State Space form

(2nd Order ODE) ⟶ 2 variables

$x_1 = \delta \quad \longleftrightarrow \quad \delta = \delta$

$x_2 = \dot{\delta} \qquad\qquad\quad \omega = \dot{\delta}$

$$\begin{cases} \dot{x}_1 = \dot{\delta} = x_2 \\ \dot{x}_2 = \ddot{\delta} = ? \text{ (refer to ODE)} \end{cases}$$

$\dot{x}_2 = \frac{1}{J}T_m - \frac{1}{J}T_e$

$\dot{x}_1 = x_2 \qquad\qquad \dot{\delta} = \omega$

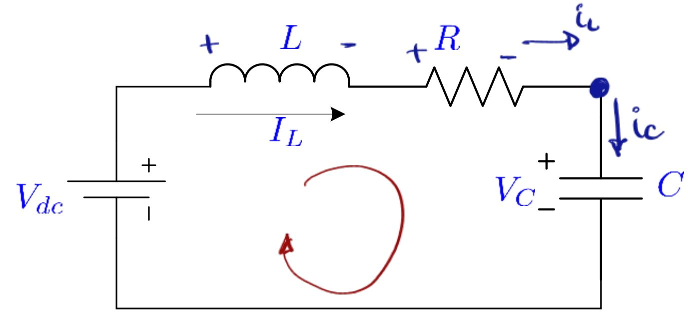
$\dot{x}_2 = \frac{1}{J}T_m - \frac{1}{J}T_e \longleftrightarrow \dot{\omega} = \frac{1}{J}T_m - \frac{1}{J}T_e$

$\dot{x} = Ax + Bu$

$$\begin{pmatrix}\dot{x}_1\\ \dot{x}_2\end{pmatrix} = \begin{pmatrix}0 & 1\\ 0 & 0\end{pmatrix}\begin{pmatrix}x_1\\ x_2\end{pmatrix} + \begin{pmatrix}0 & 0\\ \frac{1}{J} & -\frac{1}{J}\end{pmatrix}\begin{pmatrix}T_m\\ T_e\end{pmatrix}$$

Dynamic Model Example 1 (Time domain)

- Place an RLC circuit (ODE) in state space form:



(2nd order (1 inductor, 1 capacitor))

- **Method 1:** (Solve for $i_L(t) = ?$)

KVL: $-V_{dc} + L\frac{di_L}{dt} + i_L R + V_c = 0$

$\Rightarrow \left[-V_{dc} + L\frac{di_L}{dt} + i_L R + \frac{1}{C}\int i_L\, dt = 0\right]\frac{d}{dt}$

$\Rightarrow \left(-\dot{V}_{dc} + L\ddot{i}_L + \dot{i}_L R + \frac{1}{C} i_L = 0\right)$ 2nd Order ODE

$V_L = L\frac{di_L}{dt}$ $\quad$ L

$\left(i_c = C\frac{dV_c}{dt}\right)$ $\quad$ C $\quad$ $V_c = \frac{1}{C}\int i_c\, dt$

- **Method 2** 2 states or variables $\Rightarrow$ $x_1 = i_L$, $x_2 = V_c$

a) To obtain $\dot{x}_1 = \frac{di_L}{dt} = ?$ (Use KVL)

b) To " $\dot{x}_2 = \frac{dV_c}{dt} = ?$ (Use KCL)

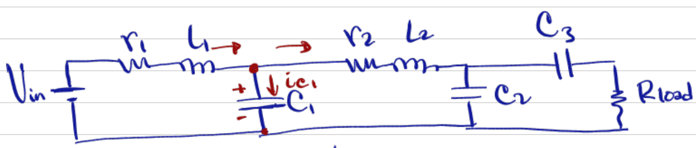
$\dot{x} = Ax + Bu$

a) $-V_{dc} + \underbrace{L\frac{di_L}{dt}}_{\dot{x}_1} + R\underbrace{i_L}_{x_1} + \underbrace{V_c}_{x_2} = 0 \longrightarrow$

b) $\sum i_{in} = \sum i_{out} \Rightarrow \underbrace{i_L}_{x_1} = i_c = C\overbrace{\frac{dV_c}{dt}}^{\dot{x}_2} \longrightarrow$

$$\begin{cases} \frac{di_L}{dt} = \dot{x}_1 = -\frac{R}{L}x_1 - \frac{1}{L}x_2 + \frac{1}{L}\underbrace{V_{dc}}_{u} \\ \frac{dV_c}{dt} = \dot{x}_2 = \frac{1}{C}x_1 \end{cases}$$

Example 2



order of ODE = order of State Space = # of states.

5th order System

Linear Time Invariant

$\dot{x} = Ax + Bu$ $\quad$ x = "states" $\quad$ u = "input" = V_{in}

$$x = \begin{pmatrix} i_{L_1} \\ i_{L_2} \\ V_{C_1} \\ V_{C_2} \\ V_{C_3} \end{pmatrix}$$

- For each inductor → do KVL ⇒ $L\frac{di}{dt} = \sum$ voltages
- " " capacitor → KCL ⇒ $C\frac{dV_c}{dt} = i_c =$ ___

$$\dot{x} = \underset{5\times5}{A}\,\underset{5\times1}{x} + B\,\underset{1\times1}{u}$$

Solution to a Linear System – General Form

- A LTI state space system of the form:

$$\dot{x} = Ax + Bu, \quad x(0) = x_0 \quad \text{where } x \in \mathbb{R}^n,\ u \in \mathbb{R}^m$$

has a solution of the form:

$$x(t) = \overbrace{e^{At}x(0)}^{\text{homogeneous}} + \overbrace{\int_0^t e^{A(t-\tau)}Bu(\tau)d\tau}^{\text{particular solution}}$$

Convolution $\left[e^{At}\right] * \left[Bu(t)\right]$

(Analytical Solution)

This $x(t)$ satisfies $\dot{x}(t) = Ax + Bu$

$$\dot{x}(t) = \frac{d}{dt}\left(e^{At}x(0) + \underbrace{\int_0^t e^{A(t-\tau)}Bu(\tau)d\tau}_{d/dt \text{ (Leibniz rule)}}\right)$$

$$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \qquad e^{At} \neq \begin{pmatrix} e^{a_{11}t} & e^{a_{12}t} \\ e^{a_{21}t} & e^{a_{22}t} \end{pmatrix}$$

Taylor Series.

- where $e^{At} \triangleq \sum_{k=0}^{\infty} \frac{1}{k!}(At)^k$ is known as the **matrix exponential**

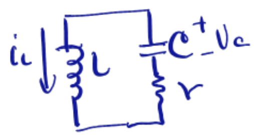
$$e^{At} = \mathcal{L}^{-1}\left\{(sI-A)^{-1}\right\}$$

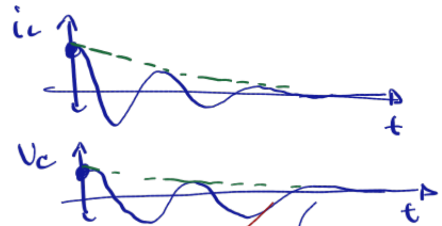
Stability of Linear Systems

- Consider a linear state space system of the form $\underbrace{\dot{x} = Ax + Bu}$, $x(0) = x_0$
- **Is it possible to know the following without obtaining an explicit solution to the ODE, $x(t)$?**
  - If the input $u = 0$ and $\underbrace{x(0) \neq 0}$, will the states decay to zero, $x(t) \to 0$?

$\dot{x} = Ax + \cancel{Bu}$

$x(0) \neq 0$



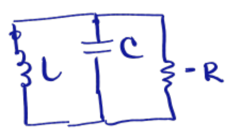


Asymptotically Stable

  - If the input $u = 0$ and $x(0) \neq 0$, will the states grow unbounded? i.e. $||x(t)||_2 \to \infty$?

$\dot{x} = Ax + \cancel{Bu}$

$x(0) \neq 0$

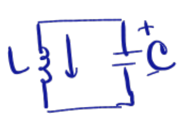


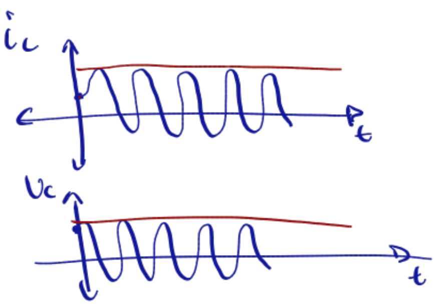
Unstable

  - If the input $u = 0$ and $\underbrace{||x(0)||_2 \leq \delta}$, will the states remain within a region? i.e. $||x(t)||_2 \leq \epsilon, \quad \forall t \geq 0$

$\dot{x} = Ax + \cancel{Bu}^{\,0}$

$x(0) \neq 0$





Marginally Stable

Block Diagram and Numerical Solution

- A linear time invariant state space system of the form:

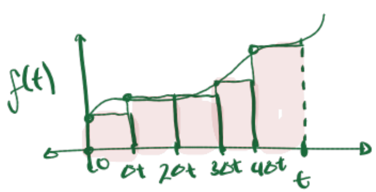
$$\dot{x} = Ax + Bu, \quad x(0) = x_0, \qquad \text{where } x \in \mathbb{R}^n,\ u \in \mathbb{R}^m$$

can be represented as a signal flow diagram of the form:

Numerical Solution

Example: 1^{st} order system $\frac{dx}{dt} = -3x + 5$. Goal: $x(t) = ?$ or $x(t) \approx$

- To find an approx. solution: $\int_0^{\Delta t} dx = \int_0^{\Delta t} (-3x+5)\,dt \Rightarrow x(\Delta t) - x(0) = \int_0^{\Delta t} (-3x(t)+5)\,dt$



$\int_0^{\Delta t} f(t)\,dt \approx \Delta t\, f(0)$ (forward Euler)

(Trapezoidal, Backward Euler)

$\Rightarrow x(\Delta t) = x(0) + \int_0^{\Delta t} (-3x(t)+5)\,dt$

$\Rightarrow x(\Delta t) \approx x(0) + \Delta t\,[-3x(0)+5]$

$\Rightarrow x(\Delta t) \approx (1 + \Delta t(-3))\,x(0) + 5\Delta t$

$\Rightarrow x(2\Delta t) \overset{?}{=} (1 + \Delta t(-3))\,x(\Delta t) + 5\Delta t$

