**Problem 5.22**  A long cylindrical conductor whose axis is coincident with the $z$-axis has a radius $a$ and carries a current characterized by a current density $J = \frac{2J_0}{r}$, where $J_0$ is a constant and $r$ is the radial distance from the cylinder’s axis. Obtain an expression for the magnetic field $H$ for

(a) $0 \leq r \leq a$

(b) $r > a$

**Solution:** This problem is very similar to Example 5-5.

(a) For $0 \leq r_1 \leq a$, the total current flowing within the contour $C_1$ is

$$I_1 = \int_0^{2\pi} \int_{r=0}^{r_1} \left( \frac{2J_0}{r} \right) \cdot (2\pi r dr d\phi) = 2\pi \int_{r=0}^{r_1} J_0 dr = 2\pi r_1 J_0.$$  

Therefore, since $I_1 = 2\pi r_1 H_1$, $H_1 = J_0$ within the wire and $H_1 = \hat{z} J_0$.

(b) For $r \geq a$, the total current flowing within the contour is the total current flowing within the wire:

$$I = \int_0^{2\pi} \int_{r=0}^{a} \left( \frac{2J_0}{r} \right) \cdot (2\pi r dr d\phi) = 2\pi \int_{r=0}^{a} J_0 dr = 2\pi a J_0.$$  

Therefore, since $I = 2\pi r H_2$, $H_2 = J_0 a/r$ within the wire and $H_2 = \hat{z} J_0 (a/r)$. 


Problem 5.27  In a given region of space, the vector magnetic potential is given by \( \mathbf{A} = \hat{x}5 \cos \pi y + \hat{z}(2 + \sin \pi x) \) (Wb/m).

(a) Determine \( \mathbf{B} \).

(b) Use Eq. (5.66) to calculate the magnetic flux passing through a square loop with 0.25-m-long edges if the loop is in the \( x-y \) plane, its center is at the origin, and its edges are parallel to the \( x- \) and \( y- \) axes.

(c) Calculate \( \Phi \) again using Eq. (5.67).

Solution:

(a) From Eq. (5.53), \( \mathbf{B} = \nabla \times \mathbf{A} = \hat{z}5 \pi \sin \pi y - \hat{y} \pi \cos \pi x \).

(b) From Eq. (5.66),

\[
\Phi = \int \int \mathbf{B} \cdot d\mathbf{s} = \int_{0.125 \text{ m}}^{-0.125 \text{ m}} \int_{0.125 \text{ m}}^{-0.125 \text{ m}} (5 \pi \cos \pi y - \hat{y} \pi \cos \pi x) \cdot (\hat{x} \, dx \, dy)
\]

\[
= \left( \left. \left( \frac{5 \pi x \cos \pi y}{\pi} \right) \right|_{x=-0.125}^{0.125} \right) \left. \left( \frac{0.125}{y=-0.125} \right) \right|_{y=-0.125}^{0.125}
\]

\[
= - \frac{5}{4} \left( \cos \left( \frac{\pi}{8} \right) - \cos \left( -\frac{\pi}{8} \right) \right) = 0.
\]

(c) From Eq. (5.67), \( \Phi = \int \int \mathbf{A} \cdot d\ell \), where \( C \) is the square loop in the \( x-y \) plane with sides of length 0.25 m centered at the origin. Thus, the integral can be written as

\[
\Phi = \int_{C} \mathbf{A} \cdot d\ell = S_{\text{front}} + S_{\text{back}} + S_{\text{left}} + S_{\text{right}},
\]

where \( S_{\text{front}}, S_{\text{back}}, S_{\text{left}}, \) and \( S_{\text{right}} \) are the sides of the loop.

