

Problem 4.31 The circular disk of radius a shown in Fig. 4-7 has uniform charge density ρ_s across its surface.

- (a) Obtain an expression for the electric potential V at a point $P = (0, 0, z)$ on the z -axis.
- (b) Use your result to find $\mathbf{E}$ and then evaluate it for $z = h$. Compare your final expression with (4.24), which was obtained on the basis of Coulomb's law.

Solution:

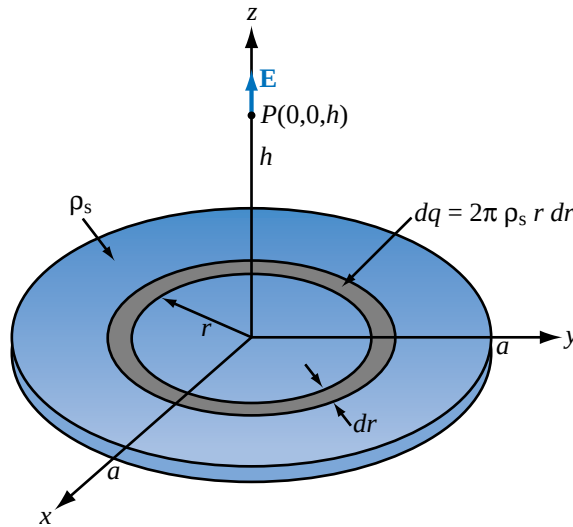


Figure P4.31: Circular disk of charge.

(a) Consider a ring of charge at a radial distance r . The charge contained in width dr is

$$dq = \rho_s(2\pi r dr) = 2\pi\rho_s r dr.$$

The potential at P is

$$dV = \frac{dq}{4\pi\epsilon_0 R} = \frac{2\pi\rho_s r dr}{4\pi\epsilon_0 (r^2 + z^2)^{1/2}}.$$

The potential due to the entire disk is

$$V = \int_0^a dV = \frac{\rho_s}{2\epsilon_0} \int_0^a \frac{r dr}{(r^2 + z^2)^{1/2}} = \frac{\rho_s}{2\epsilon_0} (r^2 + z^2)^{1/2} \Big|_0^a = \frac{\rho_s}{2\epsilon_0} \left[(a^2 + z^2)^{1/2} - z \right].$$

(b)

$$\mathbf{E} = -\nabla V = -\hat{\mathbf{x}} \frac{\partial V}{\partial x} - \hat{\mathbf{y}} \frac{\partial V}{\partial y} - \hat{\mathbf{z}} \frac{\partial V}{\partial z} = \hat{\mathbf{z}} \frac{\rho_s}{2\epsilon_0} \left[1 - \frac{z}{\sqrt{a^2 + z^2}} \right].$$

The expression for $\mathbf{E}$ reduces to Eq. (4.24) when $z = h$.

Problem 4.34 Find the electric potential V at a location a distance b from the origin in the x - y plane due to a line charge with charge density ρ_ℓ and of length l . The line charge is coincident with the z -axis and extends from $z = -l/2$ to $z = l/2$.

Solution: From Eq. (4.48c), we can find the voltage at a distance b away from a line

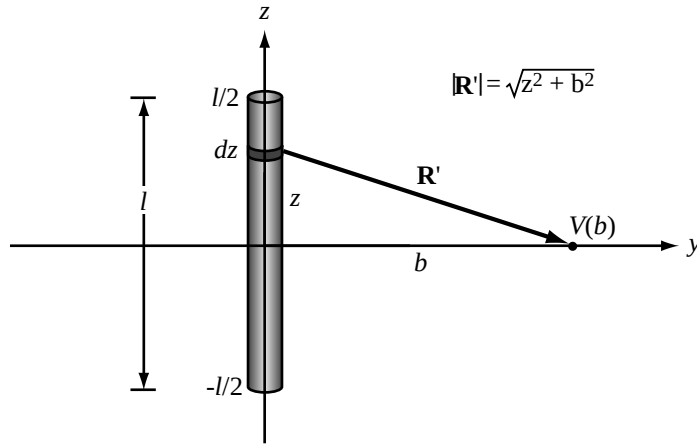


Figure P4.34: Line of charge of length ℓ .

