LAGRANGE/GIBBS F AND G SOLUTIONS

Puneet Singla

Celestial Mechanics
AERO-624
Department of Aerospace Engineering
Texas A&M University

http://people.tamu.edu/~puneet/AERO624

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planar motion ⇒ any vector in the plane can be written as the linear combination of \( r_0 \) and \( \dot{r}_0 \)

\[
\begin{align*}
\mathbf{r}(t) &= F \mathbf{r}_0 + G \dot{\mathbf{r}}_0 \\
\dot{\mathbf{r}}(t) &= \dot{F} \mathbf{r}_0 + \dot{G} \dot{\mathbf{r}}_0
\end{align*}
\]

Substitution of Eq. (1) in EOM leads to following second order differential equations for \( F \) and \( G \):

\[
\begin{align*}
\ddot{F} &= -\frac{\mu}{r^3} F \\
\ddot{G} &= -\frac{\mu}{r^3} G
\end{align*}
\]

Note:

\[
\begin{align*}
F_0 &= 1, \quad \dot{F}_0 = 0 \\
G_0 &= 0, \quad \dot{G}_0 = 1
\end{align*}
\]

\( F \) and \( G \) must depend upon time, \( \mathbf{r}_0 \) and \( \dot{\mathbf{r}}_0 \)
Let us consider the Taylor series expansion of $r(t)$

$$r(t) = r_0 + \sum_{n=1}^{\infty} \frac{(t-t_0)^n}{n!} \frac{d^n r}{dt^n}|_{t_0}$$

(6)

$$\frac{d^2 r}{dt^2} = -\frac{\mu}{r^3} r \quad \Rightarrow \quad \frac{d^2 r}{dt^2}|_{t_0} = -\frac{\mu}{r_0^3} r_0$$

$$\frac{d^3 r}{dt^3} = 3 \frac{\mu}{r^4} r \dot{r} - \frac{\mu}{r^3} \dot{r} \quad \Rightarrow \quad \frac{d^3 r}{dt^3}|_{t_0} = \left[3 \frac{\mu}{r^4} \dot{r}_0\right] r_0 - \left[\frac{\mu}{r^3}\right] \ddot{r}_0$$

$$\vdots$$

$$\frac{d^n r}{dt^n}|_{t_0} = \left[fct.(r_0, \dot{r}_0, \cdots, \frac{d^{n-1} r}{dt^{n-1}}|_{t_0})\right] r_0 + \left[fct.(r_0, \dot{r}_0, \cdots, \frac{d^{n-1} r}{dt^{n-1}}|_{t_0})\right] \ddot{r}_0$$

First three are easy to obtain and thereafter, the fun begins!

No recursion is immediately obvious
What about making use of $\sigma \triangleq \frac{r \cdot \dot{r}}{\sqrt{\mu}}$

Recall from previous lecture $\ddot{\sigma} = -\frac{\mu}{r^3} \sigma$, also $\dot{r} = \frac{\sqrt{\mu}}{r} \sigma$.

Let us consider: $\frac{d^3 r}{dt^3} = 3 \frac{\mu}{r^4} r \dot{r} - \frac{\mu}{r^3} \ddot{r} = \left[ 3 \frac{\mu^{3/2}}{r^5} \sigma \right] r - \left[ \frac{\mu}{r^3} \right] \dot{r}$

Notice,

$$\sqrt{\mu} \dot{\sigma} = v^2 + \dot{r} \cdot r, \quad \dot{r} = -\frac{\mu}{r^3} r$$

$$= v^2 - \frac{\mu}{r}, \quad v^2 = \mu \left( \frac{2}{r} - \alpha \right)$$

$$= \mu \left( \frac{1}{r} - \alpha \right)$$

Finally,

$$\dot{\sigma}_0 = \sqrt{\mu} \left( \frac{1}{r_0} - \alpha \right)$$

It can be easily shown that all derivatives can be expressed as functions of $t, r_0, \sigma_0, \dot{\sigma}_0 \text{ or } \alpha$
A much more elegant power series expansion results if we introduce
"LaGrange's Fundamental Invariants":

\[ \varepsilon \equiv \frac{\mu}{r^3} \quad \lambda \equiv \frac{\mathbf{r} \cdot \dot{\mathbf{r}}}{r^2} = \frac{\dot{r}}{r} \quad \psi \equiv \frac{\ddot{r}}{r^2} = \frac{v^2}{r^2} \]