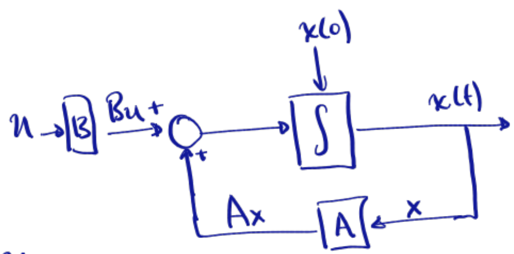
$\dot{x}(t) = Ax + Bu \xrightarrow{\text{forward Euler}} x(k+1) = (I + T_s A)x(k) + T_s B u(k)$

$x(0) = x_0 \qquad T_s = \Delta t$

Signal Flow Diagram

$\frac{dx}{dt} = Ax + Bu \rightarrow \int_0^t dx = \int_0^t Ax + Bu$

$\rightarrow x(t) = x(0) + \int_0^t Ax + Bu$

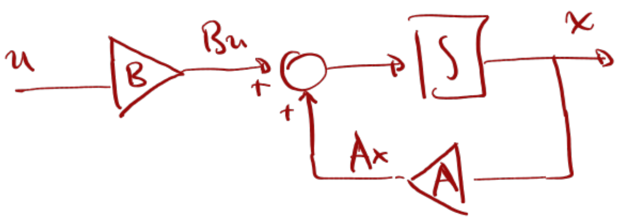


Block Diagram and Numerical Solution Example

- Derive the state space equations for the circuit shown and simulate in Matlab

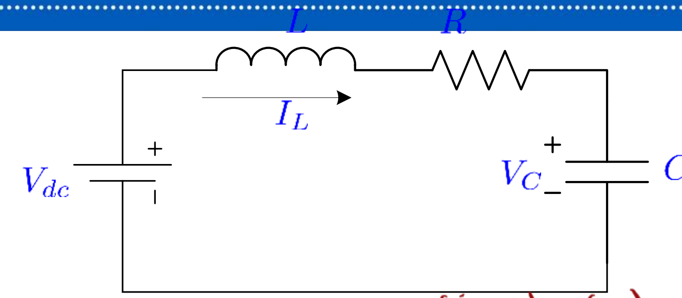
State Space (Slide 17)

$$\underbrace{\begin{bmatrix}\dot{i}_L\\ \dot{v}_C\end{bmatrix}}_{\dot{x}} = \underbrace{\begin{bmatrix}-R/L & -\frac{1}{L}\\ 1/C & 0\end{bmatrix}}_{A}\underbrace{\begin{bmatrix}i_L\\ v_C\end{bmatrix}}_{x} + \underbrace{\begin{bmatrix}1/L\\ 0\end{bmatrix}}_{B}\underbrace{V_{dc}}_{u}$$



eigenvalues of $(A) = \{-100 \pm 994.5j\}$

$\mathrm{Re}\{\lambda\} = -100$



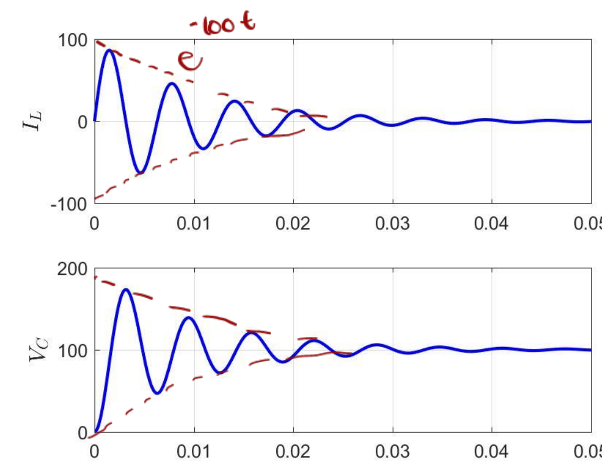
$x(0) = 0 \Rightarrow \begin{pmatrix} i_L(0)\\ v_C(0)\end{pmatrix} = \begin{pmatrix}0\\0\end{pmatrix}$

$V_{\mathrm{dc}} = 100$ V

$L = 1$ mH

$C = 1$ mF

$R = 0.2\ \Omega$



- The behavior of the linear system depends on the eigenvalues of A

Linear System Stability Definition

- Consider a general state space system: $\begin{cases} \dot{x} &= Ax + Bu \\ y &= Cx \end{cases}, \quad x(0) = x_0$

- We say that the equilibrium state x_e is:

  Marginally stable

  ☐ **Stable** if for each initial condition $x(0)$ 'close' to x_e, the corresponding trajectory $x(t)$ remains near x_e for all $t \geq 0$

eigenvalues of $A = \sigma(A) = \{\lambda_1, \lambda_2, \dots, \lambda_n\}$

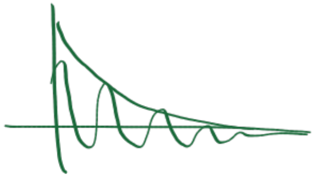
$A \in \mathbb{R}^{n \times n}$ $\quad \lambda_i \in \mathbb{C}$

$Ax_i = \lambda_i x_i$ $\quad x_i$ = eigenvector $\neq 0$ $\quad \lambda_i$ = eigenvalue

$\forall \lambda_i \quad \mathrm{Re}\{\lambda_i\} \leq 0$

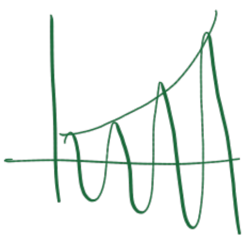
if there is λ_j s.t. $\mathrm{Re}\{\lambda_j\} = 0 \rightarrow$ alg. mult. = geom. mult.

  ☐ **Asymptotically stable** if it is stable and $x(t) \rightarrow x_e$ as $t \rightarrow \infty$



if $\forall \lambda_i \quad \underbrace{\mathrm{Re}\{\lambda_i\} < 0}$

  ☐ **Unstable** otherwise

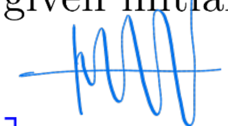


if $\exists \lambda_i$ s.t. $\mathrm{Re}\{\lambda_i\} > 0 \Rightarrow$ unstable $\quad$ or $\quad \exists \lambda_j$ s.t. $\mathrm{Re}\{\lambda_j\} = 0$ and λ_j is degenerate

at least one

Stability Example

- Which of the following systems are stable, asymptotically stable, or unstable? Plot the numerical solution for the given initial condition



System 1

$$\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\end{bmatrix} = \underbrace{\begin{bmatrix}100 & -500\\ 1000 & 0\end{bmatrix}}_{A_1}\begin{bmatrix}x_1\\ x_2\end{bmatrix} \rightarrow \dot{x} = Ax$$

$$\begin{bmatrix}x_1(0)\\ x_2(0)\end{bmatrix} = \begin{bmatrix}1\\ 0\end{bmatrix}$$

System 2

$$\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\end{bmatrix} = \underbrace{\begin{bmatrix}-100 & -500\\ 1000 & 0\end{bmatrix}}_{A_2}\begin{bmatrix}x_1\\ x_2\end{bmatrix}$$

$$\begin{bmatrix}x_1(0)\\ x_2(0)\end{bmatrix} = \begin{bmatrix}1\\ 0\end{bmatrix}$$

$$x(t) = e^{At}x(0) = \sum_{j=1}^{2} v_i e^{\lambda_i t} x_i(0)$$

$$e^{\lambda_i t} = \underline{e^{\mathrm{Re}\{\lambda_i\}t}\left(\cos(\overbrace{\mathrm{Im}(\lambda)}^{\omega}t) + j\sin(\overbrace{\mathrm{Im}(\lambda_i)}^{\omega}t)\right)}$$

$$\omega = \mathrm{Im}(\lambda)\ \left(\tfrac{\mathrm{rad}}{s}\right) = 2\pi f$$

- To find eigenvalues, - Matlab: eig(A)
- Eigenvalues satisfy $(Ax = \lambda x)$ where $x \neq 0$ (non-zero vector), x - eigenvector corresponding to eigenvalue λ

$$\Rightarrow \lambda x - Ax = 0 \rightarrow (\lambda I - A)x = 0 \xrightarrow{\lambda \in \mathbb{C}} \det(\lambda I - A) = 0$$

$=$ polynomial of n^{th} order ($n \times n$ A matrix)

$$p_A(\lambda) = \lambda^n + a_{n-1}\lambda^{n-1} + \dots + a_1\lambda + a_0 = 0$$

roots of $p_A(\lambda)$ = eigenvalues of A

eigenvalues of $A_1 = \sigma(A) = \{\underbrace{50 + j705.3}_{\lambda_1}, \underbrace{50 - j705.3}_{\lambda_2}\}$

$\mathrm{Re}\{\lambda_{1,2}\} > 0 \Rightarrow$ unstable

$\sigma(A_2) = \{-50 + j705.3, -50 - j705.3\} \Rightarrow \mathrm{Re}\{\lambda_{1,2}\} < 0$ asymp. stable.

- **Observation:** to understand how a linear system will behave we do not always need to compute its solution

Outline

- Overview of Modeling and Control
- State Space Modeling of Circuits $\{$linear state space $\dot{x} = Ax + Bu\}$.
- **Nonlinear Systems and Linearization**
- Power Electronics State Space Averaging
- Transfer Function Analysis

Nonlinear Dynamical Systems

- A nonlinear system in state space form can be written as follows:

$f_1, \cdots, f_n$ can be nonlinear function of x

$$\begin{aligned}
\dot{x}_1 &= f_1(t,\ \underbrace{x_1, \cdots, x_n}_{\text{states}},\ \underbrace{u_1, \cdots, u_m}_{\text{inputs.}}) \\
&\ \ \vdots \\
\dot{x}_n &= f_n(t,\ \underbrace{x_1, \cdots, x_n},\ \underbrace{u_1, \cdots, u_m})
\end{aligned} \qquad \Rightarrow \quad \dot{x} = f(t,\ x,\ u)$$

- When the dynamical system is **time invariant** and **autonomous**: $\underbrace{\dot{x} = f(x)}$

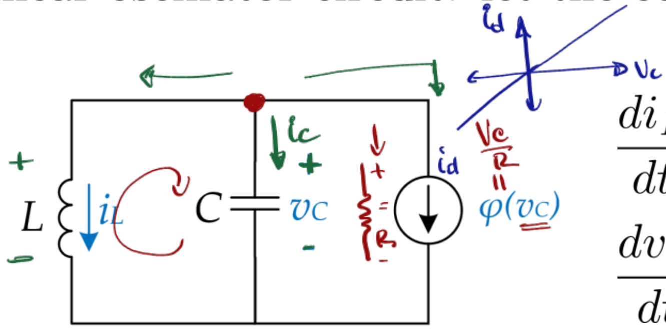
- Some differences between linear and nonlinear systems:

equilibrium points.

