
Steady-State Vehicle Optimization Using Pareto-Minimum Analysis

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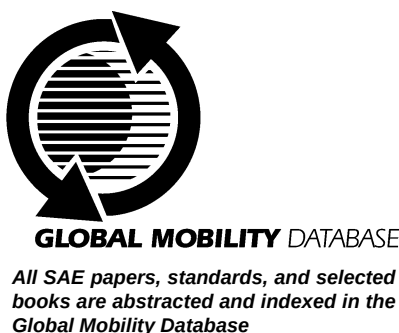
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ABSTRACT

Designing for optimal performance across a variety of situations involves compromise decisions. Through the investigation of a two-variable optimization of a vehicle for two different "races" the importance of this compromise design is underscored. The use of Pareto-minimal solution techniques, borrowed from game theory, aid in the design process by limiting the number of possible compromise designs, highlighting which solution applies for a given situation and providing some insight to the sensitivity of the design.

INTRODUCTION

The design of an automobile or race car necessarily involves a series of compromises. Such vehicles are expected to perform well in a variety of different situations. Each situation or task has its own optimum vehicle design which will maximize performance for that particular situation. It is unlikely the same design will be optimal for every task. Designers attempt to find a design which is optimal across the aggregate of situations considered. This design will be a compromise among the various individual situation optimum designs.

In mathematics, the field of game theory studies the interactions between competing "players" [1]. These players have differing goals but often control the same variables. In automotive design, each situation can be considered a "player". Each situation wants to be optimized but all situations are competing for the same vehicle design variables. Thus, the design of various aspects of an automobile can be considered a game which can be analyzed using game theory methodology. Specifically, Vincent [3] describes the Pareto-minimal analysis method which is a technique for finding optimum compromise solutions.

In this paper game theory techniques are applied to a specific design problem. Consider two designers (i.e. "players"), each of whom want to optimize vehicle performance in a particular situation. In this study, each designer wants to maximize the steady-state lateral acceleration on a circular skidpad of a certain radius. Designer A is trying to optimize the vehicle for a 100 foot (100') radius skidpad. Designer B is trying to optimize the vehicle for a 400' radius skidpad. The design variables to be determined are the ratio of front/rear roll stiffness and the longitudinal location of the center of gravity (CG). These are commonly referred to as roll stiffness distribution and weight distribution, and are two of three "magic numbers" designers need to match to their tire performance characteristics to optimize the handling of their vehicle [4]. The third "magic number", aerodynamic downforce distribution, as well as the CG height and all other vehicle parameters, are assumed to be fixed for this study.

Designers A and B will be asked to compromise between their designs so that the vehicle they agree on completes a "race" in the least amount of time. This race is defined as a single lap around a 100' skidpad and then, separately, a single lap around a 400' skidpad at maximum steady-state performance. The cumulative time for one lap on each skidpad will be the metric used to judge the design.

The race is conducted using computer simulations for the vehicle and the skidpads.

THE VEHICLE MODEL

The vehicle model used is an expansion of the classic Bicycle Model representation of the automobile [2]. The Bicycle Model was modified to include front and rear roll stiffnesses, lateral load transfer, front and rear aerody-

dynamic downforce and fully non-linear tires. While simple compared to high-end vehicle simulations, the model is sufficiently complex to accurately represent steady-state vehicle performance and provide realistic load and slip angle information for use with the tire data.

The vehicle under consideration is a recent high performance open-wheeled race car. Data on the vehicle can be found in Appendix A. Tire data corresponding to the vehicle is used, with a different tire on the front and rear axles. The tire data provides lateral force as a function of tire slip angle and normal load and was measured at a tire testing facility. Tire load sensitivity, the variation of tire friction coefficient with variations in the tire normal load, is represented in the tire data.

The design variables in this study are normalized longitudinal CG location, a' , and normalized front roll stiffness, K' . Longitudinal CG location is normalized by the wheelbase

$$a' = \frac{a}{\ell} \quad (\text{Eq. 1})$$

where ℓ is the wheelbase and a is the longitudinal distance from the front axle to the CG. A value of $a' = 0.5$ means the CG is at mid-wheelbase, $a' = 0.6$ means the CG is 60% of the way back from the front axle, etc.