of charge [Fig. P4.34]:

$$V(b) = \frac{1}{4\pi\epsilon} \int_{l'} \frac{\rho_l}{R'} dl' = \frac{\rho_l}{4\pi\epsilon} \int_{-l/2}^{l/2} \frac{dz}{\sqrt{z^2 + b^2}} = \frac{\rho_l}{4\pi\epsilon} \ln \left(\frac{l + \sqrt{l^2 + 4b^2}}{-l + \sqrt{l^2 + 4b^2}} \right).$$

Problem 4.38 Given the electric field

$$\mathbf{E} = \hat{\mathbf{R}} \frac{18}{R^2} \quad (\text{V/m})$$

find the electric potential of point A with respect to point B where A is at $+2$ m and B at -4 m, both on the z -axis.

Solution:

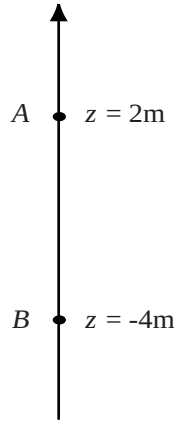


Figure P4.38: Potential between B and A .

$$V_{AB} = V_A - V_B = - \int_B^A \mathbf{E} \cdot d\mathbf{l}.$$

Along z -direction, $\hat{\mathbf{R}} = \hat{\mathbf{z}}$ and $\mathbf{E} = \hat{\mathbf{z}} \frac{18}{z^2}$ for $z \geq 0$, and $\hat{\mathbf{R}} = -\hat{\mathbf{z}}$ and $\mathbf{E} = -\hat{\mathbf{z}} \frac{18}{z^2}$ for $z \leq 0$. Hence,

$$V_{AB} = - \int_{-4}^2 \hat{\mathbf{R}} \frac{18}{z^2} \cdot \hat{\mathbf{z}} dz = - \left[\int_{-4}^0 -\hat{\mathbf{z}} \frac{18}{z^2} \cdot \hat{\mathbf{z}} dz + \int_0^2 \hat{\mathbf{z}} \frac{18}{z^2} \cdot \hat{\mathbf{z}} dz \right] = 4 \text{ V}.$$

Problem 4.48 With reference to Fig. 4-19, find $\mathbf{E}_1$ if $\mathbf{E}_2 = \hat{\mathbf{x}}3 - \hat{\mathbf{y}}2 + \hat{\mathbf{z}}2$ (V/m), $\epsilon_1 = 2\epsilon_0$, $\epsilon_2 = 18\epsilon_0$, and the boundary has a surface charge density $\rho_s = 3.54 \times 10^{-11}$ (C/m²). What angle does $\mathbf{E}_2$ make with the z-axis?

Solution: We know that $\mathbf{E}_{1t} = \mathbf{E}_{2t}$ for any 2 media. Hence, $\mathbf{E}_{1t} = \mathbf{E}_{2t} = \hat{\mathbf{x}}3 - \hat{\mathbf{y}}2$. Also, $(\mathbf{D}_1 - \mathbf{D}_2) \cdot \hat{\mathbf{n}} = \rho_s$ (from Table 4.3). Hence, $\epsilon_1(\mathbf{E}_1 \cdot \hat{\mathbf{n}}) - \epsilon_2(\mathbf{E}_2 \cdot \hat{\mathbf{n}}) = \rho_s$, which gives

$$E_{1z} = \frac{\rho_s + \epsilon_2 E_{2z}}{\epsilon_1} = \frac{3.54 \times 10^{-11}}{2\epsilon_0} + \frac{18(2)}{2} = \frac{3.54 \times 10^{-11}}{2 \times 8.85 \times 10^{-12}} + 18 = 20 \quad (\text{V/m}).$$

Hence, $\mathbf{E}_1 = \hat{\mathbf{x}}3 - \hat{\mathbf{y}}2 + \hat{\mathbf{z}}20$ (V/m). Finding the angle $\mathbf{E}_2$ makes with the z-axis:

$$\mathbf{E}_2 \cdot \hat{\mathbf{z}} = |\mathbf{E}_2| \cos \theta, \quad 2 = \sqrt{9+4+4} \cos \theta, \quad \theta = \cos^{-1} \left(\frac{2}{\sqrt{17}} \right) = 61^\circ.$$

