OPERATIONS RESEARCH
Vol. 55, No. 5, September–October 2007, pp. 976–991
ISSN 0030-364X | EISSN 1526-5463 | 07 | 5505 | 0976

DOI 10.1287/opre.1070.0434

Balancing Terrorism and Natural Disasters—Defensive Strategy with Endogenous Attacker Effort

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In this paper, we apply game theory to identify equilibrium strategies for both attacker and defender in a fully endogenous model of resource allocation for countering terrorism and natural disasters. The key features of our model include balancing protection from terrorism and natural disasters, and describing the attacker choice by a continuous level of effort rather than a discrete choice (i.e., attack or not). Interestingly, in a sequential game, increased defensive investment can lead an attacker to either increase his level of effort (to help compensate for the reduced probability of damage from an attack), or decrease his level of effort (because attacking has become less profitable). This can either reduce or increase the effectiveness of investments in protection from intentional attack, and can therefore affect the relative desirability of investing in protection from natural disasters.

Subject classifications: decision analysis: risk; games/group decisions: noncooperative; utility/preference: applications.
Area of review: Decision Analysis.
History: Received January 2006; revisions received June 2006, August 2006, September 2006; accepted September 2006.

"Our report shows that the terrorists analyze defenses. They plan accordingly."

The 9/11 Commission Report

National Commission on Terrorist Attacks upon the United States (2004, p. 383)

1. Introduction

In the aftermath of the terrorist attacks on September 11, 2001, and Hurricane Katrina in August 2005, the government is grappling with how to optimally protect the country from both terrorism and natural disasters, subject to limited resources. The all-hazards approach, which seeks protections that are effective against all types of emergency events (U.S. Government Accountability Office 2005), was originally proposed to address this kind of problem. However, this approach does not explicitly consider the terrorist's analysis and response, and tends to focus more on emergency response than on prevention.

To our knowledge, the only rigorous model to date for allocating defensive investment between terrorism and natural disasters (taking into account the endogenous nature of attacker effort) is an exploratory analysis by Powell (2007). In this paper, we propose a similar model for balancing protection from terrorism and natural disasters, to provide insights into optimal defensive strategies in a post-9/11 and post-Katrina world.

Of course, numerous researchers have studied protection from terrorism since 9/11. Much of this work has addressed the defender's optimization problem in the case of exogenous attacker effort levels (see, for example, Zhuang and Bier 2005, Bier et al. 2005, Sandler and Siqueira 2006, Zhuang et al. 2007). However, as noted by Bier (2005), protecting targets against intentional attack is fundamentally different from protecting against natural disasters because attackers can adapt their strategies in response to defensive investment. Therefore, some researchers have considered endogenous attacker decisions. Most of this work has generally allowed only discrete attacker options; e.g., attack or not (Lapan and Sandler 1988, Sandler and Lapan 1988, Sandler and Arce M. 2003, Konrad 2004, Bier et al. 2007). Similarly, Rosendorff and Sandler (2004) address the discrete choice of whether to launch a spectacular attack or an ordinary attack. Clearly, though, to efficiently protect targets from potential attack, a defender would need to predict how much effort an attacker is likely to devote to any given target, not only whether it will be attacked. Therefore, some work does allow for continuous attacker effort; see, for example, Major (2002), Lakdawalla and Zanjani (2005), Bueno de Mesquita (2005a, b), and Siqueira and Sandler (2006). Farrow (2007) provides models for attacker and defender optimization problems separately, but fails to link them.

In this work, we represent the level of attacker effort as a continuous variable. This allows us to model the probability of damage from an attack as a function of the levels of both attacker effort and defensive investment. By analogy with the laws of supply and demand governing relations between producers and consumers, one can envision attackers and defenders jointly determining their levels of attacker effort and defensive investment, in either a simultaneous or a sequential game. Game theory and the concept of Nash equilibrium have been recognized as suitable tools for studying such strategy-interaction problems for a long time. For additional applications of game theory in the security context, see Woo (2002), Harris (2004), Bier (2005), and Cavusoglu et al. (2005).

The 9/11 Commission Report (National Commission on Terrorist Attacks upon the United States 2004) clearly states that terrorists make decisions in response to the potential victim's observed strategies. Such attack decisions in principle consist of several interrelated elements: when to attack, which target(s) to attack, how much effort to allocate to the attack, etc. The decision will also presumably depend on factors such as how much the attacker values causing damage to various targets, the attacker's level of resources, and any other opportunities he has for use of those resources (see, for example, Frey and Luechinger 2003).

On the defender's side, the problem is even more complicated. First, the defender faces a similar set of choices: when to defend, which targets to defend, how much effort to allocate to defense (both against terrorism and natural disaster), etc. The decision will again depend on the defender's valuations of the various targets, the level of available defensive resources, and the other possible uses of those resources. Moreover, because many types of defensive investments are made before an attack actually occurs, the defender also in general needs to anticipate and account for possible attacker responses to her decisions (in a sequential game).

In our (simplified) model of the above decision process, the attacker and defender are described by four attributes: (a) the technologies available to the attacker and defender (represented by the probability of damage from an attack, as a function of the levels of attacker effort and defensive investment, and the probability of damage from a natural disaster, as a function of the level of defensive investment); (b) the attacker's and defender's valuations of the various potential targets; (c) the attacker's and defender's (dis)utilities with regard to the damage caused by an attack; and (d) the attacker's and defender's disutilities for attacker effort and defensive investment, respectively.

In principle, the defender can deter an attacker in numerous ways—by increasing the opportunity cost of an attack (Frey and Luechinger 2003), by making potential targets less attractive (Frey and Luechinger 2004, Perrow 2006), or by reducing the capacity of the terrorists (Keohane and Zeckhauser 2003). However, in this paper, we focus specifically on how the defender should allocate defensive investments intended to reduce the probability of damage from an attack. Similarly, although the defender can decrease damage from natural disaster in numerous ways, our focus centers on how the defender should allocate defensive investments intended to reduce the probability of damage from a natural disaster (rather than, say, to reduce the subsequent impact of that damage through emergency response, although the results in that case might be roughly similar). Variables other than the levels of attacker effort and defensive investment (such as the attractiveness of particular targets) are assumed to be exogenous in our model.

The result of this paper is a timely and rigorous model for balancing defense against terrorism and natural disasters. We view this paper as only one of several building blocks needed for a more complete understanding of strategic defense against terrorism. For example, past work has addressed the effects of more complex system structures, rather than simple series and parallel systems (Azaiez and Bier 2007), and the effects of uncertainty about attacker goals and asset valuations (Bier et al. 2007). This paper also in turn provides a solid basis for studying additional types of defenses in the future, such as all-hazards approaches and border security.

Note that this paper is intended primarily to provide qualitative (rather than quantitative) insights. We do not anticipate that our model is likely to be used to support specific decisions because in practice it is often difficult to estimate the parameters and functions needed for the model. Moreover, real systems often involve a large number of components and dynamic aspects, which are not considered fully in this paper.

The next section of this paper introduces our notation and assumptions, and formulates a basic equilibrium model for a game between an attacker and a defender. Section 3 compares the simultaneous and sequential formulations of this game, and discusses the defender's "first-mover advantage" in the sequential game. For ease of exposition, we first present the results of our model for the simple case of only a single possible target in §4. Even the single-target case is complicated enough to be interesting because it captures the fact that increasing defensive investment can either increase or decrease attacker effort, as well as the defender's trade-off between protection against terrorism and protection against natural disasters. In particular, we find the attacker and defender best-response functions for this case, and characterize the equilibria for both the simultaneous and sequential games. Section 5 then extends these results to the case of a multitarget system; we explore the effects of risk attitudes on the attacker and defender decisions, and also isnvestigate how the existence of multiple targets affects the defender's optimal strategy in balancing protection from terrorism and from natural disasters. Finally, §§6 and 7 summarize the previous sections, discuss the policy implications of our work, and provide some future research directions. An electronic companion (online appendix) to this paper is available as part of the online version that can be found at http://or.journal.informs.org/, which contains the proofs of theorems and lemmas for this paper.

2. Notation, Assumptions, and Problem Formulation

We define the parameters of our model as follows:

- Types 1 and 2 threats: Intentional threats (e.g., terrorism) and nonintentional threats (e.g., natural disaster), respectively. We use subscripts 1 and 2 throughout this paper to refer to intentional and nonintentional threats, respectively.
- N: number of possible targets.
- a_i: attacker's effort spent on target i, where $a_i \geqslant 0$ for all $i = 1, \ldots, N$. We let $a \equiv (a_1, \ldots, a_N)$.
- d_{1i} and d_{2i}: defender's investment in protecting target i from intentional and nonintentional threats, respectively, where $d_{1i} \geqslant 0$ and $d_{2i} \geqslant 0$ for all $i = 1, \ldots, N$. We let $d \equiv (d_{11}, \ldots, d_{1N}; d_{21}, \ldots, d_{2N})$.
- $\mathcal{A} \equiv \{(a_1, \ldots, a_N)\colon\ a_i \geqslant 0\ \ \forall\, i = 1, \ldots, N\}$ and $\mathcal{D} \equiv \{(d_{11}, \ldots, d_{1N}; d_{21}, \ldots, d_{2N})\colon\ d_{1i} \geqslant 0,\ d_{2i} \geqslant 0\ \forall\, i = 1, \ldots, N\}$: set of feasible strategies for a and d, respectively.
- $P_{1i}(a_i, d_{1i})$: probability of damage from an intentional threat against target i, as a function of the attacker's effort a_i and the defender's investment d_{1i}.
- $P_{2i}(d_{2i})$: probability of damage from a nonintentional threat against target i, as a function of the defender's investment d_{2i}.
- X_{1i}: Bernoulli random variable that takes on a value of one when target i is damaged as the result of an intentional threat, with probability $P_{1i}(a_i, d_{1i})$, and zero otherwise. For simplicity, the targets are assumed to be either completely damaged or not at all. (We recognize that this is a limiting case, because of course in the real world, targets might sustain partial levels of damage.)
- X_{2i}: Bernoulli random variable that takes on a value of one when target i is damaged as the result of a nonintentional threat, with probability $P_{2i}(d_{2i})$, and zero otherwise. We assume that X_{2i} is independent of X_{1i} for all $i = 1, \ldots, N$.
- w_i and v_i: attacker and defender valuations of target i, respectively.
- $w(a, d) = \sum_{i=1}^{N} w_i I_{\{X_{1i}=1\}}$, and $v(a, d) = \sum_{i=1}^{N} v_i I_{\{\max(X_{1i}, X_{2i})=1\}}$: attacker and defender valuations of all damaged targets, as a function of the strategy pair (a, d), where $I_{\{\cdot\}}$ is an indicator function. For example, these valuations could represent the economic losses or casualties associated with damage to the various targets (see O'Hanlon et al. 2002). Of course, in the real world, attacker and defender preferences are likely to depend on multiple attributes (e.g., economic losses, casualties, symbolic importance, etc.). However, limiting the model to a single measure of damage significantly increases its tractability, without dramatically restricting the applicability of the results.

We implicitly assume here that the attacker cares only about the damage from type 1 threats, while the defender cares about both. However, it is straightforward to show that the attacker and defender equilibrium strategies will remain the same (at least under the rare-event approximation), even if the attacker also cares about damage to the defender from nonintentional threats.

- $u_A[w(a, d)]$ and $u_D[v(a, d)]$: utility of total damage to the attacker and the defender, respectively. For convenience, we sometimes abbreviate $u_{Ai} \equiv u_A(w_i)$ and $u_{Di} \equiv u_D(v_i)$ for all $i = 1, \ldots, N$.
- $g_A(\sum_{i=1}^{N} a_i)$ and $g_D(\sum_{i=1}^{N} d_{1i} + d_{2i})$: disutility of total attacker effort and total defensive investment, respectively.
- $U_A(a, d)$ and $U_D(a, d)$: total expected utility of the attacker and the defender, respectively.
- $\hat{a}(d) \equiv \arg\max_{a\in\mathcal{A}} U_A(a, d)$ and $\hat{d}(a) \equiv \arg\max_{d\in\mathcal{D}} U_D(a, d)$: attacker and defender best responses, respectively.
- (a^{C*}, d^{C*}) and (a^{S*}, d^{S*}): equilibria for the simultaneous and sequential games, respectively, where $a^{C*}, a^{S*} \in \mathcal{A}$ and $d^{C*}, d^{S*} \in \mathcal{D}$. (Strictly speaking, these are Nash equilibria for the simultaneous game, and subgame-perfect Nash equilibria for the sequential game, respectively.)