Notice the derivatives of \( \{\varepsilon, \lambda, \psi\} \):

\[ \dot{\varepsilon} = -\frac{3\mu\dot{r}}{r^4} \quad \dot{\lambda} = \frac{\dot{\mathbf{r}} \cdot \dddot{\mathbf{r}}}{r^2} + \frac{\mathbf{r} \cdot \dddot{\mathbf{r}}}{r^2} - \frac{2\mathbf{r} \cdot \dddot{\mathbf{r}}}{r^3} \frac{\dot{\mathbf{r}}}{r} \quad \dot{\psi} = \frac{2\dddot{\mathbf{r}}}{r^2} - \frac{2\dot{\mathbf{r}} \cdot \dddot{\mathbf{r}}}{r^3} \frac{\dot{\mathbf{r}}}{r} \]

\[ \dot{\varepsilon} = -3\varepsilon \quad \dot{\lambda} = \psi \frac{\mu}{r^3} - \frac{2\dot{r}^2}{r^2} \quad \dot{\psi} = \frac{-2\mu \mathbf{r} \cdot \dddot{\mathbf{r}}}{r^5} - 2\lambda \psi \]

\[ \dot{\lambda} = \psi - \varepsilon - 2\lambda^2 \quad \dot{\psi} = -2\lambda \varepsilon - 2\lambda \psi \]

So \( \{\varepsilon, \lambda, \psi\} \) are "closed" wrt differentiation... notice these expressions are all polynomials... no fractions...
Battin shows that the successive derivatives of \( \mathbf{r} \) can easily be established in terms of \( \{\varepsilon, \lambda, \psi\} \):

\[
\frac{d^2 \mathbf{r}}{dt^2} = -\varepsilon \mathbf{r}
\]

\[
\frac{d^3 \mathbf{r}}{dt^3} = -\dot{\varepsilon} \mathbf{r} - \varepsilon \dot{\mathbf{r}} = 3\varepsilon \lambda \mathbf{r} - \varepsilon \ddot{\mathbf{r}}
\]

\[
\frac{d^4 \mathbf{r}}{dt^4} = \cdots = (-15\varepsilon \lambda^2 + 3\varepsilon \psi - 2\varepsilon^2) \mathbf{r} + 6\varepsilon \lambda \ddot{\mathbf{r}}
\]

\[
\frac{d^5 \mathbf{r}}{dt^5} = \cdots = (-105\varepsilon \lambda^3 - 45\varepsilon \lambda \psi + 30\varepsilon^2 \lambda) \mathbf{r}
\]

\[
\vdots + (-45 \varepsilon \lambda^2 + 9\varepsilon \psi - 8\varepsilon^2) \dot{\mathbf{r}}
\]

\[
\frac{d^n \mathbf{r}}{dt^n} = \left(\text{polynomial in } \varepsilon, \lambda, \psi\right) \mathbf{r} + \left(\text{polynomial in } \varepsilon, \psi, \lambda\right) \dot{\mathbf{r}}
\]

Polynomials are easy to differentiate, but what about recursions for the coefficients??
POWER SERIES SOLUTION FOR $F$ AND $G$: LAGRANGE’S FUNDAMENTAL INVARIANTS

To have more formal procedure let us consider a generic variable $Q$ which satisfies following differential equation:

$$\ddot{Q} + \varepsilon Q = 0$$ (7)

We seek power series solution of the following form:

$$Q = \sum_{n=0}^{\infty} Q_n(t - t_0)^n \quad \varepsilon = \sum_{n=0}^{\infty} \varepsilon_n(t - t_0)^n, \quad \varepsilon \to \lambda, \psi$$ (8)

$$\downarrow$$

$$\ddot{Q} = \sum_{n=0}^{\infty} (n + 2)(n + 1)Q(n + 2)(t - t_0)^n$$ (9)

Note:

$$\varepsilon Q = \sum_{n=0}^{\infty} \varepsilon_n(t - t_0)^n \sum_{m=0}^{\infty} Q_m(t - t_0)^m = \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \varepsilon_n Q_m(t - t_0)^{n+m}$$ (10)
POWER SERIES SOLUTION FOR $F$ AND $G$: LAGRANGE’S FUNDAMENTAL INVARIANTS

Equating coefficients of equal powers of $(t - t_0)$ leads to following recursion equations:

$$(n + 1)(n + 2)Q_{n+2} = - (\epsilon_0 Q_n + \cdots + \epsilon_n Q_0)$$

$$(n + 1)\epsilon_{n+1} = -3 (\epsilon_0 \lambda_n + \cdots + \epsilon_n \lambda_0)$$

$$(n + 1)\lambda_{n+1} = \psi_n - \epsilon_n - 2 (\lambda_0 \lambda_n + \cdots + \lambda_n \lambda_0)$$

$$(n + 1)\psi_{n+1} = -2 [\lambda_0 (\epsilon_n + \psi_n) + \cdots + \lambda_n (\epsilon_0 + \psi_0)]$$