Problem 4.55 In a dielectric medium with $\epsilon_r = 4$, the electric field is given by

$$\mathbf{E} = \hat{\mathbf{x}}(x^2 + 2z) + \hat{\mathbf{y}}x^2 - \hat{\mathbf{z}}(y + z) \quad (\text{V/m})$$

Calculate the electrostatic energy stored in the region $-1 \text{ m} \leq x \leq 1 \text{ m}$, $0 \leq y \leq 2 \text{ m}$, and $0 \leq z \leq 3 \text{ m}$.

Solution: Electrostatic potential energy is given by Eq. (4.124),

$$\begin{aligned} W_e &= \frac{1}{2} \int_V \epsilon |\mathbf{E}|^2 dV = \frac{\epsilon}{2} \int_{z=0}^3 \int_{y=0}^2 \int_{x=-1}^1 [(x^2 + 2z)^2 + x^4 + (y + z)^2] dx dy dz \\ &= \frac{4\epsilon_0}{2} \left(\left(\left(\frac{2}{5} x^5 y z + \frac{2}{3} z^2 x^3 y + \frac{4}{3} z^3 x y + \frac{1}{12} (y + z)^4 x \right) \right) \bigg|_{x=-1}^1 \right) \bigg|_{y=0}^2 \bigg|_{z=0}^3 \\ &= \frac{4\epsilon_0}{2} \left(\frac{1304}{5} \right) = 4.62 \times 10^{-9} \quad (\text{J}). \end{aligned}$$

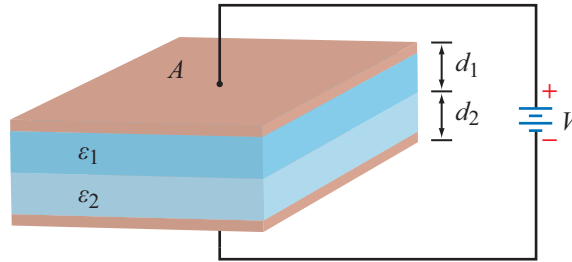
Problem 4.58 The capacitor shown in Fig. P4.58 consists of two parallel dielectric layers. Use energy considerations to show that the equivalent capacitance of the overall capacitor, C , is equal to the series combination of the capacitances of the individual layers, C_1 and C_2 , namely

$$C = \frac{C_1 C_2}{C_1 + C_2} \quad (22)$$

where

$$C_1 = \epsilon_1 \frac{A}{d_1}, \quad C_2 = \epsilon_2 \frac{A}{d_2}$$

- (a) Let V_1 and V_2 be the electric potentials across the upper and lower dielectrics, respectively. What are the corresponding electric fields E_1 and E_2 ? By applying the appropriate boundary condition at the interface between the two dielectrics, obtain explicit expressions for E_1 and E_2 in terms of ϵ_1 , ϵ_2 , V , and the indicated dimensions of the capacitor.
- (b) Calculate the energy stored in each of the dielectric layers and then use the sum to obtain an expression for C .
- (c) Show that C is given by Eq. (22).



(a)



(b)

Figure P4.58: (a) Capacitor with parallel dielectric layers, and (b) equivalent circuit (Problem 4.58).

Solution:

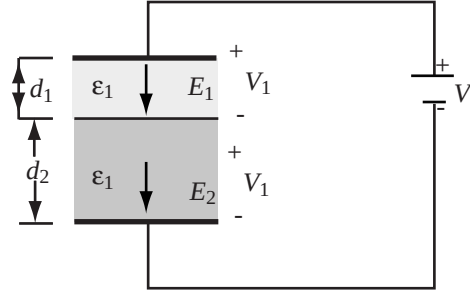


Figure P4.58: (c) Electric fields inside of capacitor.