For convenience, we also define the elements of the above vectors as follows:

$$a^{C*} \equiv (a_1^{C*}, \ldots, a_N^{C*}),$$
$$d^{C*} \equiv \big(d_{11}^{C*}, \ldots, d_{1N}^{C*}; d_{21}^{C*}, \ldots, d_{2N}^{C*}\big),$$
$$a^{S*} \equiv (a_1^{S*}, \ldots, a_N^{S*}), \quad d^{S*} \equiv \big(d_{11}^{S*}, \ldots, d_{1N}^{S*}; d_{21}^{S*}, \ldots, d_{2N}^{S*}\big),$$
$$\hat{a}(d) \equiv (\hat{a}_1(d), \ldots, \hat{a}_N(d)), \quad \text{and}$$
$$\hat{d}(a) \equiv \big(\hat{d}_{11}(a), \ldots, \hat{d}_{1N}(a); \hat{d}_{21}(a), \ldots, \hat{d}_{2N}(a)\big).$$

2.1. Assumptions About the Probability of Damage

$P_{1i}(a_i, d_{1i})$ and $P_{2i}(d_{2i})$ are assumed to be twice differentiable, and have the following properties (where superscripts denote partial derivatives) for all finite $a_i > 0$, $d_{1i} > 0$, and $d_{2i} > 0$:

- $P_{1i}(0, d_{1i}) = 0$ and $\lim_{d_{1i}\to\infty} P_{1i}(a_i, d_{1i}) = 0$. That is, the probability of damage from an intentional attack is zero when the level of attacker effort is zero, and when the level of defensive investment in protection from intentional attack goes to infinity.
- $P_{1i}^{(a_i)}(a_i, d_{1i}) \equiv \partial P_{1i}(a_i, d_{1i})/\partial a_i > 0$ and

$$P_{1i}^{(a_i, a_i)}(a_i, d_{1i}) \equiv \partial^2 P_{1i}(a_i, d_{1i})/\partial a_i^2 < 0.$$

Thus, the probability of damage from an intentional attack is increasing in the level of attacker effort, with decreasing marginal returns to attack effort.

- $P_{1i}^{(d_{1i})}(a_i, d_{1i}) \equiv \partial P_{1i}(a_i, d_{1i})/\partial d_{1i} < 0$ and

$$P_{1i}^{(d_{1i}, d_{1i})}(a_i, d_{1i}) \equiv \partial^2 P_{1i}(a_i, d_{1i})/\partial d_{1i}^2 > 0.$$

Thus, the probability of damage from an intentional attack is decreasing in the level of defensive investment, with decreasing marginal returns to investment. (Note that the returns to attacker effort are measured in terms of *increased* probability of damage, while the returns to defensive investment are measured by *decreased* probability of damage; hence, the different signs of the second-order derivatives.)

- $P_{1i}^{(a_i,\,d_{1i})}(0, d_{1i}) < 0$, where

$$P_{1i}^{(a_i,\,d_{1i})}(a_i, d_{1i}) \equiv \partial^2 P_{1i}(a_i, d_{1i})/\partial a_i d_{1i}.$$

In other words, when the attacker effort is zero, the marginal return of attacker effort, $P_{1i}^{(a_i)}(0, d_{1i})$, is decreasing in the level of defensive investment.

- $P_{2i}^{(d_{2i})}(d_{2i}) \equiv \mathrm{d}P_{2i}(d_{2i})/\mathrm{d}d_{2i} < 0$ and $P_{2i}^{(d_{2i},\,d_{2i})}(d_{2i}) \equiv \mathrm{d}^2 P_{2i}(d_{2i})/\mathrm{d}d_{2i}^2 > 0$. Thus, the probability of damage from a nonintentional threat is decreasing in the level of defensive investment, with decreasing marginal returns to investment.

The assumptions of decreasing marginal returns to both attacker effort and defensive investment are crucial to the analysis presented here. Although these assumptions are likely to be satisfied in many cases involving continuous capital investment, we recognize that it is not necessary for the marginal returns to be decreasing over all levels of effort or investment (for example, if some minimal level of investment is needed before defenses become highly cost effective). Recently, a family of probability functions called *contest success functions* (Skaperdas 1996, Hirshleifer 1989) have been studied, in which increasing marginal returns can be found in some cases.

2.2. Assumptions About Utility Functions

The attacker and defender disutility functions for attacker effort and defensive investment, g_A and g_D, are assumed to be twice differentiable and have the following properties: $g_j(0) = 0$; $g_j' > 0$; and $g_j'' \geqslant 0$ for $j = A, D$. Increasing marginal disutility of attacker effort is likely to hold in practice—e.g., due to the increasing difficulty of obtaining higher levels of attack resources, or the difficulty of implementing more demanding attacks. However, this need not always be the case; for example, if we consider both productive and appropriative activities of the attacker (Grossman and Kim 1995), then the marginal disutility of attack effort could be decreasing, provided that productive activities (i.e., opportunity costs) can be characterized by increasing marginal returns (at least over some ranges of effort levels).

Note that the functions g_A and g_D can also be used to represent (at least approximately) the effects of finite budget constraints by choosing g_A and g_D to satisfy $g_A(x) > u_A(\sum_{i=1}^N w_i)$ for all $x \geqslant B_A$ and $g_D(x) > -u_D(\sum_{i=1}^N v_i)$ for all $x \geqslant B_D$, where B_A and B_D are the available budgets of the attacker and defender, respectively. This works because neither the attacker nor the defender will devote more resources than the available budgets to attack or defense at optimality if the disutility of that investment is greater than the utility of damage to all possible targets. Therefore, it is adequate to consider an unconstrained model.

The attacker and defender utility functions for damage, u_A and u_D, are assumed to be twice differentiable with the following properties: $u_A(0) = 0$; $u_A' > 0$; $u_D(0) = 0$; and $u_D' < 0$. In other words, the attacker utility is nonnegative and increasing in total damage, while the defender utility is nonpositive and decreasing in total damage. We allow both the attacker and defender to be risk seeking, risk neutral, or risk averse, and explore the implications of different risk attitudes in §5. From utility theory, the attacker (defender) is risk seeking, risk neutral, or risk averse if and only if $u_j'' > 0$, $u_j'' = 0$, or $u_j'' < 0$, respectively, for $j = A$ $(j = D)$.

We assume that the total expected utility equals the sum of the expected utility of total damage and the disutility of attacker effort or defensive investment (i.e., both the attacker and defender utility functions satisfy additive independence). In other words, we have

$$U_A(a, d) = E\{u_A[w(a, d)]\} - g_A\left(\sum_{i=1}^N a_i\right) \quad \text{and}$$

$$U_D(a, d) = E\{u_D[v(a, d)]\} - g_D\left(\sum_{i=1}^N d_{1i} + d_{2i}\right),$$

where $E\{\cdot\}$ denotes expectation. We recognize that a general multiattribute utility function might be more appropriate than additive independence. However, the assumption of additive independence of damage and cost seems to capture the most critical features of the attacker and defender decision problems (the need for each player to trade off the level of effort expended against the probability of damage resulting from attacks), while keeping the model relatively tractable. More general multiattribute utility functions would allow effort and damage to be modeled as substitutes or complements to each other. (For example, one could easily imagine that an attacker might have greater tolerance for high effort if accompanied by high levels of damage.) However, this seems likely to be a second-order effect in most cases, especially because the costs in our model are not treated as being probabilistic.

2.3. Assumptions of Common Knowledge and Problem Formulation

As in most applications of game theory, we assume that the attacker and defender have common knowledge about the rules of the game, including the functions $P_{1i}(a_i, d_{1i})$, $P_{2i}(d_{2i})$, g_j, and u_j, and the parameters N, w_i, and v_i for all $i = 1, \ldots, N$ and $j = A, D$. We recognize that this assumption is limiting, as pointed out by Guikema (2007). However, as stated previously, we view this paper as only one of several building blocks needed for a more complete understanding of strategic defense against terrorism. For example, previous work (Bier et al. 2007) has addressed the case where the defender does not know the attacker's valuations of the targets. While this does result in some defender hedging compared to the case of common knowledge, allowing the defender to be uncertain about the attacker valuations does not radically change the nature of that game or its solution. Therefore, we would not expect reasonable relaxations of the assumption of common knowledge to result in substantial changes to the results of this paper either.

We also assume that each party knows that the other party wishes to maximize total expected utility. In other words, the goal of the attacker is to maximize his total expected utility by choosing a suitable level of attacker effort to devote to each target; that is,

$$\max_{a\in\mathcal{A}} U_A(a, d) = E\{u_A[w(a, d)]\} - g_A\left(\sum_{i=1}^{N} a_i\right). \tag{1}$$

Similarly, the goal of the defender is to maximize her total expected utility by choosing a suitable level of defensive investment for each target; that is,

$$\max_{d\in\mathcal{D}} U_D(a, d) = E\{u_D[v(a, d)]\} - g_D\left(\sum_{i=1}^{N} d_{1i} + d_{2i}\right). \tag{2}$$

3. Simultaneous vs. Sequential Games and First-Mover Advantage

We consider both simultaneous and sequential games between the attacker and the defender, as follows:

(1) *Simultaneous game*: The attacker and the defender decide on the attacker effort and defensive investment simultaneously. Note that this model can also apply even if the attacker and defender do not make their decisions at the same time, as long as neither party knows the other's decision at the time it makes its own decision. Thus, a Nash equilibrium for this simultaneous game, (a^{C*}, d^{C*}), must be a solution to optimization problems (1) and (2); i.e., it must satisfy $a^{C*} \in \hat{a}(d^{C*})$ and $d^{C*} \in \hat{d}(a^{C*})$.

(2) *Sequential game*: The defender decides on and implements a defensive investment strategy first, and then the attacker chooses his levels of attacker effort after observing the defensive investments. (The case in which the attacker moves first is not of interest here because this type of model would apply primarily to real-time defensive tactics, such as apprehending an attacker during the course of an attack, while we wish to focus on strategic interactions such as capital investments.) When the defender moves first, the attacker faces optimization problem (1). However, because the defender knows that the attacker will implement his best-response strategy $\hat{a}(d)$, the optimization problem for defender (2) takes a special form:

$$\max_{d\in\mathcal{D}} U_D[\hat{a}(d), d] = E\{u_D[v(\hat{a}(d), d)]\} - g_D\left(\sum_{i=1}^{N} d_{1i} + d_{2i}\right). \tag{3}$$

Thus, a subgame-perfect Nash equilibrium for this sequential game, (a^{S*}, d^{S*}), must be a solution to optimization problems (1) and (3); i.e., satisfying $a^{S*} = \hat{a}(d^{S*})$ and $d^{S*} = \arg\max_{d\in\mathcal{D}} U_D[\hat{a}(d), d]$. Strictly, $\hat{a}(d)$ is a set rather than a number. However, for simplicity, we also use the same notation to refer to a member of that set when the set is a singleton.

Some interesting applications of mixed-strategy equilibria are discussed in Cavusoglu and Raghunathan (2004) for the case with only a small number of discrete decision strategies. By contrast, we find in our analysis that there are frequently pure-strategy equilibria in our game. At an intuitive level, this is because we consider continuous rather than discrete decision variables (along with the fact that the success probability of an attack is convex in the defensive investment and concave in the attacker effort).

Theorem 1. *For any possible equilibrium of the simultaneous game (a^{C*}, d^{C*}), if the attacker's best-response set $\hat{a}(d^{C*})$ is a singleton (i.e., the attacker's best response to d^{C*} is unique), there exists an equilibrium of the sequential game (a^{S*}, d^{S*}) such that the defender's total expected utility in the latter equilibrium is (weakly) greater than the corresponding utility in the former one; i.e., $U_D(a^{S*}, d^{S*}) \geqslant U_D(a^{C*}, d^{C*})$.*

Proof. The result follows directly from the fact that the defender always has the option to choose d^{C*} (her equilibrium strategy in the simultaneous game) even in the sequential game. Specifically, if the defender chooses d^{C*} in the sequential game, and the attacker responds with $\hat{a}(d^{C*}) = a^{C*}$, the defender in the sequential equilibrium will have the same utility as in the simultaneous game. Therefore, if $d^{S*} \neq d^{C*}$, any such equilibrium strategy d^{S*} of the sequential game must earn at least as high a utility for the defender as d^{C*}. □

Remark. Theorem 1 indicates that the defender has a "first-mover advantage" in the sequential game as long as the attacker's best response is unique. Therefore, our results imply that the defender should in general choose a sequential game (in which she advertises her defensive investments instead of keeping them secret) to use her first-mover advantage. Similar results from attacker-defender models can be found in Bier et al. (2007). For a general discussion of first-mover advantage in the economic literature, see, for example, Lieberman and Montgomery (1988).