  - Local vs global stability (nonlinear systems can be asymp. stable around x_e, but unstable in a different x_e)
  - Multiple equilibrium points
  - Limit cycles
  - Chaos Theory.
  - $\vdots$

Linear Oscillator Example

- Linear oscillator circuit: let the current source be a resistor!



$\dot{x} = Ax$

$$\frac{di_L}{dt} = \frac{1}{L} v_C$$

$$\frac{dv_C}{dt} = -\frac{1}{C} i_L - \frac{1}{C}\left(\frac{1}{R} v_C\right)$$

derived by KCL

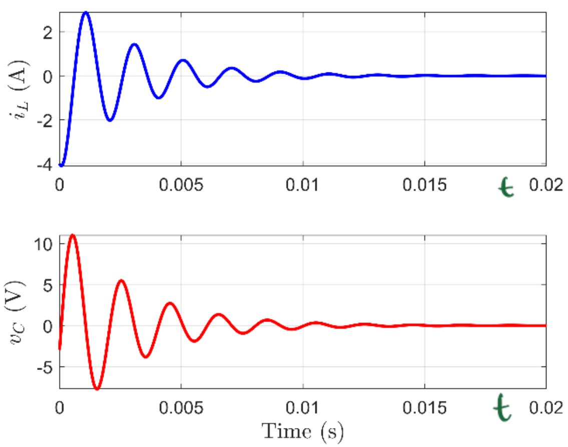
$i_C + i_L + \frac{v_C}{R} = 0$

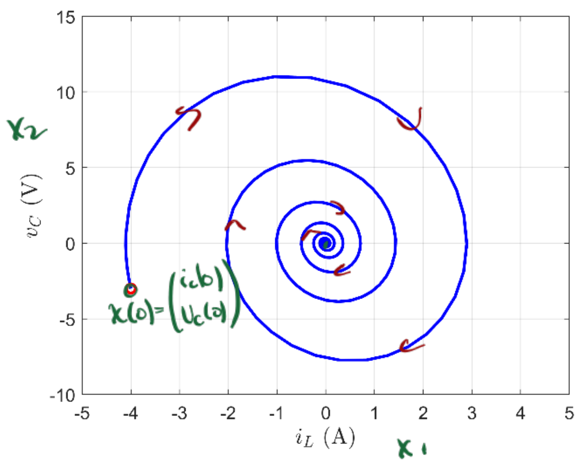
$i_C = -i_L - \frac{v_C}{R} = C\frac{dv_C}{dt}$

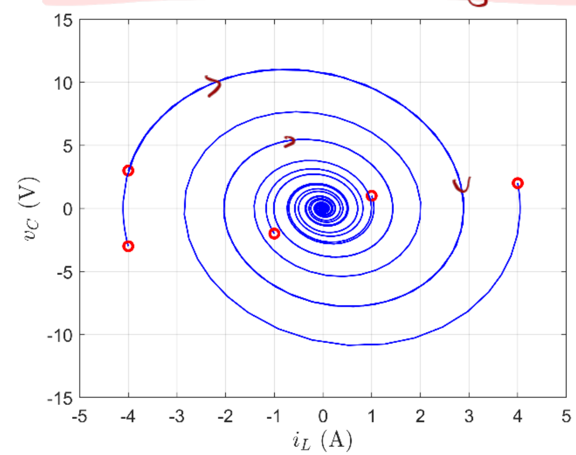
- For any initial condition, the current and voltage asymptotically decay to 0

assuming, $L, C, R > 0$

no matter which $x(0)$ we begin with.



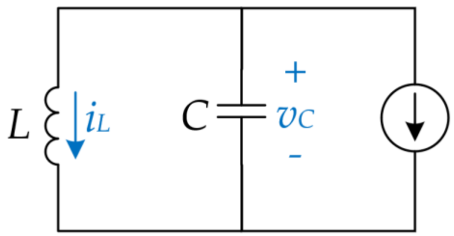


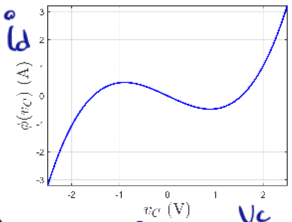


Phase portraits.

Nonlinear Oscillator Example

- Nonlinear Oscillator circuit:





id

Vc

$\varphi(v_C) = i_d$

$i_d = \phi(v_C) = \frac{1}{3}v_C^3 - Rv_C$

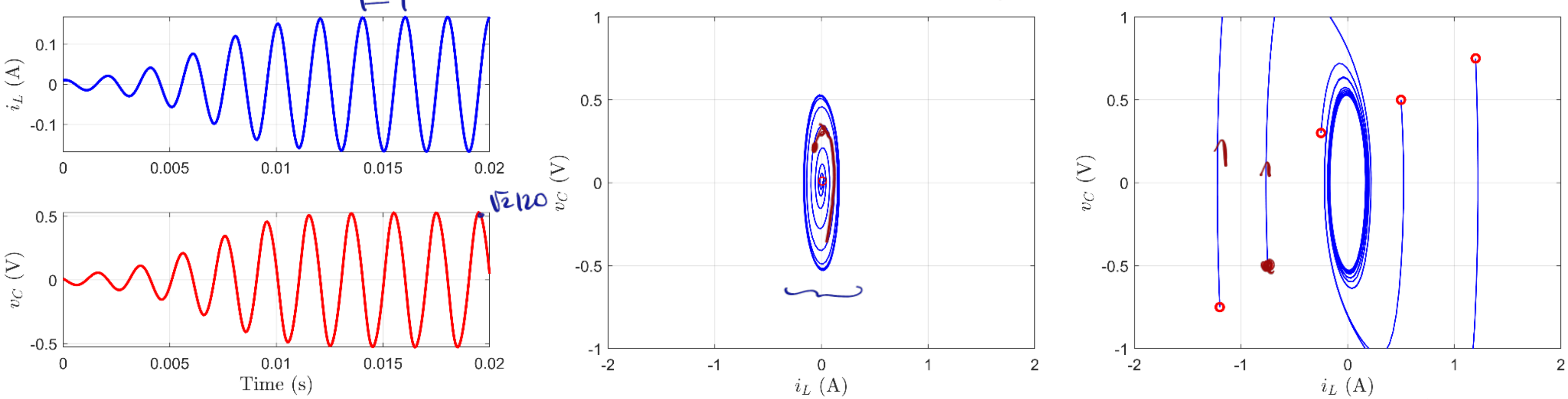
$$\left.\begin{aligned}\frac{di_L}{dt} &= \frac{1}{L}v_C \\ \frac{dv_C}{dt} &= -\frac{1}{C}i_L - \frac{1}{C}\underbrace{\left(\frac{1}{3}v_C^3 - Rv_C\right)}\end{aligned}\right\}$$

Nonlinear System

- For any initial condition, the circuit enters a 'limit cycle' (stable limit cycle)

attractive

60Hz



micrognids.

- These types of oscillators can be used to control dc/ac converters! **(Grid forming inverter)** https://johnson.ece.uw.edu/Papers/Journals/2013_TPEL-SynchronizationOfSinglePhaseInvertersWithVirtualOscillatorControl.pdf

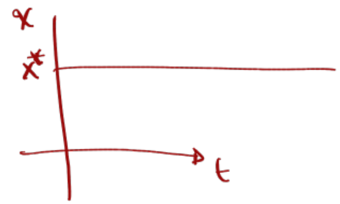
(Kuramoto Oscillators)

Equilibrium Points

- A point x^* is said to be an equilibrium point of a dynamic system ($\dot{x} = f(t,x)$) if:

$\dot{x} = f(x)$

$$0 = f(x^*) \qquad \text{if } x(0) = x^* \;\Rightarrow\; x(t) = x^* \;\forall t > 0$$



- For **nonlinear systems**, it is possible to have multiple isolated equilibrium points (steady state)

Example (Pendulum)

$x_1 = \theta \quad x_2 = \dot{\theta} = \omega$

$$\begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = -\sin(x_1) - 0.3\, x_2 \end{cases}$$

locally asymp. stable

$\theta^* = 0$ (one equilibrium point)

locally unstable

$\theta^* = \pi$

find eq. points.

x_1^*, x_2^* are eq. points if

$0 = f_1(x_1^*, x_2^*) \rightarrow 0 = x_2^*$

$0 = f_2(x_1^*, x_2^*) \qquad 0 = -\sin(x_1^*) - \cancel{0.3\, x_2^*}^{\,0}$

$-\sin(x_1^*) = 0 \qquad x_1^* = 0, \pi, 2\pi, \dots$

To find Equilibrium Points:

n-equations $\dot{x} = f(x,u) \quad x \in \mathbb{R}^n \quad u \in \mathbb{R}^m$

- Given u^* find x^* (unknowns)
- (x^*, u^*) satisfy: $0 = f(x^*, u^*)$

n-equations, n-unknowns

Linearization

- Consider a general nonlinear system with one input $\dot{x} = f(x, u)$
- Assume that x^*, u^* is an equilibrium point
  typically given
  solve
- We can study the behavior of the nonlinear system 'close' to the equilibrium point by **linearizing** the nonlinear equation $f(x, u)$ at x^*, u^*
- How can we 'linearize' $f(x, u)$? (First order Taylor Series approximation at x^*, u^*)

$$f(x,u) = \underbrace{f(x^*,u^*) + \left.\frac{\partial f}{\partial x}\right|_{x^*, u^*}(x-x^*) + \left.\frac{\partial f}{\partial u}\right|_{x^*, u^*}(u-u^*)}_{\text{linear}} + \text{Higher Order Terms}$$

H.O.T.

- We have $\dot{x} = f(x,u)$ and we know (x^*,u^*) is an eq. point $\rightarrow f(x^*,u^*) = 0$

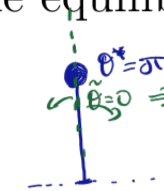
$$\dot{x} = f(x,u) \approx \cancel{f(x^*,u^*)}^{0} + \left.\frac{\partial f}{\partial x}\right|_{x^*, u^*}(x-x^*) + \left.\frac{\partial f}{\partial u}\right|_{x^*, u^*}(u-u^*)$$

- Change variables: $\tilde{x} \triangleq x - x^*$, $\tilde{u} \triangleq u - u^*$, what does $\dot{\tilde{x}}$ look like?
  $n \times n$ $n \times m$

$$\dot{\tilde{x}} = \dot{x} - \cancel{\dot{x}^*}^{0} \approx \left.\frac{\partial f}{\partial x}\right|_{x^*, u^*}\tilde{x} + \left.\frac{\partial f}{\partial u}\right|_{x^*, u^*}\tilde{u} = A\tilde{x} + B\tilde{u} \qquad A = \left.\frac{\partial f}{\partial x}\right|_{x^*, u^*} \quad B = \left.\frac{\partial f}{\partial u}\right|_{x^*, u^*}$$

Linearization

- Consider a general nonlinear system with one input $\dot{x} = f(x, u)$
- Assume that x^*, u^* is an equilibrium point
- We can study the behavior of the nonlinear system 'close' to the equilibrium point by **linearizing** the nonlinear equation $f(x,\ u)$ at $x^*,\ u^*$
- How can we 'linearize' $f(x,\ u)$?