Front roll stiffness is normalized by the total roll stiffness

$$K' = \frac{K_F}{K_F + K_R} \quad (\text{Eq. 2})$$

where K_F is the front roll stiffness and K_R is the rear roll stiffness. When $K' = 0.5$ the roll stiffness is distributed evenly between the front and rear tracks, $K' = 0.6$ means that 60% of the roll stiffness is taken on the front axle, etc.

Because this study only considers steady-state performance on a constant radius circle, lateral acceleration, forward velocity and lap time are directly related to one another and can be thought of interchangeably. Maximizing lateral acceleration or forward velocity is analogous to minimizing lap time.

INDIVIDUAL OPTIMUM DESIGNS

Before investigating the best choice of design variables (a' , K') for the race under consideration, a look into the optimal design for each radius is insightful.

Figure 1 shows a contour plot of the design space for Designer A. The abscissa plots the normalized longitudinal CG location, a' , and the ordinate plots normalized front roll stiffness, K' . To generate this plot, the design space was discretized into approximately 500 (a' , K') pairs for which the vehicle performance was calculated. Solution time for each design pair was approximately 20 seconds. The contours are lines of constant lap time for

a single lap on the 100' radius skidpad. Note that there exists a single best point on this graph--it occurs at (a' , K') = (0.68, 0.55). For the 100' radius, this vehicle configuration will produce a lap time of 8.897 seconds--the best possible on the 100' radius skidpad.

Because there is a single optimum point it can be concluded that, if the longitudinal CG location is not optimum, the deficit cannot be overcome by solely adjusting roll stiffness distribution. The finding of a single optimum point as opposed to a line of optimum points in the design space occurs on all graphs in this study and shows that, for optimum vehicle performance, longitudinal CG location is an important factor in the design.

Note, too, that the contours on Figure 1 are nearly vertical. This shows that CG location is much more important than roll stiffness distribution on this vehicle.

Designer B's design space is plotted in Figure 2 using the same discretization as in Figure 1. This is for the 400' radius skidpad. The axes of the plot are identical to Figure 1, but the contours are different. Not only are the lap times different, as expected, but the location of the optimum design point is not the same as on the 100' radius skidpad. The optimum point for Designer B is (a' , K') = (0.63, 0.39). This gives a lap time of 9.119 seconds--the best possible on the 400' radius skidpad.

Designer A and Designer B have determined the best values for the design variables on their own radius skidpads. However, the design variable values they chose do not agree. To design a single vehicle for the race under consideration a compromise solution will be necessary.

OPTIMIZING THE COMPROMISE DESIGN

Designers A and B will now try to reach an agreement on a single vehicle design which minimizes the combined time for one lap around each radius skidpad at steady-state. This race, referred to as Race 1, will be investigated for several vehicle design strategies.

POSSIBLE COMPROMISE DESIGNS – Table 1 shows the performance of several possible compromise designs which could be reached.

Strategy I: Designer A chooses the vehicle design. This is the optimum design for the 100' radius skidpad.

Strategy II: Designer B chooses the vehicle design. This is the optimum design for the 400' radius skidpad.

Strategy III: The designers average their design variable choices.

Strategy IV: The compromise design is based on a geometric argument--since Designer B's skidpad is four times as long as Designer A's the compromise design should be four parts Designer B and one part Designer A.

Depending upon which strategy is used, the elapsed time for the race varies as shown in Table 1. Of these four strategies, Strategy IV gives the best results in this case. A fifth strategy can be devised which involves an engineering analysis of the situation employing techniques borrowed from the field of game theory.

STRATEGY V: GAME THEORY ANALYSIS – To implement the game theory analysis, the lap time results of the discrete design space sampling (used to generate Figures 1 and 2) are replotted in Figure 3 in terms of resultant lap times. Each dot on the graph is a single (a', K') pair--a single vehicle design. The abscissa marks lap time for one lap on the 100' skidpad and the ordinate marks lap time for one lap on the 400' radius skidpad. The time for the race is the sum of the coordinates.