4. The Case of a Single Target

In this section, we explore the case where there is exactly one target; i.e., $N = 1$. Thus, there exist exactly two possible outcomes for the attacker:

(1) The attack succeeds, which occurs with probability $P_{11}(a_1, d_{11})$, and leads to a positive utility of damage $u_{A1} \equiv u_A(w_1)$.

(2) The attack fails, leading to a zero utility of damage for the attacker.

The attacker's optimization problem (1) thus becomes

$$\max_{a_1\geqslant 0} U_A(a_1, d_{11}) = P_{11}(a_1, d_{11})u_{A1} - g_A(a_1). \tag{4}$$

From (4), we can see that the attacker's best response depends on d_{11}, but does not depend on d_{21}. Therefore, in the single-target problem, $\hat{a}(d)$ can be reduced to $\hat{a}_1(d_{11})$.

In formulating the defender's optimization problem, we need the probability of damage to compute expected damage. Strictly speaking, the probability of damage to target i from at least one threat equals

$$1-(1-P_{1i})(1-P_{2i})=P_{1i}+P_{2i}-P_{1i}P_{2i},$$

which we will approximate by $P_{1i}+P_{2i}$ for simplicity. This rare-event approximation will be reasonably accurate when P_{1i} and P_{2i} are relatively small (e.g., less than 0.1). Under this approximation, optimization problem (2) for a simultaneous game becomes

$$\begin{aligned}&\max_{d_{11},d_{21}\geqslant 0} U_D(a_1,d_{11},d_{21})\\ &\quad=[P_{11}(a_1,d_{11})+P_{21}(d_{21})]u_{D1}-g_D(d_{11}+d_{21}),\end{aligned} \tag{5}$$

where $u_{D1}\equiv u_D(v_1)$. Similarly, the defender's optimization problem (3) for a sequential game becomes

$$\begin{aligned}&\max_{d_{11},d_{21}\geqslant 0} U_D[a_1(d_{11}),d_{11},d_{21}]\\ &\quad=\{P_{11}[\hat{a}_1(d_{11}),d_{11}]+P_{21}(d_{21})\}u_{D1}-g_D(d_{11}+d_{21}).\end{aligned} \tag{6}$$

Theorem 2. *Pure-strategy equilibria exist for both the simultaneous game characterized by Equations* (4) *and* (5), *and the sequential game characterized by Equations* (4) *and* (6).

Proof. This result follows trivially from Lemmas 1 and 2 in the online appendix, using the existence theorem for a pure-strategy Nash equilibrium (see Theorem 1 in Dasgupta and Maskin 1986; credits are given to Debreu 1952, Glicksberg 1952, and Fan 1952). □

4.1. Attacker's Best Response

Theorem 3. *The solution to optimization problem* (4) *is given by*

$$\hat{a}_1(d_{11})=\begin{cases}0 & \text{if } U_A^{(a_1)}(0,d_{11})\leqslant 0,\\ \{a_1\colon U_A^{(a_1)}(a_1,d_{11})=0\} & \\ \quad \text{if } U_A^{(a_1)}(0,d_{11})>0,\end{cases} \tag{7}$$

where

$$U_A^{(a_1)}(a_1,d_{11})\equiv\frac{\partial U_A(a_1,d_{11})}{\partial a_1}=P_{11}^{(a_1)}(a_1,d_{11})u_{A1}-g_A'(a_1)$$

is the total marginal payoff for the attacker. Note that the sign of $\mathrm{d}\hat{a}_1(d_{11})/\mathrm{d}d_{11}$ *is the same as the sign of* $P_{11}^{(a_1,d_{11})}[\hat{a}_1(d_{11}),d_{11}]$. *Moreover, we have* $\lim_{d_{11}\to\infty}\hat{a}_1(d_{11})=0$, *and* $\hat{a}_1(d_{11})=0$ *for all* $d_{11}\geqslant 0$ *if and only if* $U_A^{(a_1)}(0,0)\leqslant 0$.

Proof. See the online appendix. □

Remark. Theorem 3 above makes clear the relationship between our model and the study of strategic interactions in the literature on industrial organization (Tirole 1988, Part II). In particular, the probability of damage from terrorism, $P_{11}(a_1,d_{11})$, can be considered an analogue of a *strategic reaction function*. When the cross derivative is positive, the attacker effort and defensive investment are *strategic complements*, so that increased investment in defense will increase the level of attacker effort. A similar effect in the Cold War was described as "escalation" (Brams 1985), as opposed to deterrence. By contrast, when the cross derivative is negative, the attacker effort and defensive investment are *strategic substitutes*, so that increased investment in defense will decrease the level of attacker effort.

Theorem 3 also indicates that there are two types of targets from the attacker's perspective: (a) If $U_A^{(a_1)}(0,0)\leqslant 0$, or equivalently, $P_{11}^{(a_1)}(0,0)u_{A1}\leqslant g_A'(0)$, then target 1 never interests the attacker at all (i.e., $\hat{a}_1(d_{11})=0$ for all $d_{11}\geqslant 0$). (b) Otherwise, target 1 will interest the attacker if the level of defensive investment is sufficiently small. Obviously, no defensive investment against intentional threats should be expended if the target is of the first type. If the target is potentially of interest to the attacker, then any positive attacker best-response effort must be such that the marginal attack payoff equals the marginal disutility of attacker effort.

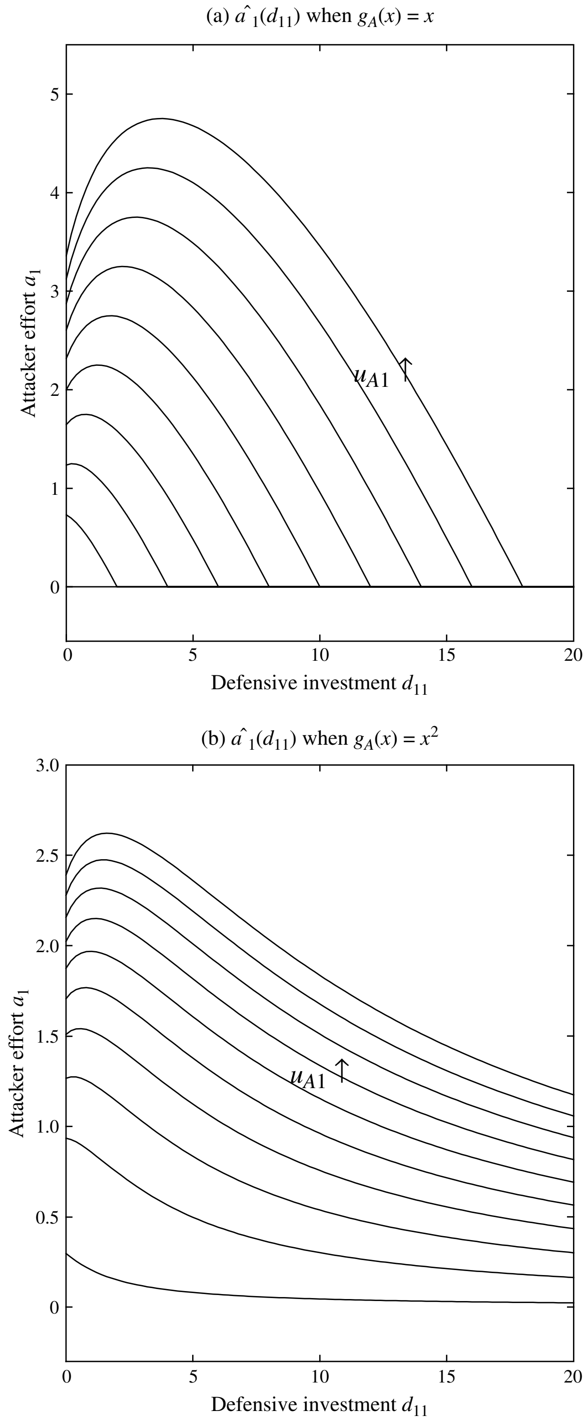
Example 1. We illustrate Theorem 3 using $P_{1i}(a_i,d_{1i})=a_i/(10(a_i+d_{1i}+c_i))$ as the probability of damage from an attack, with constants $c_i>0$ for target $i=1,\ldots,N$. (Intuitively, one can think of the constant c_i as representing the inherent security level of target i before any defensive investment has been spent.) We consider two cases—one in which the attacker preference is linear in attacker effort, and the other in which the attacker has increasing marginal disutility of attacker effort.

Case 1. $g_A(x)=x$. The attacker best-response function is

$$\hat{a}_1(d_{11})=\begin{cases}0 & \text{if } u_{A1}\leqslant 10c_1,\\ 0 & \text{if } u_{A1}>10c_1 \text{ and } d_{11}\geqslant \bar{d}_{11},\\ \sqrt{\dfrac{u_{A1}}{10}(d_{11}+c_1)}-d_{11}-c_1 & \\ \quad\text{otherwise,}\end{cases} \tag{8}$$

where $\bar{d}_{11}=u_{A1}/10-c_1$ is the minimum level of defensive investment required to deter the attacker altogether. There are three possible shapes of attacker best-response functions, illustrated in Figure 1(a) for various values of u_{A1}: (a) If the attacker's utility of damage is relatively small ($u_{A1}\leqslant 10c_1$), then $\hat{a}_1(d_{11})=0$ for all d_{11}; i.e., the target will not be attacked, regardless of the defensive investment. (b) If the attacker's utility of damage is intermediate ($10c_1<u_{A1}<40c_1$), then $\hat{a}_1(d_{11})$ will be first decreasing in d_{11} for $0\leqslant d_{11}<\bar{d}_{11}$, and then zero for $d_{11}\geqslant\bar{d}_{11}$, at

Figure 1. Possible attacker best-response functions.



which point the attacker will be completely deterred. (c) If the attacker's utility of damage is high ($u_{A1} \geqslant 40c_1$), then $\hat{a}_1(d_{11})$ will be initially increasing in d_{11} for $0 \leqslant d_{11} < u_{A1}/40 - c_1$, then decreasing in d_{11} for $u_{A1}/40 - c_1 \leqslant d_{11} < \bar{d}_{11}$, and finally zero for $d_{11} \geqslant \bar{d}_{11}$, at which point the attacker will be completely deterred.

Note that the point at which the best response of the attacker changes from increasing to decreasing in d_{11} (given by $d_{11} = u_{A1}/40 - c_1$) increases with u_{A1}. Thus, as the attacker's utility of damage grows, he becomes more willing to "redouble his effort" in the face of increased defensive investment.

Case 2. $g_A(x) = x^2$. The attacker best-response function for this case is shown in Figure 1(b) for various values of u_{A1}. There are two possible shapes for the attacker best-response function in this case: one is always decreasing in the level of defensive investment and converging to zero; and the other is initially increasing, then decreasing, and eventually converging to zero.

As seen from the above example, the attacker's best-response function $\hat{a}_1(d_{11})$ can be initially increasing in d_{11}. However, by $\lim_{d_{11}\to\infty} \hat{a}_1(d_{11}) = 0$ in Theorem 3, it must eventually converge to zero (if initially positive) as d_{11} grows. In other words, at low levels of defensive investment, increases in defensive investment might lead the attacker to allocate more effort to attacks to partially compensate for the reduced effectiveness of attacker effort. However, at high levels of defensive investment, spending more effort on attacks will no longer be cost-effective for the attacker. Thus, the attacker will eventually be deterred as the level of defensive investment increases. (The key assumption needed for this result is $\lim_{d_{1i}\to\infty} P_{1i}(a_i, d_{1i}) = 0$; i.e., the success probability of an attack must go to zero as the defensive investment goes to infinity.)