$\theta^* = \pi \qquad \tilde{\theta} = \theta - \theta^*$

$\tilde{\theta} = 0 \Rightarrow \theta = \pi$

$$\dot{\tilde{x}} = \dot{x} - \overset{0}{\cancel{\dot{x}^*}} \approx \overset{n\times n}{\frac{\partial f}{\partial x}\Big|_{\substack{x^*\\u^*}}} \tilde{x} + \overset{n\times m}{\frac{\partial f}{\partial u}\Big|_{\substack{x^*\\u^*}}} \tilde{u} = A\tilde{x} + B\tilde{u} \qquad A = \frac{\partial f}{\partial x}\Big|_{\substack{x^*\\u^*}} \quad B = \frac{\partial f}{\partial u}\Big|_{\substack{x^*\\u^*}}$$

- $\dot{x} = f(x,u) \qquad x \in \mathbb{R}^n \quad u \in \mathbb{R}^m$

$$\Rightarrow \begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \vdots \\ \dot{x}_n \end{pmatrix} = \begin{pmatrix} f_1(x_1, \dots, x_n, u_1, \dots u_m) \\ f_2(x_1, \dots, x_n, u_1, \dots u_m) \\ \vdots \\ f_n(x_1, \dots, x_n, u_1 \dots, u_m) \end{pmatrix}$$

$$\overset{n\times n}{A = \frac{\partial f}{\partial x}\Big|_{\substack{x^*\\u^*}}} = \begin{bmatrix} \frac{\partial f_1}{\partial x_1}(x_1^*, \dots, x_n^*, u_1^*, \dots, u_m^*) & \frac{\partial f_1}{\partial x_2} & \cdots & \frac{\partial f_1}{\partial x_n} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} & \cdots & \frac{\partial f_2}{\partial x_n} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial f_n}{\partial x_1} & \frac{\partial f_n}{\partial x_2} & \cdots & \frac{\partial f_n}{\partial x_n} \end{bmatrix}_{\substack{x^*\\u^*}}$$

$$\overset{n\times m}{B = \frac{\partial f}{\partial u}\Big|_{\substack{x^*\\u^*}}} = \begin{bmatrix} \frac{\partial f_1}{\partial u_1} & \frac{\partial f_1}{\partial u_2} & \cdots & \frac{\partial f_1}{\partial u_m} \\ \frac{\partial f_2}{\partial u_1} & \frac{\partial f_2}{\partial u_2} & \cdots & \frac{\partial f_2}{\partial u_m} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial f_n}{\partial u_1} & \frac{\partial f_n}{\partial u_2} & \cdots & \frac{\partial f_n}{\partial u_m} \end{bmatrix}_{\substack{x^*\\u^*}}$$

$f_1 = x_1^3 + x_2^2 + 2u_1$

$\frac{\partial f_1}{\partial x_1}\Big|_{\substack{x^*\\u^*}} = 3{x_1^*}^2$

Linearization Summary

- Consider a general nonlinear system with one input $\dot{x} = f(x, u)$
- Assume that x^*, u^* is an equilibrium point
- We can study the behavior of the nonlinear system 'close' to the equilibrium point by **linearizing** the nonlinear equation $f(x,\ u)$ close $x^*,\ u^*$

- How can we 'linearize' $f(x,\ u)$?

$$f(x,\ u) \approx \underbrace{\left.\frac{\partial f}{\partial x}\right|_{x^*,\ u^*}}_{\triangleq A}(x - x^*) + \underbrace{\left.\frac{\partial f}{\partial u}\right|_{x^*,\ u^*}}_{\triangleq B}(u - u^*)$$

- Define new coordinates $\tilde{x} = x - x^*$, $\tilde{u} = u - u^*$
- Then the behavior of the original nonlinear system close to the equilibrium point can be modeled by:

$$\dot{\tilde{x}} = A\tilde{x} + B\tilde{u}$$ (LTI State Space System)

Outline

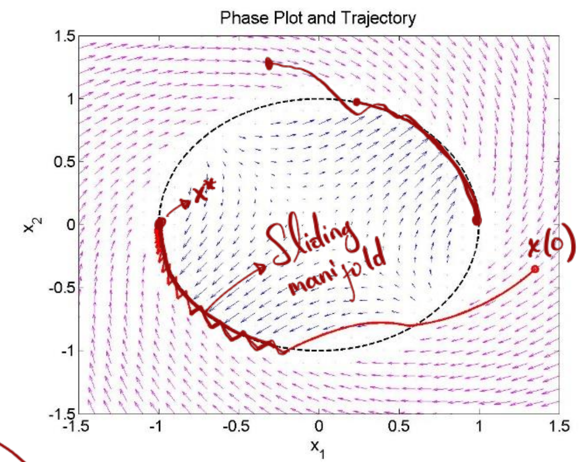
- Overview of Modeling and Control
- State Space Modeling of Circuits
- Nonlinear Systems and Linearization
- **Power Electronics State Space Averaging**
- Transfer Function Analysis

Hybrid Systems

- Hybrid systems are a class of dynamical systems that exhibit continuous and discrete (e.g. switching) dynamic behavior

- Properties associated with hybrid/switching systems:
  - Sliding mode/manifold
  - Arbitrary vs state dependent switching
  - Switching controller design

  $\vdots$



$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \begin{cases} \begin{pmatrix} -1 & 2 \\ -2 & -1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} & \text{if } x \text{ outside circle} \\ \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} & \text{if } x \text{ inside circle} \end{cases}$$

- Analysis of hybrid systems can be challenging

- Can we simplify the model for power converters? Yes

Power Electronics.

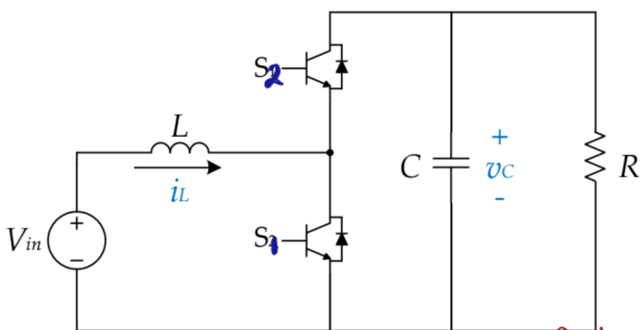
Hybrid Systems (Switching) $\xrightarrow{\text{Approx}}$ Nonlinear State Space $\xrightarrow{\text{Approx}}$ Linearized State Space.

Switching/Hybrid Dynamical Systems

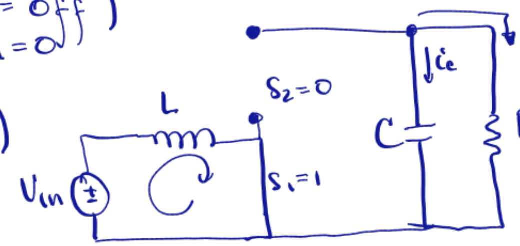
- Power electronics are nonlinear switching dynamical systems (hybrid systems)
- Modeling a power converter, requires knowledge of the switching behavior

- duty cycle = percentage of time that S_1 is on over a switching period.
- $S_2 = \overline{S_1}$ (complementary) $\longrightarrow$ 2 modes. (S_1 = on, $S_1 = 1$; S_1 = off, $S_1 = 0$)

$$-V_{in} + L\frac{di}{dt} = 0 \qquad i_c = C\frac{dv_c}{dt} = \frac{-v_c}{R} \longrightarrow \begin{pmatrix} \dot{i}_L \\ \dot{v}_c \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & \frac{-1}{RC} \end{pmatrix} \begin{pmatrix} i_L \\ v_c \end{pmatrix} + \begin{pmatrix} \frac{1}{L} \\ 0 \end{pmatrix} V_{in}$$

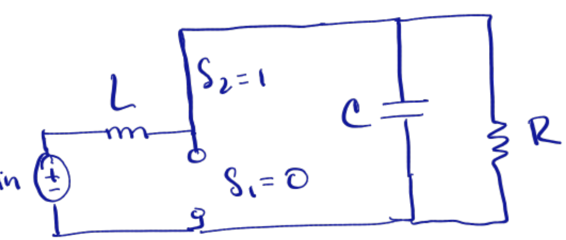


$S_1 = 1$ (on)



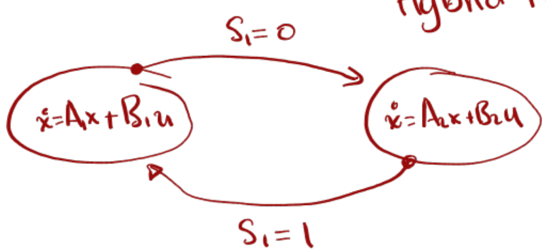
$$\dot{x} = A_1x + B_1u$$

$S_1 = 0$ (off)



$$\dot{x} = A_2x + B_2u$$

Hybrid Automaton

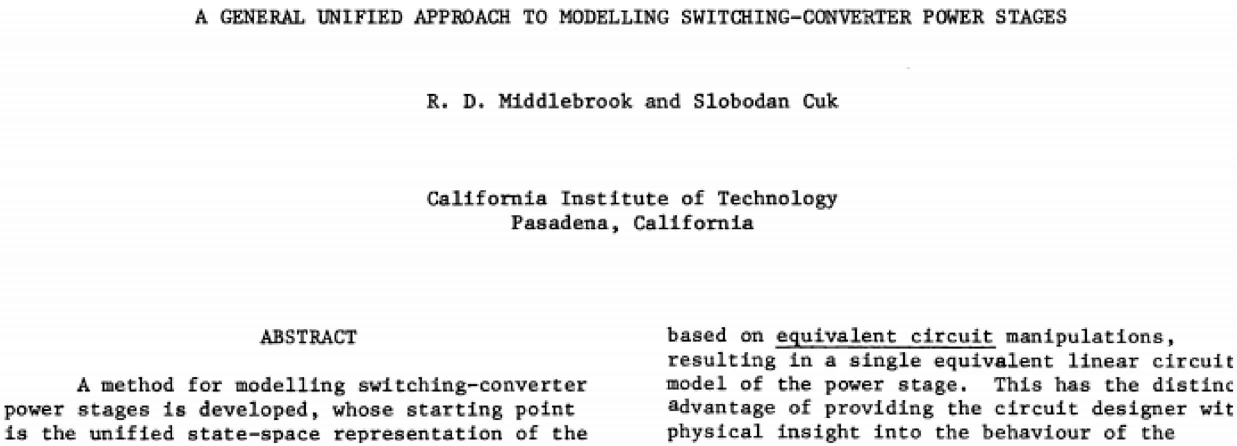


$$\dot{x} = \begin{cases} A_1x + B_1u & S_1 = 1 \\ A_2x + B_2u & S_1 = 0 \end{cases}$$

State Space Averaging

- In 1976, R. Middlebrook and S. Cuk proposed a modeling method for power converters
  The idea relies on **averaging** the switching behavior of power converters

Switching frequency is typically fixed!