Of the designs considered in Figure 3, the number of possible design choices can be narrowed significantly through the concept of Pareto-minimums [3]. A Pareto-minimal point is one in which there does not exist another design point which has a better lap time simultaneously on both axes. The Pareto-minimal design points, circled in Figure 4, have a special significance. For any race defined as "x laps on the 100' radius skidpad and y laps on the 400' radius skidpad" the optimum vehicle design will always be a point from the Pareto-minimal set. The upper left hand terminus of the line of circled Pareto-minimal points is where the 100' radius design completely dominates. The lower right hand terminus of the line of circles is where the 400' radius design completely dominates. The design points on the Pareto-minimal line between its endpoints are the only acceptable compromise designs. Which point is optimal depends on the race definition.

To choose the best point from the Pareto-minimal set the concept of "utopia point" is introduced. Game theory defines the utopia point as the theoretical best point. The designers want to minimize lap time for the race. In the limit as vehicle performance is maximized the lap time goes to zero. Thus, a "utopia point" can be defined as a 0.00 second time for the race. In Figures 3 or 4 the utopia point would occur at (0,0), the graph's origin.

The optimum design is then found by determining which of the Pareto-minimal points lies closest to the utopia point. Care must be taken that the data is plotted in a manner which places the optimum design point closest to the utopia point. For this analysis, the square root of the lap times on each skidpad are required. This allows the utopia point's method of minimum radial distance to conform with the ultimate goal of minimizing the sum of the coordinates to minimize the total race time. The data from Figure 3 is plotted in Figure 4 with the square root modification.

The solid line in Figure 4 is part of a circle centered at the utopia point. The point closest to the utopia point will lie on a circle with no elements of the Pareto-minimal set lying inside the circle. In this case, the point (3.007,

3.020) is the closest, resulting in laps of 9.042 seconds on the 100' radius and 9.119 seconds on the 400' radius for a race time of 18.161 seconds. Mapping back into the design space, this point corresponds to the point $(a', K') = (0.63, 0.39)$. This is the optimum design for the vehicle for Race 1.

Incidentally, this optimum is identical to the design Designer B located for the 400' skidpad. This indicates that the 400' skidpad dominates the design entirely. This occurs because the tires favor the aerodynamic down-force gained from the higher speeds on the longer skidpad.

ANOTHER RACE DEFINITION

As stated above, the Pareto-minimal set, once identified, contains all the possible "good" solutions to the design problem, regardless of the race definition. The optimum solution will always be a member of the same Pareto-minimal set. As the race definition changes the optimum point chosen from the Pareto-minimal set also changes.

The analysis is now repeated for second race (Race 2) defined as ten laps around the 100' radius skidpad and one lap around the 400' radius skidpad. Designers A and B reach the same conclusions they did before regarding the optimum design for the individual skidpads, so only the compromise design needs to be reexamined. Although Figures 3 and 4 could be used it is convenient to scale the axes to represent the total time on each skidpad. Figure 5 repeats Figure 3, only now the abscissa represents the time for ten laps on the 100' radius skidpad. The design points, in terms of what (a', K') pairs are members of the Pareto-minimal set, do not change. Only the scaling on the abscissa has changed.

Likewise, the definition of the utopia point does not change. Because of the additional laps on the 100' skidpad, however, the shape of Figure 6 (analogous to Figure 4 of Race 1) changes. This alters the optimum design point found by Strategy V. The new optimum is found at (9.440, 3.071) leading to lap times of 8.911 and 9.434 seconds, respectively, for a race time of 98.544 seconds. Table 1 summarizes the results of the five design strategies used in Race 2.

Converting back into the design space identifies this point as $(a', K') = (0.65, 0.37)$. This solution is not optimum for either skidpad, but it produces the minimum time for the race as defined. The design sacrifices 0.086 seconds per lap on the 100' skidpad compared to Strategy I to gain 0.288 seconds on the large skidpad. Even so, the increased importance of the smaller skidpad, because of the increase in distance on that skidpad, has shifted the optimal compromise design away from Designer B to a design not identified by any of the other strategies. Note, too, that a linear combination of the individual optimum designs (Strategies I and II) can never yield this design point.