4.2. Defender's Best Response in the Simultaneous Game

Theorem 4. *The necessary and sufficient conditions for the best-response functions $\hat{d}_{11}(a_1)$ and $\hat{d}_{21}(a_1)$ to optimization problem* (5) *are given by any of the following four cases*:

Case 1. $\hat{d}_{11} > 0$, $\hat{d}_{21} > 0$, $U_D^{(d_{11})}(a_1, \hat{d}_{11}, \hat{d}_{21}) = 0$, *and* $U_D^{(d_{21})}(\hat{d}_{11}, \hat{d}_{21}) = 0$;

Case 2. $\hat{d}_{11} = 0$, $\hat{d}_{21} = 0$, $U_D^{(d_{11})}(a_1, 0, 0) \leqslant 0$, *and* $U_D^{(d_{21})}(0, 0) \leqslant 0$;

Case 3. $\hat{d}_{11} > 0$, $\hat{d}_{21} = 0$, $U_D^{(d_{11})}(a_1, \hat{d}_{11}, 0) = 0$, *and* $U_D^{(d_{21})}(\hat{d}_{11}, 0) \leqslant 0$;

Case 4. $\hat{d}_{11} = 0$, $\hat{d}_{21} > 0$, $U_D^{(d_{11})}(a_1, 0, \hat{d}_{21}) \leqslant 0$, *and* $U_D^{(d_{21})}(0, \hat{d}_{21}) = 0$;

where

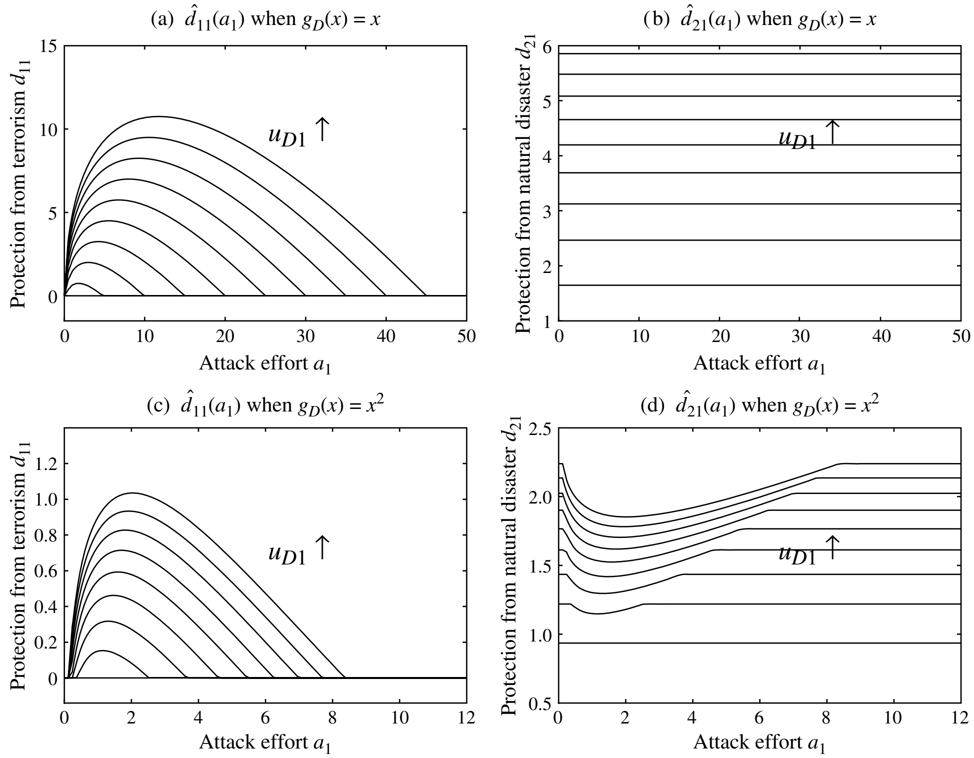
$$U_D^{(d_{11})}(a_1, d_{11}, d_{21}) \equiv \frac{\partial U_D(a_1, d_{11}, d_{21})}{\partial d_{11}} = P_{11}^{(d_{11})}(a_1, d_{11})u_{D1} - g'(d_{11} + d_{21})$$

and

$$U_D^{(d_{21})}(d_{11}, d_{21}) \equiv \frac{\partial U_D(a_1, d_{11}, d_{21})}{\partial d_{21}} = P_{21}^{(d_{21})}(d_{21})u_{D1} - g'(d_{11} + d_{21})$$

are the total marginal payoffs for the defender from protection against terrorism and natural disaster, respectively. Moreover, we have $\hat{d}_{11}(0) = 0$, *and* $\lim_{a_1\to\infty} \hat{d}_{11}(a_1) = 0$.

Figure 2. Possible defender best-response functions.



Proof. See the online appendix. □

Remark. Theorem 4 above indicates that if the defensive investments against both intentional and nonintentional threats are positive at equilibrium, then the marginal payoffs to the defender from these two types of protection must be equal, and must also equal the marginal disutility of defensive investment. Note also that the optimal defensive investment in protection against nonintentional threats depends on the level of attacker effort only indirectly, through the optimal defensive investment against intentional threats. Moreover, if the disutility function g_D of defensive investment is linear, then the optimal defensive investment in protection against nonintentional threats will be independent of both the level of attacker effort and the optimal defensive investment in protection against intentional attacks.

Example 2. We illustrate Theorem 4 using $P_{1i}(a_i, d_{1i}) = a_i/(10(a_i + d_{1i} + c_i))$ and $P_{2i}(d_{2i}) = 1/(10(d_{2i} + e_i))$ as the probability of damage from an attack, with constants $c_i > 0$ and $e_i > 0$ for target $i = 1, \ldots, N$. Similar to Example 1, we consider two cases: one in which the defender's preference is linear in defensive investment, and the other in which the defender has increasing marginal disutility of defensive investment.

Case 1. $g_D(x) = x$. The defender best-response function is

$$\hat{d}_{11}(a_1) = \begin{cases} 0 & \text{if } -u_{D1} \leqslant 40c_1, \\ 0 & \text{if } -u_{D1} > 40c_1 \text{ and } a_1 \leqslant \bar{a}_1 \text{ or } a_1 \geqslant \bar{\bar{a}}_1, \\ \sqrt{-\dfrac{u_{D1}}{10}a_1 - a_1 - c_1}, & \\ & \text{otherwise,} \end{cases} \tag{9}$$

$$\hat{d}_{21}(a_1) = \sqrt{-\frac{u_{D1}}{10}} - e_1, \tag{10}$$

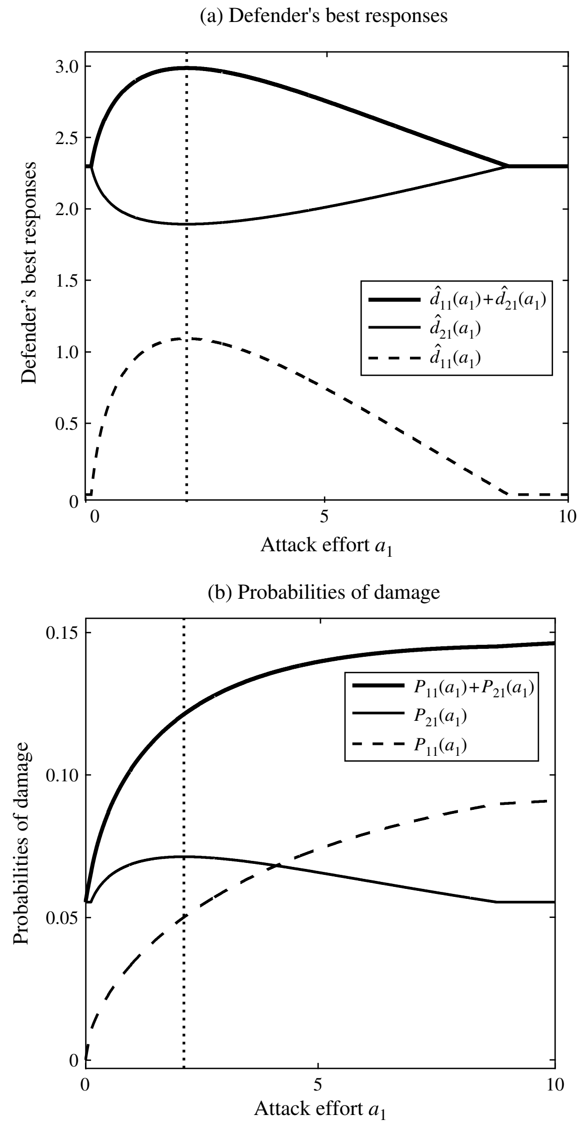
where $\bar{a}_1 = -u_{D1}/20 - c_1 - \sqrt{u_{D1}^2/400 + c_1 u_{D1}/10}$ is the minimal level of attacker effort required to induce a positive defense against terrorism; and $\bar{\bar{a}}_1 = -u_{D1}/20 - c_1 + \sqrt{u_{D1}^2/400 + c_1 u_{D1}/10}$ is the minimal level of attacker effort required to "deter" a defender from protection against terrorism. There are two possible shapes of defender best-response functions in protection against terrorism to the attacker effort, illustrated in Figure 2(a) for various values of u_{D1}: (a) If the defender's disutility of damage is relatively small ($-u_{D1} \leqslant 40c_1$), then $\hat{d}_{11}(a_1) = 0$ for all a_1; i.e., the target will not be defended against terrorism, regardless of the attacker effort. (b) If the defender's disutility of damage is high ($-u_{D1} > 40c_1$), then $\hat{d}_{11}(a_1)$ will be initially zero for small $a_1 \leqslant \bar{a}_1$, and then $\hat{d}_{11}(a_1)$ will be increasing in a_1 for $\bar{a}_1 \leqslant a_1 \leqslant -u_{D1}/40$, then be decreasing in a_1 for $-u_{D1}/40 \leqslant a_1 \leqslant \bar{\bar{a}}_1$, and eventually equal zero for $a_1 \geqslant \bar{\bar{a}}_1$. In other words, at low levels of attacker effort, increases in attacker effort might lead the defender to allocate more resources to defense against terrorism to partially compensate for the reduced effectiveness of defensive investment. However, at high levels of attacker effort, spending more on defense against terrorism will no longer be cost-effective for the defender. By contrast, Figure 2(b) shows that the best-response function $\hat{d}_{21}(a_1)$ for defensive investment in protection from natural disasters is constant in the attacker effort a_1 when the disutility of defensive investment is linear.

Note also that the point at which the defender's investment in protection from terrorism decreasing in the attacker effort a_1 (given by $a_1 = -u_{D1}/40$) is inversely proportional to u_{D1}. Thus, as the defender's disutility of damage grows, she will be willing to spend more on defense from terrorism before "giving up" in the face of overwhelming attacker effort.

Case 2. $g_D(x) = x^2$. The defender best-response function of protection from terrorism is shown in Figure 2(c) for various values of u_{D1}, which has a similar qualitative result as in Case 1 above. However, different with its counterpart in Case 1, the defender best-response function of protection from natural disaster, $\hat{d}_{21}(a_1)$, is initially decreasing in a_1, and then increases to its original level, $\hat{d}_{21}(0)$, as shown in Figure 2(d). Comparing Figures 2(c) and 2(d) suggests that protection from terrorism is a substitute for protection from natural disaster. In other words, due to the first-order conditions for the defender and the increasing marginal disutility of defensive investment, changes in attacker behavior that lead the defender to invest more against one type of threat must also lead to reduced defensive investment against the other type of threat. Thus, at low levels of attacker effort, increases in attacker effort might lead the defender to allocate more resources to defense against terrorism, as shown in Figure 2(c) (to partially compensate for the reduced effectiveness of defensive investment), but will also lead the defender to allocate less to defense against natural disaster, as shown in Figure 2(d). By contrast, at high levels of attacker effort, spending on defense against terrorism will no longer be cost-effective for the defender, so she will spend more on defense against natural disasters.

For a fixed value of defender disutility u_{D1}, Figures 3(a) and 3(b) show the defensive investment and the probability of damage, respectively, as functions of attacker effort. As shown in Figure 3(b), both the total probability of damage, as approximated by $P_{11}(a_1) + P_{21}(a_1)$, and the probability of a successful attack, $P_{11}(a_1)$, are increasing in attacker effort, even in regions where the defensive investment in protection from terrorism, $\hat{d}_{11}(a_1)$, is also increasing. Thus, the increasing defensive investment at low levels of attacker effort is insufficient to compensate for the increase in attacker effort. The probability of damage from natural disasters, $P_{21}(a_1)$, does decrease as the defender shifts her resources toward protection from natural disasters, but not enough to reduce the total probability of damage.

Figure 3. Total defensive investments and probabilities of damage as a function of attacker effort.