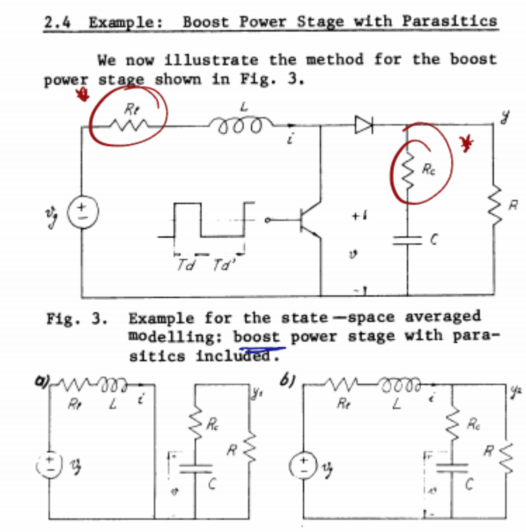
A GENERAL UNIFIED APPROACH TO MODELLING SWITCHING-CONVERTER POWER STAGES

R. D. Middlebrook and Slobodan Cuk

California Institute of Technology
Pasadena, California

ABSTRACT

A method for modelling switching-converter power stages is developed, whose starting point is the unified state-space representation of the based on equivalent circuit manipulations, resulting in a single equivalent linear circuit model of the power stage. This has the distinc advantage of providing the circuit designer wit physical insight into the behaviour of the



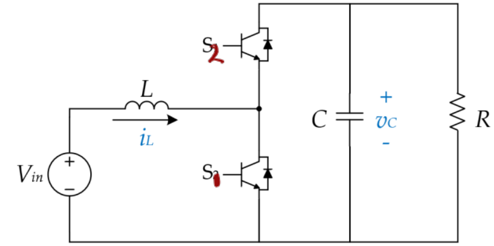
- Periodic switching systems can be approximated by an average function:

Switching Average

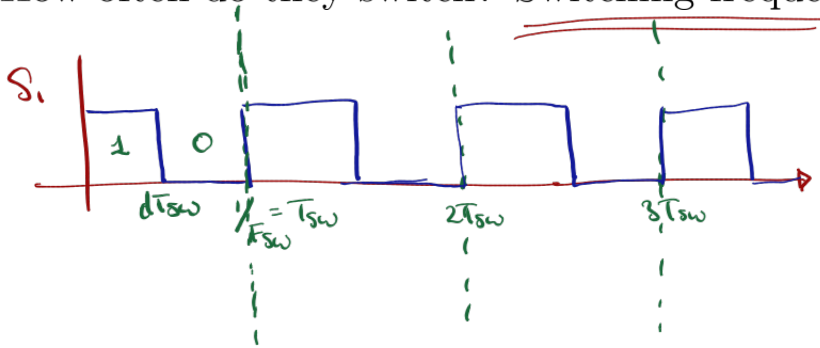
$$\dot{x} = \begin{cases} f_1(x,\ u) & \text{if } s = 1 \\ f_2(x,\ u) & \text{if } s = 0 \end{cases} \quad \Rightarrow \quad \dot{x} = f_{\text{av}}(x,\ u)$$

State Space Averaging - Definitions

- Are power electronics periodic switching systems?
  - How often do they switch? Switching frequency? F_{sw} (Hz)



$d = 0.5$



- Within a switching cycle, what decides the dynamic function/mode? S_1 (Boost example) Duty cycle?

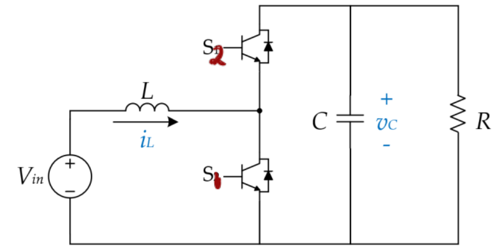
  How long we are in a certain mode over T_{sw} is decided by duty cycle.

- If each switching cycle, we have two modes, how can we obtain the state space average model? over one T_{sw}

  → Simply average the two modes (For this example)

Power Electronics State Space Average

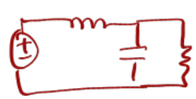
- Define the switching period as T_s, and assume there are two modes (e.g. switch on/off)
- Assume in every T_s mode 1 operates for dT_s and mode 2 operates for $(1-d)T_s$, where $d \in [0,\ 1]$
- The state space average model is then derived as follows:



$S_1 = \text{on}$

$\dot{x} = A_1 x + B_1 u$

$S_1 = \text{off}$



$\dot{x} = A_2 x + B_2 u$

Average of the two modes.

$$\dot{x} = \frac{1}{T_{sw}} \left[\int_0^{dT_{sw}} \underbrace{(A_1 x + B_1 u)}_{S_1 = 1} dt + \int_{dT_{sw}}^{T_{sw}} \underbrace{(A_2 x + B_2 u)}_{S_1 = 0} dt \right]$$

(Try it) ⋮

$$\Rightarrow \boxed{\dot{x} = d(A_1 x + B_1 u) + (1-d)(A_2 x + B_2 u)} \; *$$

State Space Average $\quad d \in [0,1]$
(non linear state space)

- Challenge $\quad$ d is also an input $\longrightarrow$ $\dot{x} = (dA_1 + (1-d)A_2)x + (dB_1 + (1-d)B_2)u$

$= \underbrace{A_1 d \cdot x}_{\text{nonlinear term}} + \dots$

Power Electronics State Space Average Summary

- Define the switching period as T_s, and assume there are two modes (e.g. switch on/off)
- Assume in every T_s mode 1 operates for dT_s and mode 2 operates for $(1-d)T_s$, where $d \in [0,\ 1]$ (duty cycle)
- Assume the modes are modeled by linear LTI systems, i.e.

$$\dot{x} = A_1 x + B_1 u \ \ \text{(switch is on)} \qquad \text{and} \qquad \dot{x} = A_2 x + B_2 u \ \ \text{(switch is off)}$$

- Then, the state space average model is given by:

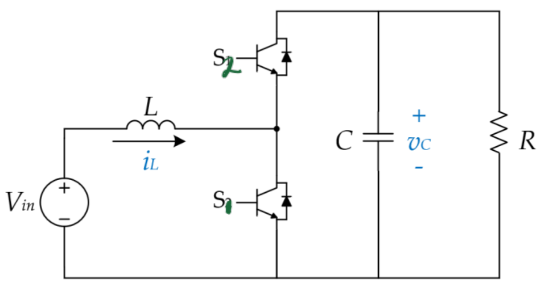
$$\dot{x} = \left(dA_1 + (1-d)A_2\right)x + \left(dB_1 + (1-d)B_2\right)u$$

$$\Rightarrow \quad \dot{x} = \underbrace{f\,(x,\ u,\ d)}$$ (nonlinear state space)

Caution: the state space average model is generally a nonlinear dynamic model with new inputs $u,\ d$

Boost Converter Example

- Derive the state space average model of a boost converter



State Space Average

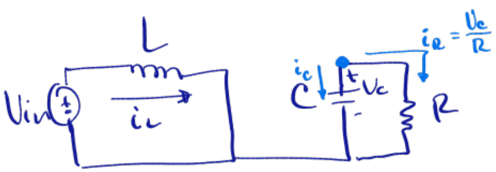
$$\dot{x} = \underbrace{\left(dA_1 + (1-d)A_2\right)}\, x + \underbrace{\left(dB_1 + (1-d)B_2\right)}\, V_{in}$$

$$d\begin{pmatrix} 0 & 0 \\ 0 & \frac{-1}{RC} \end{pmatrix} + (1-d)\begin{pmatrix} 0 & -\frac{1}{L} \\ \frac{1}{C} & \frac{-1}{RC} \end{pmatrix} = \underbrace{\begin{pmatrix} 0 & \frac{-(1-d)}{L} \\ \frac{(1-d)}{C} & \frac{-1}{RC} \end{pmatrix}}$$

$$d\begin{pmatrix} \frac{1}{L} \\ 0 \end{pmatrix} + (1-d)\begin{pmatrix} \frac{1}{L} \\ 0 \end{pmatrix} = \underbrace{\begin{pmatrix} \frac{1}{L} \\ 0 \end{pmatrix}}$$

$$d\frac{1}{L} + (1-d)\frac{1}{L} = \frac{1}{L}$$

$S_1 = 1$ (on) ⇒ $S_2 = 0$ (off) — Mode 1



$$x = \begin{pmatrix} i_L \\ V_c \end{pmatrix}$$

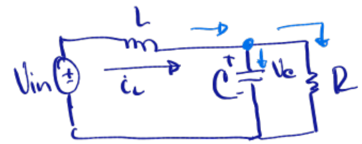
$$-V_{in} + L\frac{di_L}{dt} = 0 \longrightarrow \frac{di_L}{dt} = \frac{1}{L}V_{in}$$

$$i_c + i_R = 0 \rightarrow i_c = C\frac{dV_c}{dt} = -i_R \rightarrow \frac{dV_c}{dt} = -\frac{V_c}{RC}$$

$$\underbrace{\dot{x}} = \underbrace{\begin{pmatrix} \dot{i}_L \\ \dot{V}_c \end{pmatrix}} = \underbrace{\begin{pmatrix} 0 & 0 \\ 0 & \frac{-1}{RC} \end{pmatrix}}_{A_1} \underbrace{\begin{pmatrix} i_L \\ V_c \end{pmatrix}}_{x_1} + \underbrace{\begin{pmatrix} \frac{1}{L} \\ 0 \end{pmatrix}}_{B_1} V_{in}$$

$$\dot{x} = A_1 x_1 + B_1 V_{in}$$

$S_1 = 0$ ⇒ $S_2 = 1$ — Mode 2



$$x = \begin{pmatrix} i_L \\ V_c \end{pmatrix}$$

$$-V_{in} + L\frac{di_L}{dt} + V_c = 0 \longrightarrow \frac{di_L}{dt} = \frac{-1}{L}V_c + \frac{1}{L}V_{in}$$