(a) If V_1 is the voltage across the top layer and V_2 across the bottom layer, then

$$V = V_1 + V_2,$$

and

$$E_1 = \frac{V_1}{d_1}, \quad E_2 = \frac{V_2}{d_2}.$$

According to boundary conditions, the normal component of $\mathbf{D}$ is continuous across the boundary (in the absence of surface charge). This means that at the interface between the two dielectric layers,

$$D_{1n} = D_{2n}$$

or

$$\epsilon_1 E_1 = \epsilon_2 E_2.$$

Hence,

$$V = E_1 d_1 + E_2 d_2 = E_1 d_1 + \frac{\epsilon_1 E_1}{\epsilon_2} d_2,$$

which can be solved for E_1 :

$$E_1 = \frac{V}{d_1 + \frac{\epsilon_1}{\epsilon_2} d_2}.$$

Similarly,

$$E_2 = \frac{V}{d_2 + \frac{\epsilon_2}{\epsilon_1} d_1}.$$

(b)

$$\begin{aligned}W_{e_1} &= \frac{1}{2} \varepsilon_1 E_1^2 \cdot \gamma_1 = \frac{1}{2} \varepsilon_1 \left(\frac{V}{d_1 + \frac{\varepsilon_1}{\varepsilon_2} d_2} \right)^2 \cdot A d_1 = \frac{1}{2} V^2 \left[\frac{\varepsilon_1 \varepsilon_2^2 A d_1}{(\varepsilon_2 d_1 + \varepsilon_1 d_2)^2} \right], \\W_{e_2} &= \frac{1}{2} \varepsilon_2 E_2^2 \cdot \gamma_2 = \frac{1}{2} \varepsilon_2 \left(\frac{V}{d_2 + \frac{\varepsilon_2}{\varepsilon_1} d_1} \right)^2 \cdot A d_2 = \frac{1}{2} V^2 \left[\frac{\varepsilon_1^2 \varepsilon_2 A d_2}{(\varepsilon_1 d_2 + \varepsilon_2 d_1)^2} \right], \\W_e &= W_{e_1} + W_{e_2} = \frac{1}{2} V^2 \left[\frac{\varepsilon_1 \varepsilon_2^2 A d_1 + \varepsilon_1^2 \varepsilon_2 A d_2}{(\varepsilon_1 d_2 + \varepsilon_2 d_1)^2} \right].\end{aligned}$$

But $W_e = \frac{1}{2} C V^2$, hence,

$$C = \frac{\varepsilon_1 \varepsilon_2^2 A d_1 + \varepsilon_1^2 \varepsilon_2 A d_2}{(\varepsilon_2 d_1 + \varepsilon_1 d_2)^2} = \varepsilon_1 \varepsilon_2 A \frac{(\varepsilon_2 d_1 + \varepsilon_1 d_2)}{(\varepsilon_2 d_1 + \varepsilon_1 d_2)^2} = \frac{\varepsilon_1 \varepsilon_2 A}{\varepsilon_2 d_1 + \varepsilon_1 d_2}.$$

(c) Multiplying numerator and denominator of the expression for C by $A/d_1 d_2$, we have

$$C = \frac{\frac{\varepsilon_1 A}{d_1} \cdot \frac{\varepsilon_2 A}{d_2}}{\frac{\varepsilon_1 A}{d_1} + \frac{\varepsilon_2 A}{d_2}} = \frac{C_1 C_2}{C_1 + C_2},$$

where

$$C_1 = \frac{\varepsilon_1 A}{d_1}, \quad C_2 = \frac{\varepsilon_2 A}{d_2}.$$

Problem 4.62 Conducting wires above a conducting plane carry currents I_1 and I_2 in the directions shown in Fig. P4.62. Keeping in mind that the direction of a current is defined in terms of the movement of positive charges, what are the directions of the image currents corresponding to I_1 and I_2 ?

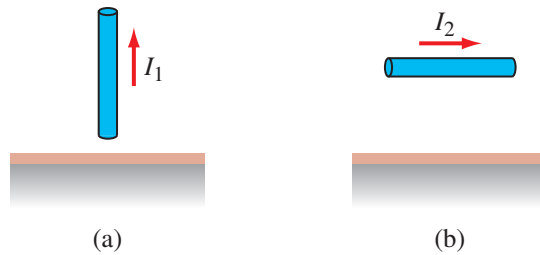


Figure P4.62: Currents above a conducting plane (Problem 4.62).

Solution:

(a) In the image current, movement of negative charges downward = movement of positive charges upward. Hence, image of I_1 is same as I_1 .