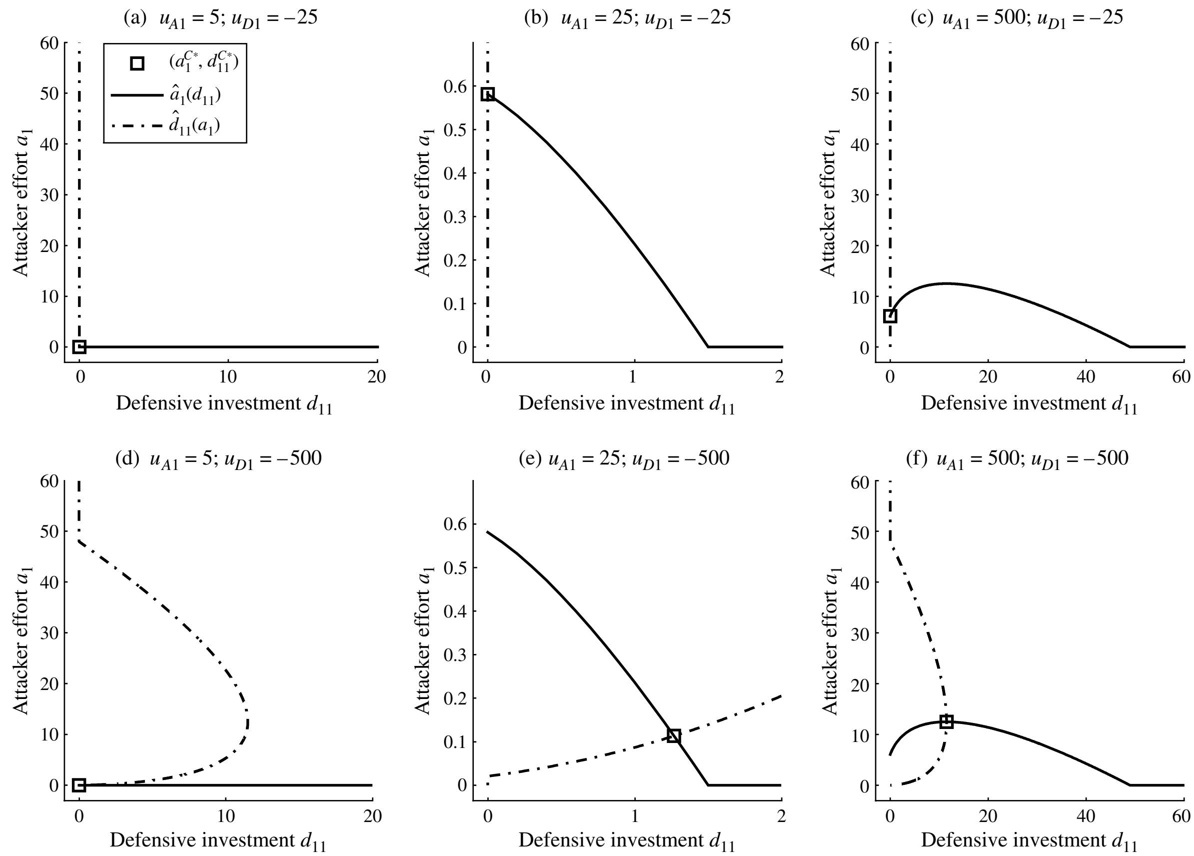
It is also interesting to know when the defender should not invest at all in protection from terrorism, and conversely when she should not invest in protection from natural disasters. Corollary 1 below, which follows directly from Theorem 4, addresses this question.

Corollary 1. *For any* a_1, *if* $P_{11}^{(d_{11})}(a_1, d_{11}) < P_{21}^{(d_{21})}(d_{21})$ *for all* $d_{11} \geqslant 0$, $d_{21} \geqslant 0$, *then* $\hat{d}_{11}(a_1) = 0$. *Similarly, if* $P_{11}^{(d_{11})}(a_1, d_{11}) > P_{21}^{(d_{21})}(d_{21})$ *for all* $d_{11} \geqslant 0$, $d_{21} \geqslant 0$, *then* $\hat{d}_{21}(a_1) = 0$ *for any* a_1.

Proof. Follows directly from Theorem 4. □

Remark. Essentially, Corollary 1 says that for a given level of attacker effort a_1, if protection from natural disasters is more cost-effective than protection from terrorism at all

Figure 4. Equilibria for the simultaneous game.



possible levels of defensive investment (d_{11} and d_{21}), then the defender should spend nothing on protection from terrorism, and vice versa. Note also that the level of attacker effort a_1 can change the relative effectiveness of protection against the two types of threats. For example, at zero attacker effort, the cost-effectiveness of protection from terrorism will be zero, which will trivially be less than the cost-effectiveness of protection from natural disaster. Therefore, the defender should spend nothing on protection from terrorism in this case. However, there may be levels of attacker effort at which protection from terrorism is more cost-effective than protection from natural disaster for all feasible levels of defensive investment d_{11} and d_{21}, in which case the defender should spend nothing on protection from natural disasters.

4.3. Equilibrium for the Simultaneous Game

Combining the results from §§4.1 and 4.2 yields the equilibrium for a simultaneous game as defined in §3. Alternatively, if we plot the attacker and defender best-response functions $\hat{a}_1(d_{11})$ and $\hat{d}_{11}(a_1)$ on a single graph, then the intersection point (a_1^{C*}, d_{11}^{C*}), together with the corresponding value of $d_{21}^{C*} = \hat{d}_{21}(a_1^{C*})$, will be the equilibrium for the simultaneous game. Figure 4 shows all six cases, corresponding to the three types of attacker best response (zero in Figures 4(a) and 4(d), decreasing in Figures 4(b) and 4(e), or initially increasing and then decreasing Figures 4(c) and 4(f)), and the two types of defender best response (zero in Figures 4(a) through 4(c), or initially increasing and then decreasing in Figures 4(d) through 4(f)).

The equilibria shown in Figure 4 are calculated using the same probability and (dis)utility functions as in Case 1 of Examples 1 and 2, with $c_1 = 1$. Because both the attacker effort and the defensive investment can be either zero or positive, there are in principle four types of equilibria for the simultaneous game:

(1) $(a_1^{C*} = 0,\ d_{11}^{C*} = 0)$, as in Figures 4(a) and 4(d): This will happen when the target does not interest the attacker regardless of the level of defensive investment because the attacker's utility of damage is too small.

(2) $(a_1^{C*} > 0,\ d_{11}^{C*} = 0)$, as in Figures 4(b) and 4(c): This will happen when the target interests the attacker, but the defender's disutility of damage is too small for her to make any positive defensive investment in protection from terrorism.

(3) $(a_1^{C*} > 0,\ d_{11}^{C*} > 0)$, as in Figures 4(e) and 4(f): This will happen when the target is sufficiently valuable to both the attacker and the defender to justify positive allocations of attacker effort and defensive investment, respectively, and neither party's (equilibrium) effort is sufficient to completely deter the other party. Note, by the way, that the defender best-response function $\hat{d}_{11}(a_1)$ in Figure 4(e) is initially increasing then decreasing, just as

in Figures 4(d) and 4(f). However, the scale of Figure 4(e) has been changed to more clearly show the behavior in the region of the equilibrium, so the decreasing section of the defender best-response function cannot be seen.

(4) ($a_1^{C*} = 0$, $d_{11}^{C*} > 0$): This outcome is impossible because by Theorem 4, we know that $\hat{d}_{11}(0) = 0$. In other words, there is no simultaneous equilibrium in which the attacker implements zero attacker effort, but the defender chooses to implement a positive defensive investment.

Remark. From the discussion above, we see that in the simultaneous game, the defender cannot completely eliminate the risk of attack at equilibrium by her defensive investment, unless the target is not of interest to the attacker in any case. Note that the third case, ($a_1^{C*} > 0, d_{11}^{C*} > 0$), shown in Figures 4(e)–4(f), will tend to be of the greatest interest in practice.

Note also that the equilibria in the above example are always unique. However, our assumptions do not preclude the existence of multiple equilibria. In that case, it may be interesting to determine which equilibrium is best for the defender, and whether the defender can take steps to ensure that her preferred equilibrium is reached.

4.4. Defender's Optimal Strategy in the Sequential Game

We define the probability of damage from an attack in a sequential game to be $\hat{P}_{11}(d_{11}) \equiv P_{11}[\hat{a}_1(d_{11}), d_{11}]$. Taking the derivative of $\hat{P}_{11}(d_{11})$ with respect to d_{11}, we get

$$\begin{aligned}\hat{P}_{11}^{(d_{11})}(d_{11}) &\equiv \frac{\mathrm{d}\hat{P}_{11}(d_{11})}{\mathrm{d}d_{11}}\\ &= P_{11}^{(d_{11})}[\hat{a}_1(d_{11}), d_{11}]\\ &\quad + P_{11}^{(a_1)}[\hat{a}_1(d_{11}), d_{11}] \cdot \frac{\mathrm{d}\hat{a}_1(d_{11})}{\mathrm{d}d_{11}}, \end{aligned} \tag{11}$$

where $\mathrm{d}\hat{a}_1(d_{11})/\mathrm{d}d_{11}$ can be derived from (7). Because we saw in §4.1 that $\mathrm{d}\hat{a}_1(d_{11})/\mathrm{d}d_{11}$ could be either positive or negative, the overall effect of defensive investment on the equilibrium attacker effort is indeterminate. By assumption, we know that $P_{11}^{(d_{11})} < 0$ and $P_{11}^{(a_1)} > 0$. Therefore, Equation (11) implies that the effect of defensive investment in protection from terrorism consists of two parts: a direct reduction in the probability of damage due to the defensive investment, reflected by $P_{11}^{(d_{11})}[\hat{a}_1(d_{11}), d_{11}]$; and an indirect increase or reduction in the probability of damage due to the attacker's modified level of effort in a sequential game, as expressed by $P_{11}^{(a_1)}[\hat{a}_1(d_{11}), d_{11}] \cdot \mathrm{d}\hat{a}_1(d_{11})/\mathrm{d}d_{11}$. Ignoring the latter effect will lead to overestimates of the effectiveness of defensive investment against terrorism whenever $\mathrm{d}\hat{a}_1(d_{11})/\mathrm{d}d_{11} > 0$.

Theorem 5. *The necessary conditions for the solution to optimization problem* (6) *are given below*:

(1) *If* $\hat{d}_{11} > 0$, $\hat{d}_{21} > 0$, *then* $\hat{U}_D^{(d_{11})}(\hat{d}_{11}, \hat{d}_{21}) = 0$ *and* $\hat{U}_D^{(d_{21})}(\hat{d}_{11}, \hat{d}_{21}) = 0$;

(2) *If* $\hat{d}_{11} = 0$, $\hat{d}_{21} = 0$, *then* $\hat{U}_D^{(d_{11})}(0,0) \leqslant 0$ *and* $\hat{U}_D^{(d_{21})}(0,0) \leqslant 0$;

(3) *If* $\hat{d}_{11} > 0$, $\hat{d}_{21} = 0$, *then* $\hat{U}_D^{(d_{11})}(\hat{d}_{11}, 0) = 0$ *and* $\hat{U}_D^{(d_{21})}(\hat{d}_{11}, 0) \leqslant 0$;

(4) *If* $\hat{d}_{11} = 0$, $\hat{d}_{21} > 0$, *then* $\hat{U}_D^{(d_{11})}(0, \hat{d}_{21}) \leqslant 0$ *and* $\hat{U}_D^{(d_{21})}(0, \hat{d}_{21}) = 0$;

where

$$\begin{aligned}\hat{U}_D^{(d_{11})}(d_{11}, d_{21}) &\equiv \frac{\partial U_D[\hat{a}_1(d_{11}), d_{11}, d_{21}]}{\partial d_{11}}\\ &= \hat{P}_{11}^{(d_{11})}(d_{11}) u_{D1} - g'(d_{11} + d_{21})\end{aligned}$$

and

$$\begin{aligned}\hat{U}_D^{(d_{21})}(d_{11}, d_{21}) &\equiv \frac{\partial U_D[\hat{a}_1(d_{11}), d_{11}, d_{21}]}{\partial d_{21}}\\ &= P_{21}^{(d_{21})}(d_{21}) u_{D1} - g'(d_{11} + d_{21})\end{aligned}$$

are the total marginal payoffs for the defender to protection from terrorism and natural disaster, respectively.