Now, assuming $Q = F$ or $Q = G$, we get series coefficients for $F$ and $G$:

$$F_0 = 1 \quad F_1 = 0 \quad F_2 = -\frac{1}{\epsilon_0} \quad F_3 = \frac{1}{2} \epsilon_0 \lambda_0 \quad \cdots \quad (11)$$

$$G_0 = 0 \quad G_1 = 1 \quad G_2 = 0 \quad G_3 = -\frac{1}{6} \epsilon_0 \quad G_4 = \frac{1}{4} \epsilon_0 \lambda_0 \quad \cdots \quad (12)$$

Problem with power series is slow convergence. It does not converge fast enough unless $(t - t_0)$ is a small fraction of an orbital period.
Let us consider:

\[ \mathbf{r}(t) = F \mathbf{r}_0 + G \mathbf{\dot{r}}_0, \quad \mathbf{\dot{r}}(t) = F \mathbf{\dot{r}}_0 + G \mathbf{\ddot{r}}_0 \]

\[ \downarrow \]

\[ \begin{cases} x(t) \\ y(t) \end{cases} = \begin{bmatrix} x_0 & \dot{x}_0 \\ y_0 & \dot{y}_0 \end{bmatrix} \begin{cases} F \\ G \end{cases} \]

\[ \Rightarrow \begin{cases} F \\ G \end{cases} = \frac{1}{(x_0 \dot{y}_0 - \dot{x}_0 y_0)} \begin{bmatrix} \dot{y}_0 & -\dot{x}_0 \\ -y_0 & x_0 \end{bmatrix} \begin{cases} x(t) \\ y(t) \end{cases} \] (13)

Note: \( x_0 \dot{y}_0 - \dot{x}_0 y_0 = h = \sqrt{\mu p} \), so

\[ F = \frac{1}{\sqrt{\mu p}} (xy_0 - y\dot{x}_0), \quad G = \frac{1}{\sqrt{\mu p}} (xy_0 - yx_0) \] (15)

Recall:

\[ x = a(\cos E - e) \quad \dot{x} = -\frac{\sqrt{\mu a}}{r} \sin E \]
\[ y = a\sqrt{1 - e^2} \sin E \quad \dot{y} = \frac{\sqrt{\mu a(1-e^2)}}{r} \cos E \] (16)
**Analytical Solution for F and G**

Now, substitution of Eq. (16) in Eq. (15) leads to following equation for $F$:

$$F = \frac{1}{\sqrt{\mu a(1-e^2)}} \left[ a(\cos E - e) \frac{\sqrt{\mu a(1-e^2)}}{r_0} \cos E_0 ight.
$$

$$+ a \sqrt{1-e^2} \sin E \frac{\sqrt{\mu a}}{r_0} \sin E_0 \right]$$

$$= \frac{a}{r_0} \left[ \cos E \cos E_0 + \sin E \sin E_0 - e \cos E_0 \right]$$

$$\cos(E-E_0) \quad 1-r_0/a \right]$$

(17)

Introduce $\hat{E} = E - E_0 =$ change in eccentric anomaly

$$F = 1 - \frac{a}{r_0} \left[ 1 - \cos(\hat{E}) \right]$$

(19)

This is an exact equation for $F$
Repeating same procedure for $G$, we get:

$$G = \frac{a^{3/2}}{\sqrt{\mu}} \left[ \sin \hat{E} - e (\sin E - \sin E_0) \right]$$ (20)

Further, from Kepler’s Equation, we have:

$$M = M_0 + \frac{\sqrt{\mu}}{a^{3/2}} (t - t_0) = E - e \sin E$$ (21)

$$M_0 = E_0 - e \sin E_0$$ (22)

subtract to obtain,

$$- e (\sin E - \sin E_0) = \frac{\sqrt{\mu}}{a^{3/2}} (t - t_0) - \hat{E}$$ (23)

substitute Eq. (23) in Eq. (20) to obtain

$$G = (t - t_0) + \frac{a^{3/2}}{\sqrt{\mu}} \left[ \sin \hat{E} - \hat{E} \right]$$ (24)
What about $\dot{F}$ and $\dot{G}$?

$$F = 1 - \frac{a}{r_0} \left[ 1 - \cos(\hat{E}) \right] \quad G = (t - t_0) + \frac{a^{3/2}}{\sqrt{\mu}} \left[ \sin \hat{E} - \hat{E} \right]$$

$$\dot{F} = -\frac{a}{r_0} \sin \hat{E} \hat{E} \quad \dot{G} = 1 + \frac{a^{3/2}}{\sqrt{\mu}} \left[ \cos \hat{E} - 1 \right] \hat{E}$$