\[
S_{\text{front}} = \int_{x=-0.125}^{0.125} \left( \hat{x} \, dx \right)\left(5 \cos \pi y + \hat{z}(2 + \sin \pi x)\right)\big|_{y=-0.125}^{0.125}
\]

\[
= \int_{x=-0.125}^{0.125} 5 \cos \pi y \, dx
\]

\[
= \left( \left. 5 \cos \pi y \right|_{x=-0.125}^{0.125} \right)\bigg/ \left. \left. \frac{0.125}{y=-0.125} \right|_{x=-0.125}^{0.125} \right) = \frac{5}{4} \cos \left( -\frac{\pi}{8} \right) = \frac{5}{4} \cos \left( \frac{\pi}{8} \right),
\]

\[
S_{\text{back}} = -\int_{x=-0.125}^{0.125} \left( \hat{x} \, dx \right)\left(5 \cos \pi y + \hat{z}(2 + \sin \pi x)\right)\big|_{y=-0.125}^{0.125}
\]

\[
= - \int_{x=-0.125}^{0.125} 5 \cos \pi y \, dx
\]

\[
= \left( \left. -5 \cos \pi y \right|_{x=-0.125}^{0.125} \right)\bigg/ \left. \left. \frac{0.125}{y=0.125} \right|_{x=-0.125}^{0.125} \right) = - \frac{5}{4} \cos \left( \frac{\pi}{8} \right),
\]
Thus,

\[
\Phi = \oint \mathbf{A} \cdot d\mathbf{l} = S_{\text{front}} + S_{\text{back}} + S_{\text{left}} + S_{\text{right}} = \frac{5}{4} \cos \left( \frac{\pi}{8} \right) - \frac{5}{4} \cos \left( \frac{\pi}{8} \right) + 0 + 0 = 0.
\]
Problem 5.28  A uniform current density given by

\[ \mathbf{J} = 2J_0 \ (A/m^2) \]

gives rise to a vector magnetic potential

\[ \mathbf{A} = -\hat{z} \frac{\mu_0 J_0}{4} (x^2 + y^2) \ (Wb/m) \]

(a) Apply the vector Poisson’s equation to confirm the above statement.

(b) Use the expression for \( \mathbf{A} \) to find \( \mathbf{H} \).

(c) Use the expression for \( \mathbf{J} \) in conjunction with Ampère’s law to find \( \mathbf{H} \). Compare your result with that obtained in part (b).

Solution:

(a)

\[ \nabla^2 \mathbf{A} = \hat{x} \nabla^2 A_x + \hat{y} \nabla^2 A_y + \hat{z} \nabla^2 A_z = \hat{z} \left( \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} \right) \left[ -\mu_0 \frac{J_0}{4} (x^2 + y^2) \right] \]

\[ = \hat{x} \mu_0 \frac{J_0}{4} (2 + 2) = -\hat{x} \mu_0 J_0. \]

Hence, \( \nabla^2 \mathbf{A} = -\mu_0 \mathbf{J} \) is verified.

(b)

\[ \mathbf{H} = \frac{1}{\mu_0} \nabla \times \mathbf{A} = \frac{1}{\mu_0} \left[ \hat{x} \left( \frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z} \right) + \hat{y} \left( \frac{\partial A_z}{\partial x} - \frac{\partial A_x}{\partial z} \right) + \hat{z} \left( \frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y} \right) \right] \]

\[ = \frac{\hat{x}}{\mu_0} \left( \frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial x} \right) \]

\[ = \frac{1}{\mu_0} \left[ \hat{x} \frac{\partial}{\partial y} \left( -\mu_0 \frac{J_0}{4} (x^2 + y^2) \right) - \hat{y} \frac{\partial}{\partial x} \left( -\mu_0 \frac{J_0}{4} (x^2 + y^2) \right) \right] \]

\[ = -\hat{x} \frac{J_0 y}{2} + \hat{y} \frac{J_0 x}{2} \ (A/m). \]

(c)

\[ \oint_C \mathbf{H} \cdot d\mathbf{l} = I = \int_S \mathbf{J} \cdot d\mathbf{s}, \]

\[ \Phi H_\phi \cdot \Phi 2\pi r = J_0 \cdot \pi r^2, \]

\[ \mathbf{H} = \Phi H_\phi = \Phi J_0 \frac{r}{2}. \]
Figure P5.28: Current cylinder of Problem 5.28.