Proof. Similar to the proof of Theorem 4, except that only necessary conditions for the solution are given. □

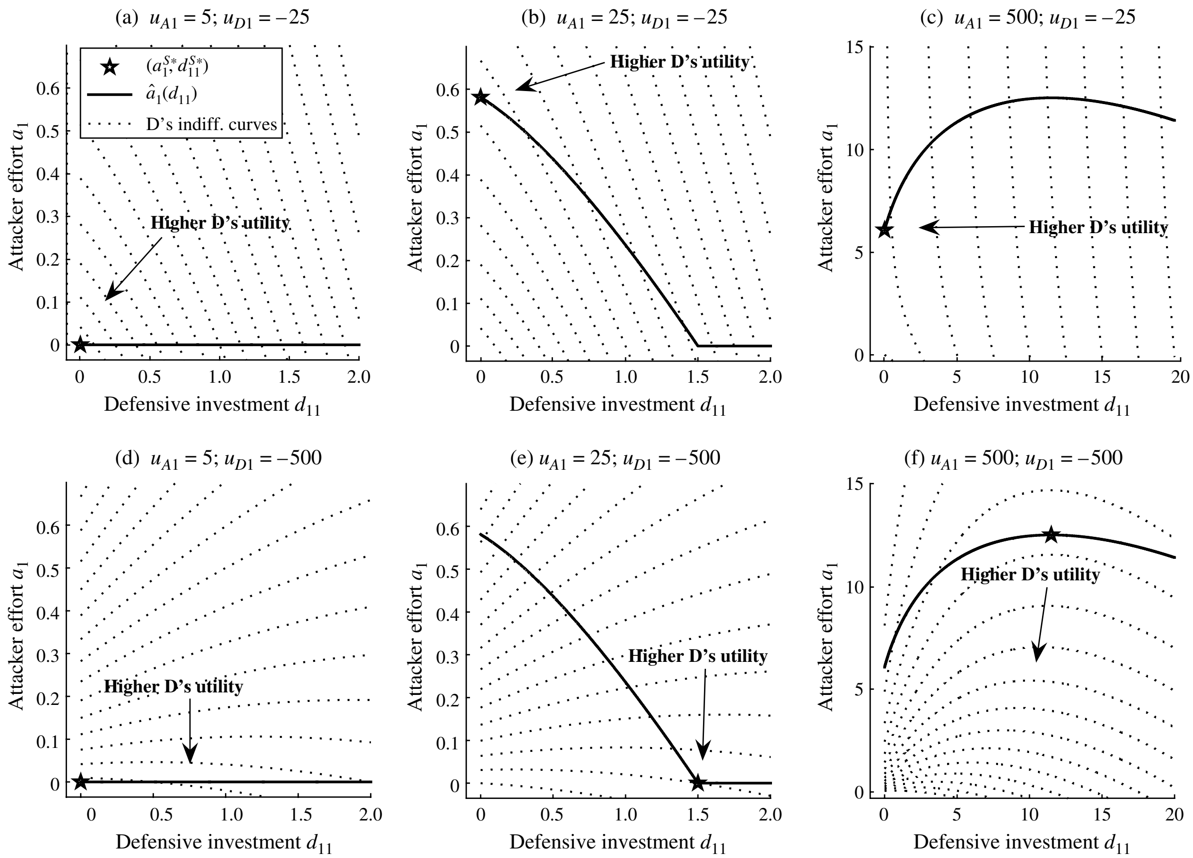
Remark. As in the simultaneous game, Theorem 5 indicates that if the defender makes positive investments in protection from both terrorism and natural disaster at equilibrium, then the marginal payoff to the defender from protection against terrorism must equal that of protection against natural disaster, and both must equal the marginal disutility of defensive investment. This result is similar to that of Powell (2007) for a sequential game. Also, if the disutility of defensive investment is linear, then the optimal defensive investment in protection against natural disaster will be independent of the defensive investment in protection against terrorism.

4.5. Equilibrium for the Sequential Game

Combining the results from §§4.1 and 4.4 gives the equilibrium for the sequential game defined in §3. Alternatively, graphical analysis provides another way to get the equilibrium to the sequential game. As shown in Figure 5, if we plot the defender's indifference curves $U_D[a_1, d_{11}, d_{21}(a_1, d_{11})]$ (because d_{21} can be viewed as a function of a_1 and d_{11}) for various values of defender utility on the same graph as the attacker's best-response function $\hat{a}_1(d_{11})$, then the defender can simply choose the indifference curve with the highest expected utility level $U_D[\hat{a}_1(d_{11}), d_{11}, d_{21}(d_{11})]$ from among those that intersect the attacker best-response function.

As in the simultaneous game, there are in principle four types of equilibria for the sequential game: ($a_1^{S*}=0$, $d_{11}^{S*}=0$); ($a_1^{S*} > 0$, $d_{11}^{S*} > 0$); ($a_1^{S*} > 0$, $d_{11}^{S*} = 0$); and ($a_1^{S*} = 0$, $d_{11}^{S*} > 0$). However, unlike in the simultaneous game, where ($a_1^{C*} = 0$, $d_{11}^{C*} > 0$) is impossible, all four cases are possible equilibria for the sequential game, as shown in Figure 5(e) (using the same parameters and functions as in Figure 4). Comparing the results in Figure 5 with those

Figure 5. Equilibria for the sequential game with defender indifference curves.



in Figure 4, we see that the sequential equilibria are identical to those of the corresponding simultaneous game in all cases except for Figure 5(e). For the cases shown in Figures 5(a) through 5(d), this is trivially so, because either the attacker's or the defender's best-response function is zero everywhere. For the case shown in Figure 5(e), the attacker is completely deterred in the sequential game, even though the corresponding simultaneous game has nonzero attacker effort at equilibrium, as shown in Figure 4(e). Thus, it is possible for the defender to completely deter the attacker in the sequential game, even when the target is sufficiently valuable to be of interest to the attacker in principle (mathematically, this is because we have $\lim_{d_{11}\to\infty} \hat{a}_1(d_{11}) = 0$ in Theorem 3). This finding reinforces the result in §3 that the defender is better off playing a sequential rather than a simultaneous game. Note, however, that it will not always be optimal for the defender to completely deter the attacker if the cost of doing so is too high, as shown in Figure 5(f). As before, note also that the situations represented in Figures 5(e) and 5(f) are likely to be of the greatest interest in practice.

5. The Case of Multiple Targets

In this section, we discuss the case where there is more than one potential target in the system; i.e., $N \geqslant 2$. We first consider the case where both the attacker and the defender are risk neutral with respect to the level of damage from an attack, and have linear disutility of attacker effort or defensive investment. Then, we explore the effects of risk attitude and convex disutility of effort (i.e., increasing marginal disutility) on the equilibria.

Note that in the case of multiple targets, we cannot guarantee that a pure-strategy equilibrium exists, nor that the defender can always constrain the attacker's choices in a sequential game. This is because the attacker and defender objective functions may not be quasiconcave, so their best responses may not be unique. (Recall that a function f defined on a convex subset U of $\Re^n$ is *quasiconcave* if for every real number a, $C_a \equiv \{x \in U: f(x) \geqslant a\}$ is a convex set.) Note, however, that the existence of a unique best response requires only that the objective function have a unique global optimum; the existence of multiple local optima does not in and of itself cause a problem. Thus, there are likely to be pure-strategy equilibria even in many multiple-target games (especially when the targets are not homogeneous), unless there are multiple local optima, with the same (global optimum) values of the attacker or defender objective functions.

5.1. Risk Neutrality for Damage and Linear Disutility of Effort

The attacker is risk neutral with respect to damage if and only if $u''_A = 0$. In that case, we will have $E\{u_A[w(a, d)]\} =$

$\sum_{i=1}^{N} P_{1i}(a_i, d_{1i})u_{Ai}$. Similarly, the attacker has linear disutility function of attack effort if and only if $g_A(\sum_{i=1}^{N} a_i) = b_A \sum_{i=1}^{N} a_i$, where b_A is a constant. Thus, optimization problem (1) becomes

$$\max_{a_i \geqslant 0 \text{ for } i=1,\ldots,N} \sum_{i=1}^{N} P_{1i}(a_i, d_{1i})u_{Ai} - b_A \sum_{i=1}^{N} a_i, \tag{12}$$

which can be reduced to N single-variable optimization problems as follows:

$$\max_{a_i \geqslant 0} P_{1i}(a_i, d_{1i})u_{Ai} - b_A a_i \quad \text{for } i = 1, \ldots, N. \tag{13}$$

Similar to the attacker's optimization problem, and using the rare-event approximation (as in §4), if the defender is also risk neutral with respect to damage and has linear disutility of defensive investment, then optimization problems (2) and (3) can be reduced to single-variable optimization problems as follows:

$$\max_{d_{1i},\, d_{2i} \geqslant 0} [P_{1i}(a_i, d_{1i}) + P_{2i}(d_{2i})]u_{Di} - b_D(d_{1i} + d_{2i}) \quad \text{for } i = 1, \ldots, N \tag{14}$$

for the simultaneous game, where b_D is a constant, and

$$\max_{d_{1i},\, d_{2i} \geqslant 0} \{P_{1i}[\hat{a}_i(d_{1i}), d_{1i}] + P_{2i}(d_{2i})\}u_{Di} - b_D(d_{1i} + d_{2i}) \quad \text{for } i = 1, \ldots, N \tag{15}$$

for the sequential game. Thus, we see that if both the defender and the attacker are risk neutral with respect to damage and have linear disutilities of attacker effort and defensive investment, the multiple-target game reduces to N independent single-target games, so the results of §4 apply in this case, implying that there will always be a pure-strategy equilibrium. This result depends critically on the assumption that the disutilities of both attacker effort and defensive investment are linear (for example, that neither the attacker nor the defender has a budget constraint, so that both the attacker and the defender can allocate as much effort as desired to one target without having to reduce their allocations to other targets).

5.2. Effects of Risk Attitude

We suggest that attackers and defenders may reasonably be modeled as risk seeking and risk averse over damage levels, respectively. Although we have not been able to find equilibria for this case in a multitarget game, the following theorems provide useful hints for policy considerations:

Theorem 6. *In the two-target game, if neither target individually would merit positive attacker effort, then neither risk-neutral nor risk-averse attackers would find it worthwhile to attack both targets, but risk-seeking attackers may do so.*

Proof. See the online appendix. □

Theorem 7. *In the two-target game, if neither target individually would merit positive defensive investment in protection from terrorism (or natural disaster), then neither risk-neutral nor risk-seeking defenders will find it worthwhile to defend both targets from terrorism (or natural disaster), but risk-averse defenders may do so.*

Proof. The proof is omitted because it is similar to that of Theorem 6. □

Remark. Note that because Theorem 6 is focused on whether the attacker is *potentially* interested in implementing an attack (i.e., whether the attacker will attack when the defensive investment is zero), it applies to both simultaneous and sequential games (and similarly Theorem 7 for defenders). We expect that results similar to those of Theorems 6 and 7 will also hold for systems consisting of more than two targets, although we have not shown this.

Theorem 6 in particular may help to explain why the terrorists involved in the 9/11 tragedy chose to attack four targets simultaneously. Similar examples include: the four attacks on the London public transport system on July 7, 2005; the four attempted attacks on the London transport system on July 21, 2005; the 10 commuter-train explosions in Madrid on March 11, 2004; and the long series of suicide bombings in both Iraq and Israel. In some of these cases, any one target by itself may conceivably not have been sufficiently attractive individually to be worth attacking, but the prospect of being able to cause larger amounts of damage by attacking multiple targets could have been sufficient to motivate the attacker. Similarly, a risk-averse defender may optimally defend more targets than a risk-neutral or risk-seeking defender (possibly even including some targets for which the defensive cost is greater than the expected loss due to an attack on that target alone) because of the effect of risk aversion.

5.3. Convex Disutility of Effort

Theorems 4 and 5 indicate that if the disutility of defensive investment is linear, then the optimal level of defensive investment in protection from nonintentional threats at equilibrium will be independent of both the attacker effort and the defensive investment in protection from intentional threats. However, this is not true for $N \geqslant 2$ when the disutility of effort is convex (as seems likely to be the case in practice), as illustrated in the following example.

Example 3. Consider a two-target game with the following parameters: $u_{A1} = 100$ and $u_{A2} = 3{,}000$ (the attacker's utilities of damage to targets 1 and 2, respectively); $u_{D1} = -200$ and $u_{D2} = -150$ (the defender's disutilities of damage to targets 1 and 2, respectively); $P_{11}(a_1, d_{11}) = a_1/(10(a_1 + d_{11} + 1))$ and $P_{12}(a_2, d_{12}) = a_2/(10(a_2 + d_{12} + 1))$ (the probabilities of damage to targets 1 and 2, respectively, from an intentional attack); $P_{21}(d_{21}) = 1/(10(3 + d_{21}))$ and $P_{22}(d_{22}) = 1/(10(100 + d_{22}))$ (the probabilities of damage to targets 1 and 2, respectively, from natural disaster); and $g_A(x) = g_D(x) = x^2$ (nonlinear disutilities of total attacker effort and total defensive investment, respectively). We also assume that both the attacker and the defender are risk neutral with respect to the level of damage from an attack. Numerical computation yields the simultaneous equilibrium for this game as $a_1^{C*} = 0$, $a_2^{C*} = 5$, $d_{11}^{C*} = 0$, $d_{21}^{C*} = 0.4$, $d_{12}^{C*} = 0.5$, and $d_{22}^{C*} = 0$. In other words, the defender protects target 1 from natural disaster and target 2 from terrorism (knowing that target 2 is more attractive to the attacker than target 1). However, in a single-target game involving only the first target, the simultaneous equilibrium would be $a_1^{C*} = 1$, $d_{11}^{C*} = 1$, and $d_{21}^{C*} = 0$. Thus, the existence of the second target causes the defender to switch from optimally protecting target 1 against terrorism to protecting it against natural disaster at optimality (and also causes the defender to reduce her total defensive investment from 1 to $0.4 + 0.5 = 0.9$).

