Random Vectors (Continued)

Independent Random Variables:

Two random variables *X* and *Y* are said to be (statistically) independent i*f:*

Discrete case:

$p_{X|Y}(x_i|y_j) = p_X(x_i)$. And since $p_{X|Y}(x_i|y_j) = \dfrac{p_{X,Y}(x_i, y_j)}{p_Y(y_j)}$,

then $p_{X,Y}(x_i, y_j) = p_X(x_i)p_Y(y_j)$ results, and could be shown to be equivalent to condition of independence.

Continuous case:

$f_{X|Y}(x|y) = f_X(x)$, which is equivalent to $f_{X,Y}(x, y) = f_X(x)f_Y(y)$. Similarly $F_{X,Y}(x, y) = F_X(x)F_Y(y)$.

Generally, n random variables are independent iff

$$f_{X_1, X_2, \ldots, X_n}(x_1, x_2, \ldots, x_n) = f_{X_1}(x_1)f_{X_2}(x_2)\ldots f_{X_n}(x_n)$$

Example: Gas in thermal equilibrium at temperature *T*.

Experiment: Pick a molecule at random. define three random variables:

$V_X(s) = v_x, V_Y(s) = v_y, V_Z(s) = v_z$, which map sample point (picked molecule) into its three velocities along the three axes *x,y,* and *z*.

From statistical thermodynamics, $f_{V_x}(v_x) = \sqrt{\frac{m}{2\pi kT}}e^{-mv_x^2/2kT}$, same for *y, z.* $V_X(s)V_Y(s), V_Z$ are independent R.V's, then

$$f_{V_x, V_y, V_z}(v_x, v_y, v_z) = \left(\frac{m}{2\pi kT}\right)^{3/2} e^{-m(v_x^2 + v_y^2 + v_z^2)/2kT}$$, Maxwell-Boltzman distribution.

Kinetic energy $E = \frac{m}{2}(V_x^2 + V_y^2 + V_z^2)$, $E(s) = e$, so $e = \frac{m}{2}(v_x^2 + v_y^2 + v_z^2)$.

$$F_E(e_0) = P(\{E \le e_0\}) = \underset{v_x^2 + v_y^2 + v_z^2 \le 2e_0/m}{\int dv_x \int dv_y \int dv_z} \left(\frac{m}{2\pi kT}\right)^{3/2} e^{-m(v_x^2 + v_y^2 + v_z^2)/2kT}$$

Transform to spherical coordinates. Element volume $4\pi r^2 dr$ is a shell volume, then

$$F_E(e_0) = \int_0^{\sqrt{2e_0/m}} 4\pi r^2 dr\left(\frac{m}{2\pi kT}\right)^{3/2} e^{-mr^2/2kT}$$. Since $f_E(e_0) = \frac{d}{de_0}F_E(e_0)$, then we differentiate directly

without solving integral, to get $f_E(e_0) = \frac{2}{\sqrt{\pi}(kT)^{3/2}}e_0^{1/2}e^{-e_0/kT}$

Functions of Random Variables

Let $Z \equiv g(X)$ be a real-valued function of a real variable. Let *X* be a random variable defined on *S*, where $X(s) = x$. Then $Z \equiv g(X)$ is a random variable on *S*, defined by $Z(s) = g(X(s)) = g(x) = z$.

Statement of Problem:

If $Z = g(X)$ and $f_X(x)$ is known, find $f_Z(z)$.