We need to convert the expression from cylindrical to Cartesian coordinates. From Table 3-2,

\[ \hat{\mathbf{f}} = -\hat{x}\sin \phi + \hat{y}\cos \phi = -\hat{x}\frac{y}{\sqrt{x^2 + y^2}} + \hat{y}\frac{x}{\sqrt{x^2 + y^2}}, \]

\[ r = \sqrt{x^2 + y^2}. \]

Hence

\[ \mathbf{H} = \left( -\hat{x}\frac{y}{\sqrt{x^2 + y^2}} + \hat{y}\frac{x}{\sqrt{x^2 + y^2}} \right) \cdot \frac{J_0}{2} \sqrt{x^2 + y^2} = -\hat{x}\frac{yJ_0}{2} + \hat{y}\frac{xJ_0}{2}, \]

which is identical with the result of part (b).
Problem 5.32  The \(x-y\) plane separates two magnetic media with magnetic permeabilities \(\mu_1\) and \(\mu_2\) (Fig. P5.32). If there is no surface current at the interface and the magnetic field in medium 1 is

\[
\mathbf{H}_1 = \hat{x}H_{1x} + \hat{y}H_{1y} + \hat{z}H_{1z}
\]

find:

(a) \(\mathbf{H}_2\)

(b) \(\theta_1\) and \(\theta_2\)

(c) Evaluate \(\mathbf{H}_2, \theta_1,\) and \(\theta_2\) for \(H_{1x} = 2 \text{ (A/m)}, \ H_{1y} = 0, \ H_{1z} = 4 \text{ (A/m)}, \ \mu_1 = \mu_0,\) and \(\mu_2 = 4\mu_0\)

![Figure P5.32: Adjacent magnetic media (Problem 5.32).](image)

**Solution:**

(a) From (5.80),

\[
\mu_1 H_{1n} = \mu_2 H_{2n},
\]

and in the absence of surface currents at the interface, (5.85) states

\[
H_{1t} = H_{2t}.
\]

In this case, \(H_{1z} = H_{1n}\), and \(H_{1x}\) and \(H_{1y}\) are tangential fields. Hence,

\[
\mu_1 H_{1z} = \mu_2 H_{2z},
\]

\[
H_{1x} = H_{2x},
\]

\[
H_{1y} = H_{2y},
\]

and

\[
\mathbf{H}_2 = \hat{x}H_{1x} + \hat{y}H_{1y} + \hat{z} \frac{\mu_1}{\mu_2} H_{1z}.
\]
(b) 

\[ H_{1t} = \sqrt{H_{1x}^2 + H_{1y}^2}, \]

\[ \tan \theta_1 = \frac{H_{1t}}{H_{1z}} = \frac{\sqrt{H_{1x}^2 + H_{1y}^2}}{H_{1z}}, \]

\[ \tan \theta_2 = \frac{H_{2t}}{H_{2z}} = \frac{\sqrt{H_{2x}^2 + H_{2y}^2}}{H_{2z}} = \frac{\mu_2}{\mu_1} \tan \theta_1. \]

(c) 

\[ H_2 = \hat{x} 2 + \hat{z} \frac{1}{4} \cdot 4 = \hat{x} 2 + \hat{z} \quad (\text{A/m}), \]

\[ \theta_1 = \tan^{-1} \left( \frac{2}{4} \right) = 26.56^\circ, \]

\[ \theta_2 = \tan^{-1} \left( \frac{2}{1} \right) = 63.44^\circ. \]
**Problem 5.33**  Given that a current sheet with surface current density $J_s = \hat{x}8$ (A/m) exists at $y = 0$, the interface between two magnetic media, and $H_1 = \hat{z}11$ (A/m) in medium 1 ($y > 0$), determine $H_2$ in medium 2 ($y < 0$).

**Solution:**

![Figure P5.33: Adjacent magnetic media with $J_s$ on boundary.](image)

$J_s = \hat{x}8$ A/m,

$H_1 = \hat{z}11$ A/m.

$H_1$ is tangential to the boundary, and therefore $H_2$ is also. With $\hat{n}_2 = \hat{y}$, from Eq. (5.84), we have

$$\hat{n}_2 \times (H_1 - H_2) = J_s,$$

$$\hat{y} \times (\hat{z}11 - H_2) = \hat{x}8,$$

$$\hat{x}11 - \hat{y} \times H_2 = \hat{x}8,$$

or

$$\hat{y} \times H_2 = \hat{x}3,$$

which implies that $H_2$ does not have an $x$-component. Also, since $\mu_1 H_{1y} = \mu_2 H_{2y}$ and $H_1$ does not have a $y$-component, it follows that $H_2$ does not have a $y$-component either. Consequently, we conclude that

$$H_2 = \hat{z}3.$$
Problem 5.39  In terms of the dc current $I$, how much magnetic energy is stored in the insulating medium of a 3-m-long, air-filled section of a coaxial transmission line, given that the radius of the inner conductor is 5 cm and the inner radius of the outer conductor is 10 cm?