Remark. The above example indicates that even if protection from terrorism is more cost-effective than protection from natural disasters for a single target, this may no longer be true when additional targets are considered. This is because the terrorist can now redirect his effort among the possible targets in response to the defender's investments. This drastically reduces the defender's ability to allocate her investments in protection from terrorism to take advantage of those targets that are most cost-effective to defend (Bier et al. 2005), if those are not also the most attractive targets to the attacker. In particular, this phenomenon can reduce or eliminate the desirability of protecting less attractive targets (such as relatively small cities) from terrorism.

6. Future Research Directions

In our model, the only decision variable for the defender is the level of defensive investment. Of course, in practice, other possible defender strategies might include deceiving the attacker into mistaken valuations of the various possible targets, increasing the disutility of attack effort (Frey and Luechinger 2003), or taking preemptive action (Sandler and Siqueira 2006). Models addressing the trade-offs between such defender options and others addressed in this paper would be desirable to pursue.

Further research on the multitarget case would also be desirable (possibly using computed equilibria if analytical equilibria prove difficult to find). Note in particular that as in Bier et al. (2007), the targets of greatest interest to the attacker will tend to receive not only the greatest attacker effort at equilibrium, but also the greatest defensive investment. Thus, in many cases, it may be adequate to analyze a reduced game involving a relatively small number of high-value targets, rather than a more complete game involving much larger numbers of targets.

Bayesian methods and signaling games make it possible to relax the assumption of perfect information and allow the defender to have some private information. Note that Powell (2006) has addressed the case in which the defender has private information about the target vulnerability. Extending this to other types of private information would be desired (e.g., the valuation of the target). Moreover, dynamic games (with more than two stages) and population games (with more than one attacker and/or defender) would also be of interest.

Finally, we are interested in exploring the possibility of a "contract" between the attacker and defender, in which the defender gives the attacker "rent" in return for the attacker giving up his attack efforts. To prevent other agents from masquerading as attackers and claiming this rent, a mechanism that yields a separating equilibrium is necessary. For example, the attacker may be required to surrender weapons or other attack resources to claim the rent. This idea still requires further development to investigate its applicability, however.

7. Summary and Conclusions

In the single-target case, our results indicate that increased defensive investment can lead the attacker to either increase his level of effort (to help compensate for the reduced probability of damage from an attack), or decrease his level of effort (because attacking is less profitable at high levels of defensive investment). This can either reduce or increase the effectiveness of investments in protection from intentional attack, and will therefore affect the defender's optimal allocation of resources between protection from intentional attacks and from natural disasters. In particular, this implies that when increased defensive investment causes the attacker to redouble his efforts, defensive investment against terrorism will not decrease the probability of a successful attack as much as the defender might have expected based on an exogenous model of attacker effort.

Thus, the assumption of endogenous attacker effort in this work is critical to capturing important insights into the nature of equilibrium defensive strategies. In particular, therefore, our results emphasize the importance of intelligence in counter-terrorism—to anticipate not only the attacker's choice of targets, but also the likely attacker responses to defensive investments.

Note that protection from terrorism will tend to become less cost-effective for the defender as the number of targets grows, due to the ability of the attacker to redirect his attack effort to less defended targets. Thus, even a target that would have been worth protecting from terrorism in a single-target game may no longer be worth defending from terrorism in a multitarget game. This will in general tend to reduce the effectiveness of protecting large numbers of targets against intentional attacks, and therefore increase the relative desirability of protection from natural disaster (and of all-hazards approaches; see, for example, Woo 2006). This suggests, for example, that the strong emphasis on terrorism defense over natural disaster preparedness at the U.S. Department of Homeland Security may have been misplaced. (Note, however, that our results do not call into question the cost-effectiveness of overarching measures such as intelligence or border security, only the effectiveness of hardening large numbers of targets as a strategy for protection from terrorism.)

Finally, recall that the results of this paper are intended to provide mainly qualitative insights because in practice, it might be difficult to estimate the parameters and functions used in our model. Therefore, we do not currently intend that our model be used in a numeric manner in support of specific decisions. However, the work of Beitel et al. (2004) suggests that the needed estimation tasks may not be altogether impractical, opening up the possibility of applying this type of model to give quantitative as well as qualitative results in the future.

8. Electronic Companion

An electronic companion to this paper is available as part of the online version that can be found at http://or.journal.informs.org/.

Acknowledgments

This research was supported by the United States Department of Homeland Security through the Center for Risk and Economic Analysis of Terrorism Events (CREATE) under grant N00014-05-0630, and the U.S. Army Research Laboratory and the U.S. Army Research Office under grant DAAD19-01-1-0502. Any opinions, findings, and conclusions or recommendations expressed in this document are those of the authors and do not necessarily reflect views of the sponsors. The authors thank Santiago Oliveros (University of California, Berkeley), Todd Sandler (University of Texas at Dallas), William H. Sandholm (University of Wisconsin–Madison), Area Editor Gordon Hazen (Northwestern University), one anonymous associate editor, and two anonymous referees for their helpful comments. The authors assume responsibility for any errors.

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Operations Research
DOI 10.1287/opre.1070.0434ec
pp. ec1–ec3

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Appendix. Proof of Theorems

In order to prove the theorems in this paper, we first prove the following lemmas:

Lemma 1. *The sets of feasible attacker and defender strategies, $\mathcal{A}$ and $\mathcal{D}$, can be replaced by smaller compact and convex sets that still contain any equilibria that may exist.*

Proof for Lemma 1. For the attacker's optimization problem (1), note that the objective function has an upper bound, $U_A(a, d) \leqslant u_A(\sum_{i=1}^N w_i)$, equal to the utility if all targets are successfully attacked. Therefore, we must have $g_A(a_i) \leqslant u_A(\sum_{i=1}^N w_i)$, or $a_i \leqslant g_A^{-1}[u_A(\sum_{i=1}^N w_i)] \equiv \bar{a}\ \forall\, i = 1, \ldots, N$. (Note that our assumptions about g_A guarantee the existence and monotonicity of g_A^{-1}.) Thus, any optimal attacker strategy must be contained in a compact and convex subset of $\mathcal{A}$, given by $\mathcal{A}' = \{(a_1, \ldots, a_N)\colon 0 \leqslant a_i \leqslant \bar{a}\ \forall\, i = 1, \ldots, N\}$. We can use a similar argument to get a compact and convex set $\mathcal{D}' \subset \mathcal{D}$, which must contain the optimal defender strategy. $\square$

Lemma 2. *In single-target game, the attacker and defender objective functions, as defined in optimization problems* (4) *and* (5), *respectively, are strictly concave. Moreover, best-response functions $\hat{a}_1(d_{11})$ and $\hat{d}_{11}(a_1)$ exist uniquely.*

Proof for Lemma 2. First, the attacker's objective function in (4) is strictly concave since

$$\frac{\partial^2 U_A(a_1, d_{11})}{\partial a_1^2} = P_{11}^{(a_1, a_1)}[\hat{a}_1(d_{11}), d_{11}]u_{A1} - g_A''[\hat{a}_1(d_{11})] < 0$$

Second, we show that the defender's objective function in (5) is strictly concave (i.e., the Hessian matrix is negative definite) by calculating the following components in the Hessian matrix:

$$U_D^{(d_{11}, d_{11})} \equiv \frac{\partial^2 U_D(a_1, d_{11}, d_{21})}{\partial d_{11}^2} = P_{11}^{(d_{11}, d_{11})} u_{D1} - g_D'' < 0$$

$$U_D^{(d_{21}, d_{21})} \equiv \frac{\partial^2 U_D(a_1, d_{11}, d_{21})}{\partial d_{21}^2} = P_{21}^{(d_{21}, d_{21})} u_{D1} - g_D'' < 0$$

$$U_D^{(d_{11}, d_{21})} \equiv \frac{\partial^2 U_D(a_1, d_{11}, d_{21})}{\partial d_{11}\partial d_{21}} = -g_D''.$$

Therefore, we have

$$\begin{aligned} U_D^{(d_{11}, d_{11})} U_D^{(d_{21}, d_{21})} - [U_D^{(d_{11}, d_{21})}]^2 &= (P_{11}^{(d_{11}, d_{11})} u_{D1} - g_D'')(P_{21}^{(d_{21}, d_{21})} u_{D1} - g_D'') - (g_D'')^2 \\ &= P_{11}^{(d_{11}, d_{11})} P_{21}^{(d_{21}, d_{21})} u_{D1}^2 - (P_{11}^{(d_{11}, d_{11})} + P_{21}^{(d_{21}, d_{21})}) u_{D1} g_D'' > 0. \end{aligned}$$

Finally, since both the attacker and defender objective functions are strictly concave and twice differentiable, and their domains can be reduced to compact and convex sets (see Lemma 1), $\hat{a}_1(d_{11})$ and $\hat{d}_{11}(a_1)$ must exist uniquely. $\square$

Lemma 3. $(a_i, d_{1i}) = (0, 0)$ *maximizes* $P_{1i}^{(a_i)}(a_i, d_{1i}) - g_A'(a_i)$.

Proof for Lemma 3. Since we assumed that $P_{1i}^{(a_i, d_{1i})}(0, d_{1i}) < 0$, we will have $P_{1i}^{(a_i)}(0,0) \geqslant P_{1i}^{(a_i)}(0, d_{1i})$ for any $d_{1i} \geqslant 0$. Similarly, since we assumed that $P_{1i}^{(a_i, a_i)}(a_i, d_{1i}) < 0$, we will have $P_{1i}^{(a_i)}(0, d_{1i}) \geqslant P_{1i}^{(a_i)}(a_i, d_{1i})$ for any $a_i \geqslant 0$ and $d_{1i} \geqslant 0$. Therefore, we will have $P_{1i}^{(a_i)}(0,0) \geqslant P_{1i}^{(a_i)}(0, d_{1i}) \geqslant P_{1i}^{(a_i)}(a_i, d_{1i})$ for any $a_i \geqslant 0$ and $d_{1i} \geqslant 0$. That is, $(a_i, d_{1i}) = (0,0)$ maximizes $P_{1i}^{(a_i)}(a_i, d_{1i})$. Finally, since we assumed that $g_A'' \geqslant 0$, we will have $a_i = 0$ maximizes $-g_A'(a_i)$. Combining these two results finishes the proof. □

Lemma 4. $\lim_{a_i \to \infty} P_{1i}^{(a_i)}(a_i, d_{1i}) = \lim_{a_i \to \infty} P_{1i}^{(d_{1i})}(a_i, d_{1i}) = 0.$

Proof for Lemma 4. Since we assumed that $P_{1i}^{(a_i)}(a_i, d_{1i}) > 0$ and $P_{1i}^{(a_i, a_i)}(a_i, d_{1i}) < 0$, we know that $P_{1i}^{(a_i)}(a_i, d_{1i})$ is decreasing in a_i but must always be positive. Therefore, we must have $\lim_{a_i \to \infty} P_{1i}^{(a_i)}(a_i, d_{1i}) = L$ for some $L \geqslant 0$ (i.e., the limit must exist and be non-negative). Suppose that $L > 0$. Then there must exist $\acute{a}_i \geqslant 0$ such that $|P_{1i}^{(a_i)}(a_i, d_{1i}) - L| < L/2$ for all $a_i \geqslant \acute{a}_i$. Thus, we will have $P_{1i}^{(a_i)}(a_i, d_{1i}) = |P_{1i}^{(a_i)}(a_i, d_{1i}) - L + L| \geqslant L - |P_{1i}^{(a_i)}(a_i, d_{1i}) - L| > L - L/2 = L/2$ for all $a_i \geqslant \acute{a}_i$.