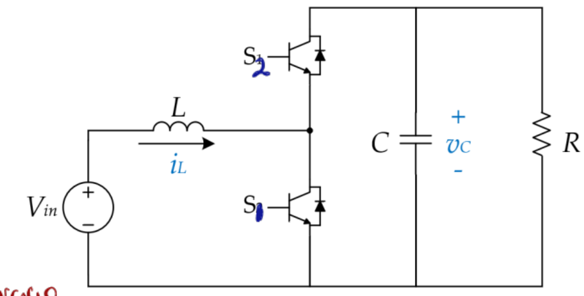
$$i_L = i_c + i_R \rightarrow i_c = C\frac{dV_c}{dt} = i_L - \frac{V_c}{R} \rightarrow \frac{dV_c}{dt} = \frac{1}{C}i_L - \frac{1}{RC}V_c$$

$$\dot{x} = A_2 x + B_2 V_{in}$$

$$\begin{pmatrix} \dot{i}_L \\ \dot{V}_c \end{pmatrix} = \underbrace{\begin{pmatrix} 0 & \frac{-1}{L} \\ \frac{1}{C} & \frac{-1}{RC} \end{pmatrix}}_{A_2} \begin{pmatrix} i_L \\ V_c \end{pmatrix} + \underbrace{\begin{pmatrix} \frac{1}{L} \\ 0 \end{pmatrix}}_{B_2} V_{in}$$

Boost Converter Example Simulation Results

- Derive the state space average model of a boost converter



$$\dot{x} = (dA_1 + (1-d)A_2)x + (dB_1 + (1-d)B_2)V_{in}$$

$d = u$ (input)

$$* \quad \dot{x} = \begin{pmatrix} \dot{i}_L \\ \dot{V}_c \end{pmatrix} = \underbrace{\begin{pmatrix} 0 & \frac{-(1-d)}{L} \\ \frac{1-d}{C} & \frac{-1}{RC} \end{pmatrix}}_{A_{avg}(d)} \begin{pmatrix} i_L \\ V_c \end{pmatrix} + \begin{pmatrix} \frac{1}{L} \\ 0 \end{pmatrix} V_{in}$$

* State Space Average

(UB Learns) $V_{out} = \frac{1}{1-d} V_{in}$

- State Space Average model can also be written in nonlinear form: $\dot{x} = f(x, V_{in}, d)$

$V_{in} = 270V$

$L = 1mH$

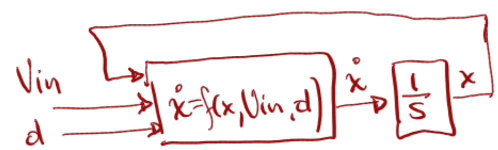
$C = 5mF$

$F_{sw} = 5kHz$

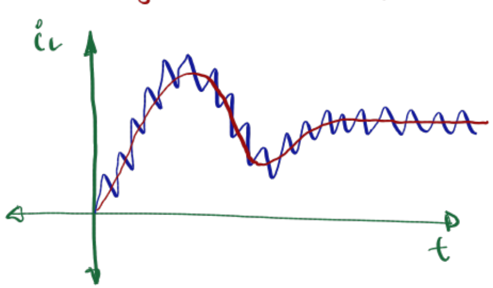
$R = 2\Omega$

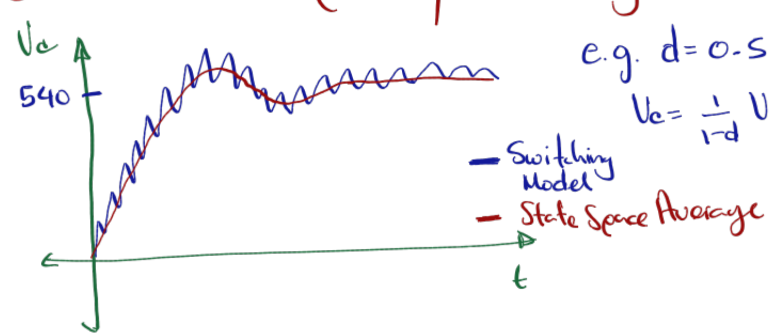
$$* \quad \dot{i}_L = \frac{-(1-d)}{L} V_c + \frac{1}{L} V_{in} = f_1(x, V_{in}, d)$$

$$** \quad \dot{V}_c = \frac{(1-d)}{C} i_L - \frac{1}{RC} V_c = f_2(x, V_{in}, d)$$



Simulate "Average" behavior of boost Converter and compare w/ Switching Model





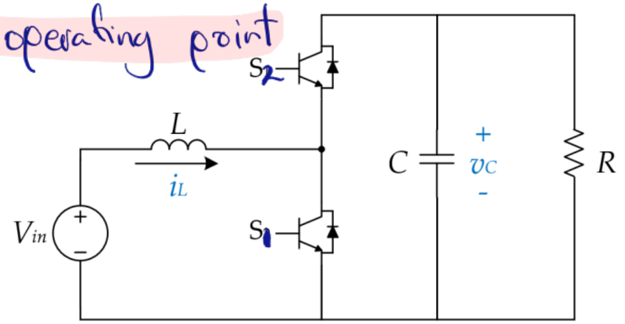
e.g. $d = 0.5$

$V_c = \frac{1}{1-d} V_{in}$

Boost Converter Example - Linearization

- Derive the state space average model of a boost converter $\dot{x} = f(x,u,d)$

- Obtain a linear approximation (linearize) around a certain operating point



$\Rightarrow$ first we need to find our "operating" points

$d = d^*$ $V_{in} = V_{in}^*$ (Given, fixed) $\rightarrow$ find equilibrium points (Steady State)

i_L^*, V_c^* or x^*

$V_{out} = V_c^* = \frac{1}{1-d^*} V_{in}^*$ $d \in [0,1)$

- Remember: at operating/equilibrium points $0 = f(x^*, V_{in}^*, d^*)$

- find x^* given d^*, V_{in}^*

$$L\left(0 = \frac{-(1-d^*)}{L} V_c^* + \frac{1}{L} V_{in}^* \right) \rightarrow \boxed{V_c^* = \frac{1}{(1-d^*)} V_{in}^*}$$

SAME!

$$C\left(0 = \frac{(1-d^*)}{C} i_L^* - \frac{1}{RC} V_c^* \right) \rightarrow \boxed{i_L^* = \frac{1}{(1-d^*)} \frac{V_c^*}{R} = \frac{1}{(1-d^*)^2} \frac{V_{in}^*}{R}}$$

- Obtain linearized model around x^*, V_{in}^*, d^*

$\tilde{x} = x - x^*$, $\tilde{V}_{in} = V_{in} - V_{in}^*$, $\tilde{d} = d - d^*$ $\longrightarrow$ $\dot{\tilde{x}} = A\tilde{x} + B_v \tilde{V}_{in} + B_d \tilde{d}$ (linearized)

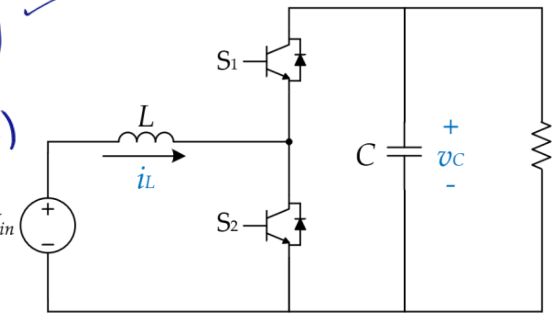
$$A = \left. \begin{pmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} \end{pmatrix} \right|_{x^*, V_{in}^*, d^*} \quad B_v = \left. \begin{pmatrix} \frac{\partial f_1}{\partial V_{in}} \\ \frac{\partial f_2}{\partial V_{in}} \end{pmatrix} \right|_{x^*, V_{in}^*, d^*} \quad B_d = \left. \begin{pmatrix} \frac{\partial f_1}{\partial d} \\ \frac{\partial f_2}{\partial d} \end{pmatrix} \right|_{x^*, V_{in}^*, d^*}$$

Boost Converter Example - Linearization

- Derive the state space average model of a boost converter

$x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} i_L \\ v_C \end{pmatrix}$

- find the linear approximation around x^*, d^*, V_{in}^* $\quad x^* = \begin{pmatrix} i_L^* \\ v_C^* \end{pmatrix}$



- Average Model: (nonlinear)

$$\frac{di_L}{dt} = \frac{-(1-d)}{L} v_C + \frac{1}{L} V_{in} = \frac{-1}{L} v_C + \frac{d}{L} v_C + \frac{1}{L} V_{in} = f_1(x, d, V_{in})$$

$$\frac{dv_C}{dt} = \frac{(1-d)}{C} i_L - \frac{1}{RC} v_C = \frac{i_L}{C} - \frac{d}{C} i_L - \frac{1}{RC} v_C = f_2(x, d, V_{in})$$

- find linearized model around $\tilde{x} = x - x^* \quad \tilde{V}_{in} = V_{in} - V_{in}^* \quad \tilde{d} = d - d^*$

$$\dot{\tilde{x}} = A\tilde{x} + B_v \tilde{V}_{in} + B_d \tilde{d}$$

- find $A = \begin{pmatrix} \frac{\partial f_1}{\partial i_L} & \frac{\partial f_1}{\partial v_C} \\ \frac{\partial f_2}{\partial i_L} & \frac{\partial f_2}{\partial v_C} \end{pmatrix}_{i_L^*, V_{in}^*, v_C^*, d^*} = \begin{bmatrix} 0 & \frac{-(1-d^*)}{L} \\ \frac{(1-d^*)}{C} & \frac{-1}{RC} \end{bmatrix}_{i_L^*, v_C^*, d^*, V_{in}^*}$, $\quad B_v = \begin{pmatrix} \frac{\partial f_1}{\partial V_{in}} \\ \frac{\partial f_2}{\partial V_{in}} \end{pmatrix}_{i_L^*, v_C^*, d^*, V_{in}^*} = \begin{pmatrix} \frac{1}{L} \\ 0 \end{pmatrix}$,

$B_d = \begin{pmatrix} \frac{\partial f_1}{\partial d} \\ \frac{\partial f_2}{\partial d} \end{pmatrix}_{i_L^*, v_C^*, d^*, V_{in}^*} = \begin{bmatrix} \frac{v_C^*}{L} \\ \frac{-i_L^*}{C} \end{bmatrix}$

$A = \begin{pmatrix} \frac{\partial f_1}{\partial x_1} & \cdots \\ \vdots & \ddots \end{pmatrix}$ allows us to analyze the local stability of the boost conv. @ x^*, V_{in}^*, d^*

→ by computing eigenvalues of A

ideally we cant $\mathrm{Re}\{\lambda_i\} < 0 \quad \forall i$

↓

loc. asymp. stable

Linearized Model

$$\begin{bmatrix} \dot{\tilde{i}}_L \\ \dot{\tilde{v}}_C \end{bmatrix} = \begin{bmatrix} 0 & \frac{-(1-d^*)}{L} \\ \frac{(1-d^*)}{C} & \frac{-1}{RC} \end{bmatrix} \begin{bmatrix} \tilde{i}_L \\ \tilde{v}_C \end{bmatrix} + \begin{bmatrix} \frac{1}{L} \\ 0 \end{bmatrix} \tilde{V}_{in} + \begin{bmatrix} \frac{v_C^*}{L} \\ \frac{-i_L^*}{C} \end{bmatrix} \tilde{d}$$

Keep in mind: $\tilde{x} = x - x^*$

→ if $\tilde{x} = 0 \Rightarrow x = x^*$

if $\tilde{d} = 0 \Rightarrow d = d^*$

Outline

- Overview of Modeling and Control

⇒ Modern Control

- State Space Modeling of Circuits
- Nonlinear Systems and Linearization
- Power Electronics State Space Averaging

- **Transfer Function Analysis** (Laplace Transforms)

⇒ Classical Control

PI, PID controllers, Root locus, Nyquist diagram, Bode plots, ...