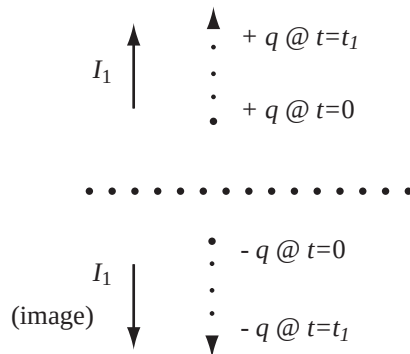


Figure P4.62(a): Solution for part (a).

(b) In the image current, movement of negative charges to right = movement of positive charges to left.

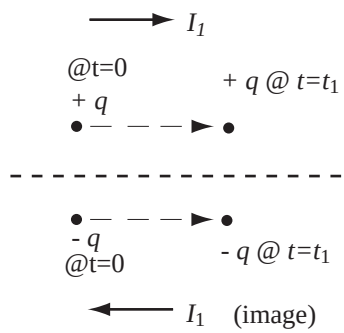


Figure P4.62(b): Solution for part (b).

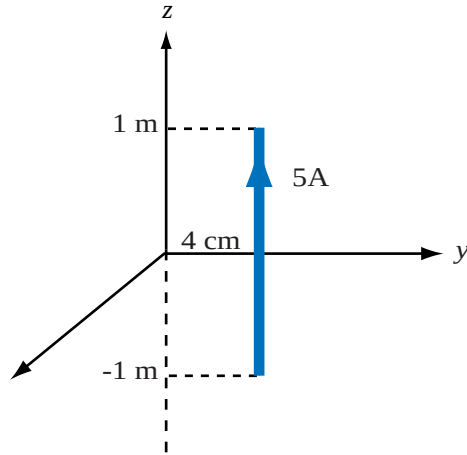


Figure P5.5: Problem 5.5.

Problem 5.5 In a cylindrical coordinate system, a 2-m-long straight wire carrying a current of 5 A in the positive z -direction is located at $r = 4$ cm, $\phi = \pi/2$, and $-1 \text{ m} \leq z \leq 1 \text{ m}$.

- (a) If $\mathbf{B} = \hat{\mathbf{r}}0.2 \cos \phi$ (T), what is the magnetic force acting on the wire?
- (b) How much work is required to rotate the wire once about the z -axis in the negative ϕ -direction (while maintaining $r = 4$ cm)?
- (c) At what angle ϕ is the force a maximum?

Solution:

(a)

$$\begin{aligned}\mathbf{F} &= I\ell \times \mathbf{B} \\ &= 5\hat{\mathbf{z}}2 \times [\hat{\mathbf{r}}0.2 \cos \phi] \\ &= \hat{\boldsymbol{\phi}}2 \cos \phi.\end{aligned}$$

At $\phi = \pi/2$, $\hat{\boldsymbol{\phi}} = -\hat{\mathbf{x}}$. Hence,

$$\mathbf{F} = -\hat{\mathbf{x}}2 \cos(\pi/2) = 0.$$

(b)

$$\begin{aligned}W &= \int_{\phi=0}^{2\pi} \mathbf{F} \cdot d\mathbf{l} = \int_0^{2\pi} \hat{\boldsymbol{\phi}}[2 \cos \phi] \cdot (-\hat{\boldsymbol{\phi}})r d\phi \Big|_{r=4 \text{ cm}} \\ &= -2r \int_0^{2\pi} \cos \phi d\phi \Big|_{r=4 \text{ cm}} = -8 \times 10^{-2} [\sin \phi]_0^{2\pi} = 0.\end{aligned}$$

The force is in the $+\hat{\phi}$ -direction, which means that rotating it in the $-\hat{\phi}$ -direction would require work. However, the force varies as $\cos\phi$, which means it is positive when $-\pi/2 \leq \phi \leq \pi/2$ and negative over the second half of the circle. Thus, work is provided by the force between $\phi = \pi/2$ and $\phi = -\pi/2$ (when rotated in the $-\hat{\phi}$ -direction), and work is supplied for the second half of the rotation, resulting in a net work of zero.

(c) The force $\mathbf{F}$ is maximum when $\cos\phi = 1$, or $\phi = 0$.