By the mean-value theorem, we know that there exists $\grave{a}_i \in (\acute{a}_i, a_i)$ such that $P_{1i}(a_i, d_{1i}) = P_{1i}(\acute{a}_i, d_{1i}) + (a_i - \acute{a}_i) P_{1i}^{(a_i)}(\grave{a}_i, d_{1i})$. Therefore, we have $P_{1i}(a_i, d_{1i}) \geqslant (a_i - \acute{a}_i) P_{1i}^{(a_i)}(\grave{a}_i, d_{1i}) > (a_i - \acute{a}_i)(L/2)$ for all $a_i \geqslant \acute{a}_i$. Hence $P_{1i}(a_i, d_{1i})$ must go to infinity as a_i goes to infinity. However, this contradicts the fact that a probability must be less than or equal to one. Since a contradiction has been found by supposing $L > 0$, we must have $\lim_{a_i \to \infty} P_{1i}^{(a_i)}(a_i, d_{1i}) = L = 0$. (The second part of this lemma can be proved similarly, so the proof is omitted.) □

Lemma 5. *If $u_j(0) = 0$ for $j = A, D$, then we have $u_j'' > 0$, $u_j'' = 0$, or $u_j'' < 0$, respectively, if and only if $u_j(c_1 + c_2) - u_j(c_1) - u_j(c_2) > 0$, $u_j(c_1 + c_2) - u_j(c_1) - u_j(c_2) = 0$, or $u_j(c_1 + c_2) - u_j(c_1) - u_j(c_2) < 0$ for all $c_1, c_2 > 0$.*

Proof for Lemma 5. By the intermediate-value theorem, we have $u_j(c_1 + c_2) = u_j(c_1) + c_2 u_j'(c_3)$ and $u_j(c_2) = u_j(0) + c_2 u_A'(c_4) = c_2 u_j'(c_4)$, where $0 < c_4 < c_2 < c_3 < c_1 + c_2$. Therefore, we have $u_j(c_1 + c_2) - u_j(c_1) - u_j(c_2) = c_2[u_j'(c_3) - u_j'(c_4)] > 0$ for all c_1, c_2 if and only if $u_j'' > 0$. Other parts of this lemma can be proved similarly. □

Proof for Theorem 3. Since the attacker's objective function in (4) is strictly concave (Lemma 2), and the set of feasible strategies $\mathscr{A}$ can be replaced by a smaller convex set (Lemma 1), the following first-order condition for an interior solution $\hat{a}_1(d_{11}) > 0$,

$$U_A^{(a_1)}[\hat{a}_1(d_{11}), d_{11}] = P_{11}^{(a_1)}[\hat{a}_1(d_{11}), d_{11}] u_{A1} - g_A'[\hat{a}_1(d_{11})] = 0 \tag{EC1}$$

is indeed a necessary and sufficient condition for an interior solution $\hat{a}_1(d_{11}) > 0$. There are two possibilities:

1. $U_A^{(a_1)}(0, d_{11}) \leqslant 0$. Since $U_A^{(a_1)}(a_1, d_{11})$ is decreasing in a_1 for any given value of d_{11}, (EC1) will not hold for any $\hat{a}_1(d_{11}) > 0$. Therefore, we must have a corner solution $\hat{a}_1(d_{11}) = 0$.

2. $U_A^{(a_1)}(0, d_{11}) > 0$. Since $g_A'' \geqslant 0$ and $g_A > 0$, we must have $\lim_{a_1 \to \infty} g_A'(a_1) > 0$. Note also that we have $\lim_{a_1 \to \infty} P_{11}^{(a_1)}(a_1, d_{11}) = 0$ (from Lemma 4).s Therefore, we have $\lim_{a_1 \to \infty} U_A^{(a_1)}(a_1, d_{11}) = \lim_{a_1 \to \infty} P_{11}^{(a_1)}(a_1, d_{11}) u_{A1} - \lim_{a_1 \to \infty} g_A'(a_1) = 0 - \lim_{a_1 \to \infty} g_A'(a_1) < 0$. Recalling that $U_A^{(a_1)}(a_1, d_{11})$ is continuous and decreasing in a_1, there must exist $a_1 > 0$ such that $U_A^{(a_1)}(a_1, d_{11}) = 0$. Therefore, we get an interior solution, $\{a_1\colon U_A^{(a_1)}(a_1, d_{11}) = 0\}$.

The slope $\mathrm{d}\hat{a}_1(d_{11})/\mathrm{d}d_{11}$ is obtained by differentiating Equation (EC1):

$$P_{11}^{(a_1, d_{11})}[\hat{a}_1(d_{11}), d_{11}] u_{A1} \mathrm{d}d_{11} + \{P_{11}^{(a_1, a_1)}[\hat{a}_1(d_{11}), d_{11}] u_{A1} - g_A''[\hat{a}_1(d_{11})]\} \mathrm{d}\hat{a}_1(d_{11}) = 0$$

$$\Rightarrow \frac{\mathrm{d}\hat{a}_1(d_{11})}{\mathrm{d}d_{11}} = \frac{P_{11}^{(a_1, d_{11})}[\hat{a}_1(d_{11}), d_{11}]}{g_A''[\hat{a}_1(d_{11})] - P_{11}^{(a_1, a_1)}[\hat{a}_1(d_{11}), d_{11}] u_{A1}}.$$

Since we assume that $g_A'' > 0$ and $P_{11}^{(a_1, a_1)} < 0$, we have the sign of $\mathrm{d}\hat{a}_1(d_{11})/\mathrm{d}d_{11}$ the same as the sign of $P_{11}^{(a_1, d_{11})}[\hat{a}_1(d_{11}), d_{11}]$.

Moreover, since we have assumed that $\lim_{d_{1i} \to \infty} P_{1i}(a_i, d_{1i}) = 0$, $g_A(0) = 0$, and $g_A' > 0$, we must have $\lim_{d_{11} \to \infty} U_A(a_1, d_{11}) = \lim_{d_{11} \to \infty} P_{11}(a_1, d_{11}) - g_A(a_1) = -g_A(a_1) \leqslant -g_A(0) = 0$. This implies that the attacker best-response function must satisfy $\lim_{d_{11} \to \infty} \hat{a}_1(d_{11}) = 0$ in order to maximize the attacker's overall utility as the defense against terrorism grows to infinity. Finally, since we have $\partial U_A^{(a_1)}(0, d_{11})/\partial d_{11} = P_{11}^{(a_1, d_{11})}(0, d_{11}) u_{A1} < 0$, we will have that $U_A^{(a_1)}(0, d_{11}) \leqslant 0$, and therefore $\hat{a}_1(d_{11}) = 0$ for any $d_{11} > 0$ (by applying the result from the first case above in this proof) if and only if $U_A^{(a_1)}(0, 0) \leqslant 0$. □

Proof for Theorem 4. Since the defender's objective function in (5) is concave (Lemma 2) and the set of feasible strategies $\mathscr{D}$ can be replaced by a smaller convex set (see Lemma 1), the necessary and sufficient conditions to have interior solutions $\hat{d}_{11} > 0$ and $\hat{d}_{21} > 0$ are $U_D^{(d_{11})}(a_1, \hat{d}_{11}, \hat{d}_{21}) = 0$ and $U_D^{(d_{21})}(\hat{d}_{11}, \hat{d}_{21}) = 0$, respectively.

Moreover, we have $\hat{d}_{11} = 0$ if and only if $U_D^{(d_{11})}(a_1, 0, \hat{d}_{21}) \leqslant 0$ for any $\hat{d}_{21} \geqslant 0$. This statement is proved by contradiction as follows: (1) If $\hat{d}_{11} = 0$, but we suppose $U_D^{(d_{11})}(a_1, 0, \hat{d}_{21}) > 0$, then there must exist $\hat{d}'_{11} > 0$ such that $U_D^{(d_{11})}(a_1, \hat{d}'_{11}, \hat{d}_{21}) = 0$ (since $U_D^{(d_{11}, d_{11})} < 0$), so $\hat{d}'_{11} > 0$ would be an interior solution, which is a contradiction. (b) If $U_D^{(d_{11})}(a_1, 0, \hat{d}_{21}) \leqslant 0$, but we suppose $\hat{d}_{11} > 0$, then by the first-order condition we will have $U_D^{(d_{11})}(a_1, \hat{d}_{11}, \hat{d}_{21}) = 0$. This is a contradiction since we have shown that $U_D^{(d_{11}, d_{11})} < 0$ in the proof of Lemma 2, which means that $U_D^{(d_{11})}(a_1, d_{11}, \hat{d}_{21})$ must be decreasing in d_{11}. Similarly, we can show that we have $\hat{d}_{21} = 0$ if and only if $U_D^{(d_{21})}(\hat{d}_{11}, 0) \leqslant 0$ for any $\hat{d}_{11} \geqslant 0$.

Finally, we prove that $\hat{d}_{11}(0) = 0$ and $\lim_{a_1 \to \infty} \hat{d}_{11}(a_1) = 0$ by showing that $U_D^{(d_{11})}(0, 0, \hat{d}_{21}) \leqslant 0$ and $\lim_{a_1 \to \infty} U_D^{(d_{11})}(a_1, 0, \hat{d}_{21}) \leqslant 0$ for all $d_{21} \geqslant 0$. The first is because $P_{11}(0, d_{11}) = 0 \Rightarrow P_{11}^{(d_{11})}(0, d_{11}) = 0$ for all $d_{11} \geqslant 0$. The second follows from the fact that $\lim_{a_1 \to \infty} P_{11}^{(d_{11})}(a_1, d_{11}) = 0$. (See Lemma 4.) □

Proof for Theorem 6. For convenience, in this proof, we let $P_{11} \equiv P_{11}(a_1, d_{11})$ and $P_{12} \equiv P_{12}(a_2, d_{12})$. There exist exactly four possible outcomes for the attacker when $N = 2$:

1. Attacks succeed against both targets. This occurs with probability $P_{11}P_{12}$, and leads to a positive damage utility of $u_A(w_1 + w_2)$.
2. The attack against target 1 succeeds, but the attack against target 2 fails. This occurs with probability $P_{11}(1 - P_{12})$, and leads to a positive damage utility of u_{A1}.
3. The attack against target 2 succeeds, but the attack against target 1 fails. This occurs with probability $P_{12}(1 - P_{11})$, and leads to a positive damage utility of u_{A2}.
4. Attacks fail against both targets 1 and 2, leading to a zero utility of damage.

Therefore, the attacker's optimization problem (1) becomes:

$$\max_{a_1, a_2 \geqslant 0} P_{11}P_{12}u_A(w_1 + w_2) + P_{11}(1 - P_{12})u_{A1} + (1 - P_{11})P_{12}u_{A2} - g_A(a_1 + a_2). \tag{EC2}$$

By Theorem 3, we know that neither target individually would ever merit positive attacker effort if $U_A^{(a_i)}(0, 0) \leqslant 0$, or equivalently if

$$P_{1i}^{(a_i)}(0, 0)u_{Ai} - g'_A(0) \leqslant 0 \quad \text{for } i = 1, 2. \tag{EC3}$$

In order to prove that a risk neutral or risk averse attacker would not attack both targets, without loss of generality, it is sufficient to show that he would not attack target 2 given any positive level of attack effort on target 1. We calculate the marginal total expected utility by taking the derivative of (EC2) with respect to a_2, yielding the following marginal total payoff to the attacker:

$$\begin{aligned} U_A^{(a_2)}(a_2, d_{12}) &\equiv P_{11}P_{12}^{(a_2)}u_A(w_1 + w_2) + P_{11}(-P_{12}^{(a_2)})u_{A1} + (1 - P_{11})P_{12}^{(a_2)}u_{A2} - g'_A(a_1 + a_2) \\ &= P_{12}^{(a_2)}u_{A2} + P_{12}^{(a_2)}P_{11}[u_A(w_1 + w_2) - u_{A1} - u_{A2}] - g'_A(a_1 + a_2). \end{aligned}$$

Similar to the reasoning in the proof of Theorem 3, the attacker will not be interested in attacking target 2 (even the defense is zero) if and only if $U_A^{(a_2)}(0, 0) \leqslant 0$, or, equivalently,

$$U_A^{(a_2)}(0, 0) = P_{12}^{(a_2)}(0, 0)u_{A2} + P_{12}^{(a_2)}(0, 0)P_{11}\,[u_A(w_1 + w_2) - u_{A1} - u_{A2}] - g'_A(a_1) \leqslant 0. \tag{EC4}$$

By comparing (EC3) with (EC4) and noting that $P_{12}^{(a_2)}(0, 0)P_{11} > 0$ (see the proof for Lemma 3), and $g'_A(a_1) \geqslant g_A(0)$, we see that (EC4) will hold if $u_A(w_1 + w_2) - u_A(w_1) - u_A(w_2) \leqslant 0$ (in other words, if the attacker is risk neutral or risk averse; see Lemma 5). However, for a risk seeking attacker, (EC4) will not necessarily hold even if (EC3) is satisfied, since $u_A(w_1 + w_2)$ may be substantially greater than $u_A(w_1) + u_A(w_2)$. □