Review of Laplace Transform

- The **Laplace Transform** $\mathcal{L}[f(t)]$ of $f(t)$ is the function defined as follows:

$$\mathcal{L}[f(t)] = F(s) = \int_0^\infty e^{-st} f(t)dt, \quad \forall s \in \mathbb{C} \ \text{ for which the integral exists}$$

$s = j\omega \Rightarrow$ frequency domain analysis.

- Find the Laplace Transform for the following functions
  - $e^{at} = f(t)$ $\longrightarrow$ $F(s) = \mathcal{L}\{f(t)\} = \int_0^\infty e^{-st} f(t)\,dt = \int_0^\infty e^{-st} e^{at}\,dt = \int_0^\infty e^{-(s-a)t}\,dt = \dots$

    $F(s) = \mathcal{L}\{e^{at}\} = \frac{1}{s-a}$

    Most of the time we can use Laplace transform tables
  - $\cos(\omega t)$

    $\mathcal{L}\{\cos(\omega t)\} = \frac{s}{s^2+\omega^2}$
  - $\sin(\omega t)$

    $\mathcal{L}\{\sin(\omega t)\} = \frac{\omega}{s^2+\omega^2}$
  - $e^{-at}\cos(\omega t)$ $\longrightarrow$ $\mathcal{L}\{e^{-at}\cos(\omega t)\} = \frac{s+a}{(s+a)^2+\omega^2}$
  - $e^{-at}\sin(\omega t)$ $\longrightarrow$ $\mathcal{L}\{e^{-at}\sin(\omega t)\} = \frac{\omega}{(s+a)^2+\omega^2}$

Properties of Laplace Transform

- The **convolution** $h(t) = f(t) * g(t)$ of two signals $f(t)$ and $g(t)$ is given by:

$$h(t) = \underbrace{\int_0^t f(\tau)g(t-\tau)d\tau}$$

- The Laplace Transform of $\mathcal{L}[h(t)]$ as defined above has an easier representation:

$$\mathcal{L}[h(t)] = \mathcal{L}[f(t) * g(t)] = \mathcal{L}[f(t)]\,\mathcal{L}[g(t)] = \underbrace{F(s)G(s)}$$

- Convolution in time domain ~~Laplace transform~~ is essentially multiplication! in frequency Domain.

- The Laplace Transform of a **time derivative** is $\underbrace{\mathcal{L}\left[\frac{df(t)}{dt}\right]} = \underline{sF(s) - f(0)}$

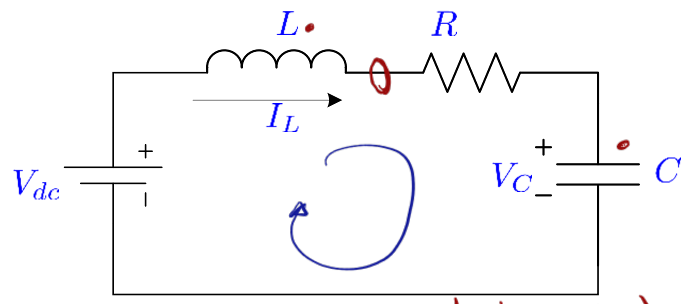
- The Laplace Transform of an **integral** is $\mathcal{L}\left[\int_0^t f(\tau)d\tau\right] = \frac{F(s)}{s}$

- Classical control methods (in Laplace/freq. Domain) are very useful for Single Input Single Output systems. (SISO)

Analyzing Electrical Circuits Using Laplace Transform

- The state space equations for the circuit are: (slide 29)

$$\begin{pmatrix} \dot{i}_L \\ \dot{v}_C \end{pmatrix} = \begin{pmatrix} \frac{-R}{L} & \frac{-1}{L} \\ \frac{1}{C} & 0 \end{pmatrix} \begin{pmatrix} i_L \\ v_C \end{pmatrix} + \begin{pmatrix} \frac{1}{L} \\ 0 \end{pmatrix} V_{\mathrm{dc}}$$



(RLC circuit, 2nd order system)

- How can we analyze this circuit using Laplace Transform?

- Single output ⇒ $i_L(t)$ (measure one state)

KVL $\mathcal{L}\left[-V_{dc} + L\frac{di_L}{dt} + Ri_L + \underbrace{\frac{1}{C}\int i_L dt}_{V_C(t)} = 0\right] \rightarrow -V_{dc}(s) + \underbrace{LsI_L(s) - \cancel{Li_L(0)}}_{\mathcal{L}\{L\frac{di_L}{dt}\}} + RI_L(s) + \frac{1}{C}\frac{I_L(s)}{s} = 0$

Assume $i_L(0) = 0$ ⇒ $\left[-V_{dc}(s) + LsI_L(s) + RI_L(s) + \frac{1}{C}\frac{I_L(s)}{s} = 0\right] s$

⇒ $-sV_{dc}(s) + Ls^2I_L(s) + RsI_L(s) + \frac{1}{C}I_L(s) = 0$

⋮ Simplify ⋮ ⋮

$I_L(s)\left[s^2L + Rs + \frac{1}{C}\right] = sV_{dc}(s)$ ⇒ $\boxed{I_L(s) = \left[\frac{s}{s^2L + Rs + \frac{1}{C}}\right] V_{dc}(s)}$

Transfer function

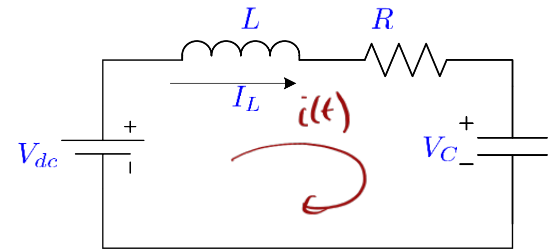
output input

$Y(s) = H(s)\,U(s)$

$\boxed{\frac{I_L(s)}{V_{dc}(s)} = \frac{s}{s^2L + Rs + \frac{1}{C}} = H(s)}$ I/O Transfer function

Analyzing Electrical Circuits Using Laplace Transform

- The state space equations for the circuit are: (slide 29)



$L = 1$ mH
$C = 1$ mF
$R = 0.2\ \Omega$

$$\begin{pmatrix} \dot{i}_L \\ \dot{v}_C \end{pmatrix} = \begin{pmatrix} \frac{-R}{L} & \frac{-1}{L} \\ \frac{1}{C} & 0 \end{pmatrix} \begin{pmatrix} i_L \\ v_C \end{pmatrix} + \begin{pmatrix} \frac{1}{L} \\ 0 \end{pmatrix} V_{\text{dc}}$$

→ Single output

- Assuming we are interested in finding the current, $I(t)$, the transfer function is now:

$$H(s) = \frac{I(s)}{V_{\text{dc}}(s)} = \frac{\frac{1}{L}s}{s^2 + \frac{R}{L}s + \frac{1}{LC}}$$

- If $V_{\text{dc}}(s) = 1$, how can we find $i(t)$? ⟶ $\frac{I(s)}{1} = I(s) = \frac{\frac{1}{L}s}{s^2+\frac{R}{L}s+\frac{1}{LC}} \Rightarrow i(t) = \mathcal{L}^{-1}\{I(s)\}$.

$\mathcal{L}^{-1}\downarrow \quad \uparrow\mathcal{L}$

$V_{dc}(t) = \delta(t)$ impulse function → impulse response.

$$i(t) = \mathcal{L}^{-1}\{I(s)\} = \mathcal{L}^{-1}\left\{\frac{\frac{1}{L}s}{s^2+\frac{R}{L}s+\frac{1}{LC}}\right\} \xrightarrow[\text{Decompe.}]{\text{Using Partial fraction}} \mathcal{L}^{-1}\left\{\frac{a_1}{s-p_1} + \frac{a_2}{s-p_2}\right\}$$

(find a_1, a_2, p_1, p_2)

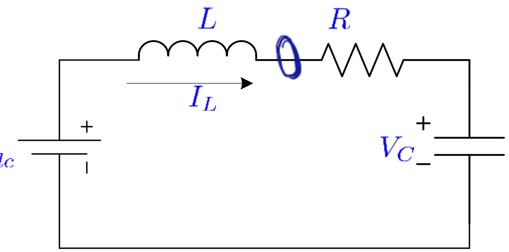
$$\mathcal{L}^{-1}\left\{\frac{\frac{1}{L}s}{(s-p_1)(s-p_2)}\right\}$$

p_1, p_2 are the roots of the denominator ⟺ poles of the transfer function

Analyzing Electrical Circuits Using Laplace Transform

- The state space equations for the circuit are: (slide 29)

$$\begin{pmatrix} \dot{i}_L \\ \dot{v}_C \end{pmatrix} = \underbrace{\begin{pmatrix} \frac{-R}{L} & \frac{-1}{L} \\ \frac{1}{C} & 0 \end{pmatrix}}_{A} \begin{pmatrix} i_L \\ v_C \end{pmatrix} + \begin{pmatrix} \frac{1}{L} \\ 0 \end{pmatrix} V_{\rm dc}, \quad y = (i_L(t)) = \overbrace{\begin{pmatrix} 1 & 0 \end{pmatrix}}^{C} \begin{pmatrix} i_L \\ v_C \end{pmatrix}$$