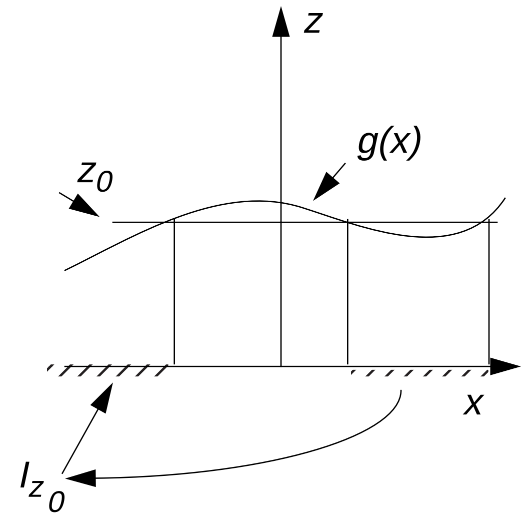
Step 1:

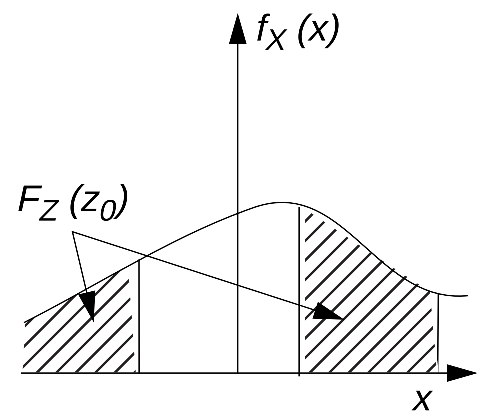
Find Cumulative Distribution Function (C.D.F.) of *Z., i.e.* $F_Z(z)$.

$$F_Z(z_0) = P\{Z \le z_0\} = P\{g(X) \le z_0\}$$

Define $I_{z_0} = \{x:(g(x) \le z_0)\}$, which is a collection of intervals on the *x* axis.

Then $F_Z(z_0) = P\{X \in I_{z_0}\} = \int_{I_{z_0}} f_X(x)dx$.

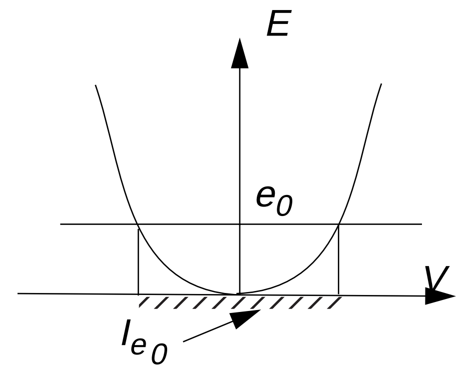




Step 2:

$$f_Z(z_0) = \frac{d}{dz_0}F_Z(z_0)$$

Examples:



1- $\boxed{E = V^2}$

Given $f_V(v)$, find $f_E(e)$.

Step 1:

$$F_E(e_0) = P\{E \le e_0\} = P\left\{V^2 \le e_0\right\} = P\{V \in I_{e_0}\} = P\{-\sqrt{e_0} < V < \sqrt{e_0}\} = \begin{cases} \int_{-\sqrt{e_0}}^{\sqrt{e_0}} f_V(v)dv, \text{ for } e_0 > 0 \\ 0, \text{ for } e_0 < 0 \end{cases}$$

Step 2:

$$f_E(e_0) = \frac{d}{de_0}\int_{-\sqrt{e_0}}^{\sqrt{e_0}} f_V(v)dv = \begin{cases} f_V(\sqrt{e_0})\frac{d}{de_0}\sqrt{e_0} - f_V(-\sqrt{e_0})\frac{d}{de_0}(-\sqrt{e_0}), e_0 > 0 \\ 0, e_0 < 0 \end{cases}$$

Then
$$\boxed{f_E(e_0) = \begin{cases} \frac{1}{2\sqrt{e_0}}[f_V(\sqrt{e_0}) + f_V(-\sqrt{e_0})], \ e_0 > 0 \\ 0, e_0 < 0 \end{cases}}$$

2- $Y = \sin\Theta$

Given $f_\Theta(\theta) = \begin{cases} \frac{1}{2\pi}, & -\pi \le \theta < \pi \\ 0, & \text{else} \end{cases}$, find $f_Y(y)$.