A SECOND VEHICLE EXAMPLE

The same two races are reexamined using a different vehicle. The vehicle now under consideration is a recent mid-size sedan with special, high performance tires. Data for this vehicle is listed in Appendix A.

Figures 7 and 8 mimic Figures 1 and 2. They are the lap time contour plots for Designers A and B on their own individual skidpads. In Figure 7, Designer A locates the optimum for the 100' skidpad at $(a', K') = (0.61, 0.46)$. This gives a lap time of 10.768 seconds--the best possible at this radius. In Figure 8, Designer B locates the optimum for the 400' skidpad at $(a', K') = (0.52, 0.56)$. This gives a lap time of 20.821 seconds--the best possible at this radius.

Note that for this vehicle, the issue of longitudinal CG location is less crucial than with the previous vehicle. The lap time contours are oriented more-or-less horizontally as opposed to the near-vertical contours seen before. For the sedan, weight distribution is somewhat less important than roll stiffness distribution.

The contours in Figures 7 and 8 do not fill the entire design space. This is because the vehicle model used in this study does not allow for the lifting of a wheel off the ground. The upper boundary of the contours is where the inside front wheel lifts off the ground. The lower boundary is where the inside rear wheel lifts off the ground.

Figure 9 shows the lap times for the sedan on these radii. After applying the square root correction, Figure 10 shows the members of the Pareto-minimal set as circled points. Any race defined as a combination of steady-state lap times on these two radii will have an optimal vehicle design which is a member of the Pareto-minimal set. The utopia point remains at (0,0). For Race 1, defined as one lap on each radius, the optimal solution gives a lap time of 11.026 seconds on the 100' skidpad and 20.821 seconds on the 400' radius skidpad. This is the point in Figure 10 intersected by the line which is part of a circle centered at the utopia point. The resulting race time is 31.847 seconds and is produced by setting the design variables to $(a', K') = (0.52, 0.56)$. Table 2 lists compares the five design strategies examined before, with Strategy V again being the one produced by game theory analysis.

Note that once again the optimum design given by Strategy V is identical to that given by Strategy II. The 400' radius design dominates the design for the race completely.

Table 2 also compares the same strategies for Race 2, defined as ten laps on the 100' radius skidpad and one lap on the 400' radius skidpad. Figure 11 depicts the reoriented Pareto-minimal set. The optimum is now at (10.768, 21.282) for a race time of 128.962 seconds. This occurs when the design variables are $(a', K') = (0.61, 0.46)$ --the same as Strategy I. Revising the race defini-

tion has now caused the 100' radius design--Designer A's design--to become completely dominant.

COMMENTS ON THE PARETO-MINIMAL SET

Thus far, the Pareto-minimal set has only been examined in graphs showing lap time. This is the domain in which the set is determined. Mapping the Pareto-minimal set back into the design space is highly insightful, but is not a trivial endeavor. Discrete point function evaluations, such as used in this study, cannot fully depict the line made by the Pareto-minimal set--just as any discrete representation of a continuous function cannot give a complete description of the function. Additionally, the trace of the Pareto-minimal set in the design space is often complex, further complicating the issue.

The complexity of the line means that the Pareto-minimal set is rarely a straight line. It traces a line which has as its endpoints the optimum solutions for Designer A and Designer B. Between these points, the set meanders about the design space. Thus, any linear combination of the designs proposed by Designers A and B as the best for their individual radii (such as Strategies III and IV) are unlikely to intersect the Pareto-minimal set, thereby practically insuring a non-optimal design.

To illustrate this, Figure 12 plots the estimated Pareto-minimal set in the design space for the open-wheeled vehicle. While this graph is associated with Figures 1 and 2, the contours have been omitted for clarity. Strategies I-V are labeled on this plot. Note where the optimal design points, Strategy II and V (identical) for Race 1 and Strategy V for Race 2 lie. Also note how non-linear the trace is.

In Figure 4 the Pareto-minimal set for the open-wheeled car was plotted in the lap time domain. The shape of the set is concave away from the utopia point at the origin. In Figure 10 the Pareto-minimal set for the sedan was plotted. This set is concave toward the utopia point. The direction of concavity has a large influence on the sensitivity of the design.