Problem 5.12 Two infinitely long, parallel wires are carrying 6-A currents in opposite directions. Determine the magnetic flux density at point P in Fig. P5.12.

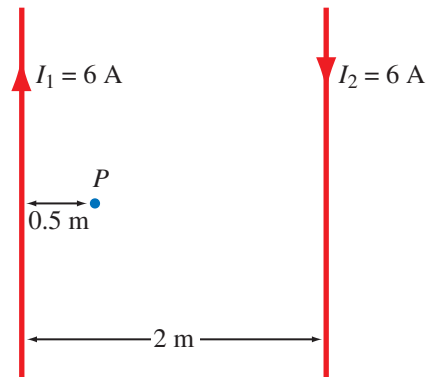


Figure P5.12: Arrangement for Problem 5.12.

Solution:

$$\mathbf{B} = \hat{\phi} \frac{\mu_0 I_1}{2\pi(0.5)} + \hat{\phi} \frac{\mu_0 I_2}{2\pi(1.5)} = \hat{\phi} \frac{\mu_0}{\pi} (6 + 2) = \hat{\phi} \frac{8\mu_0}{\pi} \quad (\text{T}).$$

Problem 5.19 Three long, parallel wires are arranged as shown in Fig. P5.19. Determine the force per unit length acting on the wire carrying I_3 .

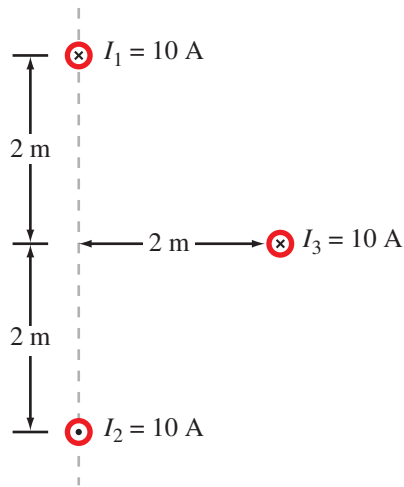


Figure P5.19: Three parallel wires of Problem 5.19.

Solution: Since I_1 and I_2 are equal in magnitude and opposite in direction, and

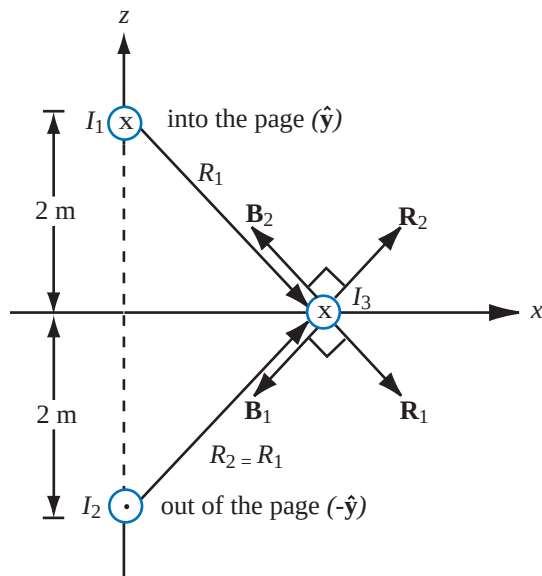


Figure P5.19: (a) $\mathbf{B}$ fields due to I_1 and I_2 at location of I_3 .

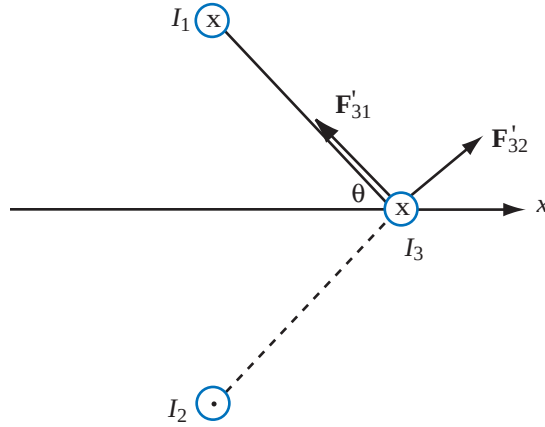


Figure P5.19: (b) Forces acting on I_3 .