Step 1:

$$F_Y(y_0) = P\{Y \le y_0\} = P\{\sin\Theta \le y_0\} = P\{\Theta \in I_{y_0}\}$$

$$F(y_0) = \begin{cases} 1, y_0 > 1 \\ \int_{I_{y_0}} f_\Theta(\theta) d\theta = \frac{1}{2\pi}(\pi + 2\operatorname{asin} y_0) = \frac{1}{2} + \frac{1}{\pi}\operatorname{asin} y_0, & -1 < y_0 < 1 \\ 0, y_0 < -1 \end{cases}$$

Step 2:

$$f_Y(y_0) = \frac{d}{dy_0} F_Y(y_0) = \begin{cases} 0, y_0 > 1, y_0 < 1 \\ \frac{1}{\pi\sqrt{1 - y_0^2}}, -1 \le y_0 \le 1 \end{cases}$$

Functions of Two Random Variables

Let $Z \equiv g(X, Y)$ be a real-valued function of two real variable. Let *X* and *Y* be joint-random variable defined on *S*. Then Z is a random variable on *S*, defined by $Z(s) = g(X(s), Y(s)) = g(x, y) = z$.

Statement of Problem:

Given $Z = g(X, Y)$ and $f_{X,Y}(x, y)$, find $f_Z(z)$.

Step 1:

Find $F_Z(z_0) = P\{Z \le z_0\} = P\{g(X, Y) \le z_0\} = \iint_{R_{z_0}} f_{X,Y}(x, y)dxdy$

where $R_{z_0} = \{x, y : g(x, y) \le z_0\}$

Step 2:

$$f_Z(z_0) = \frac{d}{dz_0}F_Z(z_0)$$

Sum of Two Random Variables

This is a very important example of a function of two random variables.

$Z = X + Y$

Step 1:

$$F_Z(z_0) = P\{Z \le z_0\} = P\{X + Y \le z_0\} = \int\int_{R_{z_0}} f_{X,Y}(x, y)dxdy = \int_{-\infty}^{\infty} dx \int_{-\infty}^{z_0 - x} dy\, f_{X,Y}(x, y)$$

Step 2:

$$\boxed{f_Z(z_0) = \frac{d}{dz_0}F_Z(z_0) = \int_{-\infty}^{\infty} dx \frac{d}{dz_0}\int_{-\infty}^{z_0 - x} dy\, f_{X,Y}(x, y) = \int_{-\infty}^{\infty} dx\, f_{X,Y}(x, (z_0 - x))}$$

Which is the general formula of the pdf of the sum of two R.V.'s

Example: Two resistors in series with uniform pdf's. find pdf of series combination. Assume two R. V's to be independent.

Let R_1 and R_2 be two random variables with identical pdf,s

$$f_{R_1}(r_1) = \begin{cases} 0.01, & 950 \le r_1 \le 1050 \\ 0, & \text{else} \end{cases}, \text{ and } f_{R_2}(r_2) = \begin{cases} 0.01, & 950 \le r_2 \le 1050 \\ 0, & \text{else} \end{cases}.$$

and let there sum be $R_T = R_1 + R_2$. Since independent, $f_{R_1, R_2}(r_1, r_2) = f_{R_1}(r_1) f_{R_2}(r_2)$

Step1:

Step 2:

Another Method:

General formula $f_{R_T}(r_T) = \int_{-\infty}^{\infty} dr\, f_{R_1, R_2}(r, (r_T - r))$.

Since independent, then $f_{R_T}(r_T) = \int_{-\infty}^{\infty} dr\, f_{R_1}(r) f_{R_2}(r_T - r)$,

which is a ***convolution***.

Theorem:

If *Z* is the sum of two

independent R. V's *X* and *Y*, then;

$$\boxed{f_Z(z) = \int_{-\infty}^{\infty} d\xi\, f_X(\xi) f_Y(z - \xi) = f_X(z) * f_Y(z)}$$