Recall that the point from the Pareto-minimal set chosen as the best design for a given race is the one which is closest to the utopia point after the axes are rescaled according to the number of laps on each radii. With a concave inward Pareto-minimal set, the two endpoints of the set are most likely to be closest to the utopia point. Except when the optimum switches from one endpoint to the other the design is influenced very little by changes in race definition. One or the other endpoint is the optimum design. This was seen in the sedan example.

The concave outward shape of the open-wheeled car's Pareto-minimal set allows any interior point on the Pareto line to be a possible solution. Depending on the race definition, the solution will gradually shift along the Pareto-minimal set in the lap time domain from one endpoint to the other. This vehicle needs constant adjustment to keep up with changes in race definition.

ONGOING RESEARCH

Research in the near future is focusing on determining more efficient and reliable ways to represent the Pareto-minimal set in both the lap time domain and the design space. Numerically-based design sensitivity criteria from the shape of the Pareto-minimal set, inclusion of the third "magic number" (aerodynamic downforce distribution) as a design variable and ways to change vehicle parameters to improve the shape of the Pareto-minimal function while not degrading performance are topics under consideration. The potential exists for this method to be adapted for use with high-end vehicle simulations as a design and optimization tool.

CONCLUSIONS

The use of contours of constant lap time help to visualize the influence of design variables on vehicle performance as well as the sensitivity of a design to changes in the design variables.

By determining the Pareto-minimal set the designers of a vehicle can achieve many benefits. First, the number of design variable value combinations which need to be considered are limited to those in the Pareto-minimal set for any race definition which corresponds to the definition format which created the Pareto-minimal set. When plotted in the design space, the Pareto-minimal set does not plot as a straight line between the individual optimum designs for the two radius skidpads. Thus, any linear combination of individual optimum design variable values is unlikely to be optimal for the race under consideration.

Additionally, the shape of the Pareto-minimal set has important implications in assessing the robustness of the design to changes in race definition. Sets which are concave away from the utopia point in the lap time domain are less robust than those which are concave toward the utopia point.

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APPENDIX A - VEHICLE DATA

HIGH-PERFORMANCE OPEN-WHEELED RACE CAR

Mass	41.7 slugs
Wheelbase	9.67 ft
Front Track	5.5 ft
Rear Track	5.25 ft
CG Height	1.167 ft
Frontal Area	10 ft ²
Downforce Coefficients	
Front: 1.78 (downforce)	Rear 3.22 (downforce)

MID-SIZED SEDAN

Mass	78.33 slugs
Wheelbase	8.44 ft
Front Track	4.66 ft
Rear Track	4.62 ft
CG Height	1.79 ft
Frontal Area	18.43 ft ²
Downforce Coefficients	
Front: 0.4 (downforce)	Rear 0.1(downforce)

Table 1: Open-wheeled vehicle. Race 1: One lap on each radius skidpad. Race 2: Ten laps on 100' radius, one lap on 400' radius

Strategy	a'	K'	100' Radius	400' Radius	Race 1	Race 2
			Time (sec)	Time (sec)	Time (sec)	Time (sec)
I	0.68	0.55	8.897	9.722	18.619	98.692
II	0.63	0.39	9.042	9.119	18.161	99.539
III	0.655	0.47	8.971	9.464	18.435	99.174
IV	0.64	0.422	9.015	9.23	18.245	99.38
V (Race 1)	0.63	0.39	9.042	9.119	18.161	NA
V (Race 2)	0.65	0.37	8.911	9.434	NA	98.544

Table 2: Sedan. Race 1: One lap on each radius skidpad. Race 2: Ten laps on 100' radius, one lap on 400' radius

Strategy	a'	K'	100' Radius	400' Radius	Race 1	Race 2
			Time (sec)	Time (sec)	Time (sec)	Time (sec)
I	0.61	0.46	10.768	21.282	32.05	128.962
II	0.52	0.56	11.026	20.821	31.847	131.081
III	0.565	0.51	10.905	21.067	31.972	130.117
IV	0.538	0.54	10.978	20.925	31.903	130.705
V (Race 1)	0.52	0.56	11.026	20.821	31.847	NA
V (Race 2)	0.61	0.46	10.768	21.282	NA	128.962