equidistant from I_3 , our intuitive answer might be that the net force on I_3 is zero. As we will see, that's not the correct answer. The field due to I_1 (which is along $\hat{\mathbf{y}}$) at location of I_3 is

$$\mathbf{B}_1 = \hat{\mathbf{b}}_1 \frac{\mu_0 I_1}{2\pi R_1}$$

where $\hat{\mathbf{b}}_1$ is the unit vector in the direction of $\mathbf{B}_1$ shown in the figure, which is perpendicular to $\hat{\mathbf{r}}_1$. The force per unit length exerted on I_3 is

$$\mathbf{F}'_{31} = \frac{\mu_0 I_1 I_3}{2\pi R_1} (\hat{\mathbf{y}} \times \hat{\mathbf{b}}_1) = -\hat{\mathbf{r}}_1 \frac{\mu_0 I_1 I_3}{2\pi R_1}.$$

Similarly, the force per unit length excited on I_3 by the field due to I_2 (which is along $-\hat{\mathbf{y}}$) is

$$\mathbf{F}'_{32} = \hat{\mathbf{r}}_2 \frac{\mu_0 I_2 I_3}{2\pi R_2}.$$

The two forces have opposite components along $\hat{\mathbf{x}}$ and equal components along $\hat{\mathbf{z}}$. Hence, with $R_1 = R_2 = \sqrt{8}$ m and $\theta = \sin^{-1}(2/\sqrt{8}) = \sin^{-1}(1/\sqrt{2}) = 45^\circ$,

$$\begin{aligned} \mathbf{F}'_3 &= \mathbf{F}'_{31} + \mathbf{F}'_{32} = \hat{\mathbf{z}} \left(\frac{\mu_0 I_1 I_3}{2\pi R_1} + \frac{\mu_0 I_2 I_3}{2\pi R_2} \right) \sin \theta \\ &= \hat{\mathbf{z}} 2 \left(\frac{4\pi \times 10^{-7} \times 10 \times 20}{2\pi \times \sqrt{8}} \right) \times \frac{1}{\sqrt{2}} = \hat{\mathbf{z}} 2 \times 10^{-5} \text{ N/m}. \end{aligned}$$

Problem 5.20 A square loop placed as shown in Fig. P5.20 has 2-m sides and carries a current $I_1 = 5$ A. If a straight, long conductor carrying a current $I_2 = 10$ A is introduced and placed just above the midpoints of two of the loop's sides, determine the net force acting on the loop.

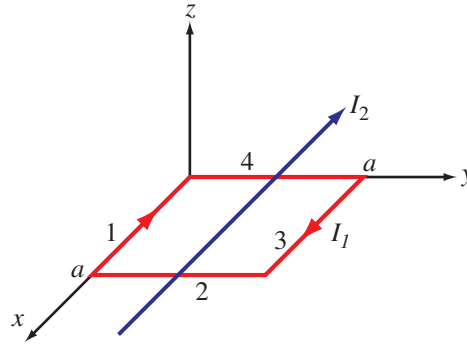


Figure P5.20: Long wire carrying current I_2 , just above a square loop carrying I_1 (Problem 5.20).

Solution: Since I_2 is just barely above the loop, we can treat it as if it's in the same plane as the loop. For side 1, I_1 and I_2 are in the same direction, hence the force on side 1 is attractive. That is,

$$\mathbf{F}_1 = \hat{\mathbf{y}} \frac{\mu_0 I_1 I_2 a}{2\pi(a/2)} = \hat{\mathbf{y}} \frac{4\pi \times 10^{-7} \times 5 \times 10 \times 2}{2\pi \times 1} = \hat{\mathbf{y}} 2 \times 10^{-5} \text{ N}.$$

I_1 and I_2 are in opposite directions for side 3. Hence, the force on side 3 is repulsive, which means it is also along $\hat{\mathbf{y}}$. That is, $\mathbf{F}_3 = \mathbf{F}_1$.

The net forces on sides 2 and 4 are zero. Total net force on the loop is

$$\mathbf{F} = 2\mathbf{F}_1 = \hat{\mathbf{y}} 4 \times 10^{-5} \text{ N}.$$