$L = 1$ mH

$C = 1$ mF

$R = 0.2\ \Omega$

- Assuming we are interested in finding the current, $I(t)$, the transfer function is now:

Transfer function $\quad \underline{H(s)} = \dfrac{I(s)}{V_{\rm dc}(s)} = \dfrac{\frac{1}{L}s}{s^2 + \frac{R}{L}s + \frac{1}{LC}}$

if $V_{dc}(s) = 1 \Rightarrow$ impulse response $\dfrac{I(s)}{1} = H(s)$

- What are the poles of $H(s)$? ~~What are the eigenvalues of A?~~

- Solve for impulse response of the current in time domain.

$$i(t) = \mathcal{L}^{-1}\{I(s)\} = \mathcal{L}^{-1}\left\{\frac{\frac{1}{L}s}{s^2 + \frac{R}{L}s + \frac{1}{LC}}\right\} = \mathcal{L}^{-1}\left\{\frac{\frac{1}{L}s}{(s-p_1)(s-p_2)}\right\} \overset{\text{PFD}}{=} \mathcal{L}^{-1}\left\{\frac{a_1}{s-p_1} + \frac{a_2}{s-p_2}\right\}$$

$$\Rightarrow i(t) = \mathcal{L}^{-1}\left\{\frac{a_1}{s-p_1}\right\} + \mathcal{L}^{-1}\left\{\frac{a_2}{s-p_2}\right\} = a_1 e^{p_1 t} + a_2 e^{p_2 t}$$

$$e^{j\omega_i t} = \cos(\omega_i t) + j\sin(\omega_i t)$$

$$\boxed{\text{impulse response} \quad i(t) = a_1 e^{p_1 t} + a_2 e^{p_2 t}}$$

$p_1, p_2 \in \mathbb{C} \quad p_i = a_i + j\omega_i \quad i = 1, 2.$

exp. decay $= \mathrm{Re}\{p_i\} = a_i \quad$ oscillation: $\mathrm{Im}(p_i) = \omega_i$

Laplace Transform of State Space Systems

- Consider the following linear state space system:

$$\begin{cases} \dot{x} = Ax + Bu \\ y = Cx \end{cases}$$

$x \in \mathbb{R}^n$ (states), $u \in \mathbb{R}^1$ (input)
n = order of system
$y \in \mathbb{R}^1$ (output) or measurement

- We can also solve for $x(t)$ using Laplace Transforms!

- And we can also obtain $\frac{Y(s)}{U(s)} = H(s)$ using Laplace from state space model

Time domain → Freq. Domain

(Modern Control) $\dot{x} = Ax + Bu$, $y = Cx$ → $\frac{Y(s)}{U(s)} = H(s) = \underbrace{C(sI-A)^{-1}B}_{1\times 1}$ (SISO) (Classical Control)

- find $\frac{Y(s)}{U(s)}$ from state space model

- $\mathcal{L}\{\dot{x} = Ax + Bu\} \Rightarrow sX(s) - x(0) = AX(s) + BU(s) \longrightarrow sX(s) - AX(s) = x(0) + BU(s)$

$\Rightarrow (sI - A)X(s) = x(0) + BU(s)$

$$\boxed{X(s) = (sI-A)^{-1}x(0) + (sI-A)^{-1}BU(s)}$$

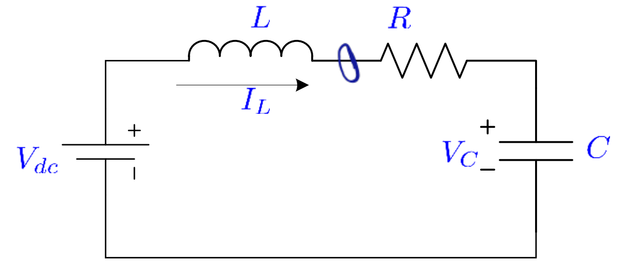
- $\mathcal{L}\{y(t) = Cx(t)\} \Rightarrow Y(s) = CX(s) = C(sI-A)^{-1}x(0) + C(sI-A)^{-1}BU(s)$ if $x(0) = 0$ $\boxed{\frac{Y(s)}{U(s)} = C(sI-A)^{-1}B}$

- The state space equations for the circuit are: (slide 29)

$$\begin{pmatrix} \dot{i}_L \\ \dot{v}_C \end{pmatrix} = \overbrace{\begin{pmatrix} \frac{-R}{L} & \frac{-1}{L} \\ \frac{1}{C} & 0 \end{pmatrix}}^{A} \begin{pmatrix} i_L \\ v_C \end{pmatrix} + \overbrace{\begin{pmatrix} \frac{1}{L} \\ 0 \end{pmatrix}}^{B} V_{\mathrm{dc}}$$

$$i_L = \underbrace{\begin{pmatrix} 1 & 0 \end{pmatrix}}_{C} \begin{pmatrix} i_L \\ v_C \end{pmatrix}$$

Single input
Single output (SISO)



- Obtain the transfer function $H(s) = I(s)/V_{\mathrm{dc}}(s)$ from the state space model

$$\frac{I(s)}{V_{dc}(s)} = C(sI-A)^{-1}B = \underset{1\times 2}{[1 \quad 0]} \underbrace{\left(? \right)}_{\substack{(sI-A)^{-1} \\ 2\times 2}} \underset{2\times 1}{\begin{bmatrix} \frac{1}{L} \\ 0 \end{bmatrix}} = [1 \quad 0] \frac{\mathrm{adj}(sI-A)}{\det(sI-A)} \begin{bmatrix} \frac{1}{L} \\ 0 \end{bmatrix}$$

$$= [1 \quad 0] \frac{\begin{bmatrix} s & -\frac{1}{L} \\ \frac{1}{C} & s+\frac{R}{L} \end{bmatrix}}{s^2 + \frac{R}{L}s + \frac{1}{LC}} \begin{bmatrix} \frac{1}{L} \\ 0 \end{bmatrix}$$

$$\boxed{\frac{I(s)}{V_{dc}(s)} = \frac{1}{\det(sI-A)} [1 \quad 0] \begin{bmatrix} \frac{s}{L} \\ \frac{1}{LC} \end{bmatrix} = \frac{s/L}{s^2 + \frac{R}{L}s + \frac{1}{LC}}}$$

$$(sI-A)^{-1} = \frac{\mathrm{adj}(sI-A)}{\det(sI-A)}$$

$$(sI-A) = s\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} -\frac{R}{L} & -\frac{1}{L} \\ \frac{1}{C} & 0 \end{bmatrix} = \begin{bmatrix} s+\frac{R}{L} & \frac{1}{L} \\ -\frac{1}{C} & s \end{bmatrix}$$

$$\mathrm{adj}(sI-A) = \begin{bmatrix} s & -\frac{1}{L} \\ \frac{1}{C} & s+\frac{R}{L} \end{bmatrix}$$

$$\det(sI-A) = \left(s+\frac{R}{L}\right)s + \frac{1}{LC} = s^2 + \frac{R}{L}s + \frac{1}{LC}$$

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} = A$$

$$\mathrm{adj}(A) = \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

Comparison of State Space vs Transfer Function

- Consider the following linear state space system:

$$\dot{x} = Ax + Bu$$
$$y = Cx$$

- The output in s domain can be obtained as follows:

$$\frac{Y(s)}{U(s)} = C\,(sI - A)^{-1}\,B = C\frac{\text{adj}(sI - A)}{\det(sI - A)}B$$

- $\frac{Y(s)}{U(s)} = H(s) = \frac{\text{numerator}(s)}{\text{denominator}(s)} \Rightarrow$ poles = roots of the denominator.

$\overset{PFD}{=} \frac{a_1}{s-p_1} + \frac{a_2}{s-p_2} + \cdots + \frac{a_n}{s-p_n}$

- $\frac{Y(s)}{U(s)} = C(sI-A)^{-1}B = C\frac{\text{adj}(sI-A)}{\det(sI-A)}B$

$\Rightarrow$ denominator$(s) = \det(sI-A)$. roots of $\det(sI-A)$ = eigenvalues of A

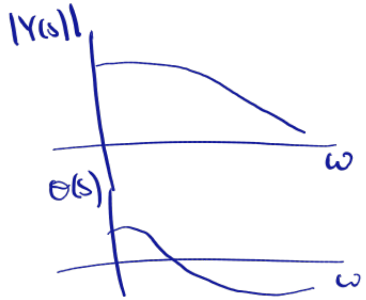
if $\text{Re}\{\lambda_i\} = \text{Re}\{p_i\} < 0 \;\forall i \Rightarrow$ asymptotically stable.

Summary of Transfer Functions

- Consider the following linear state space system:

$$\dot{x} = Ax + Bu$$
$$y = Cx$$

- The output in s domain can be obtained as follows:



$$\frac{Y(s)}{U(s)} = C\,(sI - A)^{-1}\,B = C\frac{\operatorname{adj}(sI - A)}{\det(sI - A)}B$$

- **In state space, we can use the entire set of states (or multiple outputs) as well as include multiple inputs: Multiple Input Multiple Output (MIMO)**

- **A transfer function gives information only on a Single Input Single Output (SISO)**

- **When we design controllers with transfer functions (e.g. PI, PID) , we only have information on a single output (disadvantage)**

- In frequency domain, we can analyze the stability of the system using bode plots or Nyquist criteria

Key Points

- Why is it important to control power electronics?
- To use model based controller design, it is necessary to have a dynamic model of the converter
- Power electronics are switching (hybrid) dynamic systems
- The dynamic model of power converters can be simplified by using **state space average**
- However, many average models of power converters are nonlinear
- Linearization can then be used to simplify the analysis further

Next Topic

- Contoller design for power electronics using State Space techniques!
  - State feedback ($u = Kx$)
  - Integral control (current/voltage tracking)
  - Observers*
- Many more we may not cover (e.g. optimal control/linear quadratic regulator, model predictive control, feedback linearization, etc.)

References

- Useful references in control (state space) and applications in power electronics

[1] Friedland, Bernard. Control system design: an introduction to state-space methods. Courier Corporation, 2012. (Modern control)

[2] Dorf, Richard C., and Robert H. Bishop. Modern control systems. Pearson, 2011. (classical book, more focus on transfer functions)

[3] H. K. Khalil, Nonlinear Systems. New York: Macmillan, 1992 (nonlinear control)

[4] Erickson, Robert W., and Dragan Maksimovic. Fundamentals of power electronics. Springer Science & Business Media, 2007 (Classical control for power electronics)

[5] Power Electronics: Converters, Applications, and Design, 3rd Edition by Ned Mohan, Tore M. Undeland, William P. Robbins; October 2002

Modern Control