Note: $\dot{\hat{E}} = \dot{E} = \sqrt{\frac{\mu}{a}} \frac{1}{r}$

$$\dot{F} = -\frac{\sqrt{\mu a}}{rr_0} \sin \hat{E} \quad \dot{G} = 1 - \frac{a}{r} \left[ 1 - \cos \hat{E} \right]$$

Note: $E = \hat{E} + E_0$

$$\cos E = \cos E_0 \cos \hat{E} - \sin E_0 \sin \hat{E}$$

$$\sin E = \sin E_0 \cos \hat{E} + \cos E_0 \sin \hat{E}$$

$$r = a \left( 1 - e \cos E \right) \Rightarrow r = a \left( 1 - e \cos E_0 \cos \hat{E} + e \sin E_0 \sin \hat{E} \right)$$

$$r = a + (r_0 - a) \cos \hat{E} + \sqrt{a \sigma_0} \sin \hat{E}$$
Recall: \[ \frac{d\hat{E}}{dt} = \sqrt{\frac{\mu}{a}} \]

\[ rd\hat{E} = \sqrt{\frac{\mu}{a}} dt \]

\[ (a + (r_0 - a) \cos \hat{E} + \sqrt{a}\sigma_0 \sin \hat{E}) \, d\hat{E} = \sqrt{\frac{\mu}{a}} dt \]

Integration of the above Eq. results:

\[ a\hat{E} + (r_0 - a) \sin \hat{E} - \sqrt{a}\sigma_0 \cos \hat{E} = \sqrt{\frac{\mu}{a}} (t - t_0) \]

\[ \hat{E} - \left( 1 - \frac{r_0}{a} \right) \sin \hat{E} - \frac{\sigma_0}{\sqrt{a}} \cos \hat{E} = \sqrt{\frac{\mu}{a^3/2}} (t - t_0) \]

Newton’s Method can be used to solve the above Eq.
\( \mathbf{r} = F \mathbf{r}(t_0) + G \dot{\mathbf{r}}(t_0) \quad \dot{\mathbf{r}} = \mathbf{\dot{F}} \mathbf{r}(t_0) + \mathbf{\dot{G}} \dot{\mathbf{r}}(t_0) \)

**Equations in Order of Solution**

\[
\begin{align*}
\mathbf{r}_0^2 &= \mathbf{r}(t_0) \cdot \mathbf{r}(t_0) = X_0^2 + Y_0^2 + Z_0^2 \\
\mathbf{v}_0^2 &= \mathbf{\dot{r}}(t_0) \cdot \mathbf{\dot{r}}(t_0) = \dot{X}_0^2 + \dot{Y}_0^2 + \dot{Z}_0^2 \\
\sqrt{\mu} \quad \sigma_0 &= \mathbf{r}(t_0) \cdot \mathbf{\dot{r}}(t_0) = X_0 \dot{X}_0 + Y_0 \dot{Y}_0 + Z_0 \dot{Z}_0 \\
\alpha &= \frac{1}{a} = \frac{2}{r_0} - \frac{\mathbf{v}_0^2}{\mu} \\
\sqrt{\frac{\mu}{a^{3/2}}} (t - t_0) &= \hat{E} - \left(1 - \frac{r_0}{a}\right) \sin \hat{E} - \frac{\sigma_0}{\sqrt{a}} \left(\cos \hat{E} - 1\right) \\
r &= a + (r_0 - a) \cos \hat{E} + \sqrt{a} \quad \sigma_0 \sin \hat{E} \\
F &= 1 - \frac{a}{r_0} \left(1 - \cos \hat{E}\right), \quad G = (t - t_0) + \frac{a^{3/2}}{\sqrt{\mu}} \left(\sin \hat{E} - \hat{E}\right) \\
\dot{F} &= -\frac{\sqrt{\mu} a}{r} \sin \hat{E}, \quad \dot{G} = 1 - \frac{a}{r} \left(1 - \cos \hat{E}\right)
\end{align*}
\]

\[
\begin{bmatrix} X \\ Y \\ Z \end{bmatrix} = F \begin{bmatrix} X(t_0) \\ Y(t_0) \\ Z(t_0) \end{bmatrix} + G \begin{bmatrix} \dot{X}(t_0) \\ \dot{Y}(t_0) \\ \dot{Z}(t_0) \end{bmatrix}, \quad \begin{bmatrix} \dot{X} \\ \dot{Y} \\ \dot{Z} \end{bmatrix} = \mathbf{\dot{F}} \begin{bmatrix} X(t_0) \\ Y(t_0) \\ Z(t_0) \end{bmatrix} + \mathbf{\dot{G}} \begin{bmatrix} \dot{X}(t_0) \\ \dot{Y}(t_0) \\ \dot{Z}(t_0) \end{bmatrix} \Rightarrow \mathbf{r}(t), \quad \dot{\mathbf{r}}(t)
\]

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