**Solution:** From Eq. (5.99), the inductance per unit length of an air-filled coaxial cable is given by

$$L' = \frac{\mu_0}{2\pi} \ln \left( \frac{b}{a} \right) \quad \text{(H/m)}. $$

Over a length of 2 m, the inductance is

$$L = 2L' = \frac{3 \times 4\pi \times 10^{-7}}{2\pi} \ln \left( \frac{10}{5} \right) = 416 \times 10^{-9} \quad \text{(H)}. $$

From Eq. (5.104), $W_m = LI^2/2 = 208I^2$ (nJ), where $W_m$ is in nanojoules when $I$ is in amperes. Alternatively, we can use Eq. (5.106) to compute $W_m$:

$$W_m = \frac{1}{2} \int \mu_0 H^2 \, dV. $$

From Eq. (5.97), $H = B/\mu_0 = I/2\pi r$, and

$$W_m = \frac{1}{2} \int_{z=0}^{3m} \int_{\phi=0}^{2\pi} \int_{r=a}^{b} \mu_0 \left( \frac{I}{2\pi r} \right)^2 r \, dr \, d\phi \, dz = 208I^2 \quad \text{(nJ)}. $$
Problem 5.40  The rectangular loop shown in Fig. P5.40 is coplanar with the long, straight wire carrying the current $I = 20$ A. Determine the magnetic flux through the loop.

![Figure P5.40: Loop and wire arrangement for Problem 5.40.](image)

Solution: The field due to the long wire is, from Eq. (5.30),

$$B = \hat{\phi} \frac{\mu_0 I}{2\pi r} = -\hat{x} \frac{\mu_0 I}{2\pi y} = -\hat{x} \frac{\mu_0 I}{2\pi y},$$

where in the plane of the loop, $\hat{\phi}$ becomes $-\hat{x}$ and $r$ becomes $y$.

The flux through the loop is along $-\hat{x}$, and the magnitude of the flux is

$$\Phi = \int_S B \cdot ds = \frac{\mu_0 I}{2\pi} \int_{5 \text{ cm}}^{20 \text{ cm}} \frac{-\hat{x}}{y} \cdot -\hat{x} (30 \text{ cm} \times dy)$$

$$= \frac{\mu_0 I}{2\pi} \times 0.3 \int_{0.05}^{0.2} \frac{dy}{y}$$

$$= \frac{0.3 \mu_0}{2\pi} \times 20 \times \ln \left( \frac{0.2}{0.05} \right) = 1.66 \times 10^{-6} \text{ (Wb)}.$$
**Problem 5.41**  Determine the mutual inductance between the circular loop and the linear current shown in Fig. P5.41.

![Figure P5.41: Linear conductor with current $I_1$ next to a circular loop of radius $a$ at distance $d$ (Problem 5.41).](image)

**Solution:** To calculate the magnetic flux through the loop due to the current in the conductor, we consider a thin strip of thickness $dy$ at location $y$, as shown. The magnetic field is the same at all points across the strip because they are all equidistant (at $r = d + y$) from the linear conductor. The magnetic flux through the strip is

$$d\Phi_{12} = B(y) \cdot ds = 2 \frac{\mu_0 I}{2\pi(d + y)} \cdot 2\pi(a^2 - y^2)^{1/2} dy$$

$$= \frac{\mu_0 I}{\pi(d + y)} (a^2 - y^2)^{1/2} dy$$

$$L_{12} = \frac{1}{I} \int_S d\Phi_{12}$$

$$= \frac{\mu_0}{\pi} \int_{y = -a}^{a} \frac{(a^2 - y^2)^{1/2}}{(d + y)} dy$$

Let $z = d + y \to dz = dy$. Hence,

$$L_{12} = \frac{\mu_0}{\pi} \int_{z = d - a}^{d + a} \frac{\sqrt{a^2 - (z - d)^2}}{z} dz$$

$$= \frac{\mu_0}{\pi} \int_{d - a}^{d + a} \sqrt{(a^2 - d^2) + 2dz - z^2} dz$$

$$= \frac{\mu_0}{\pi} \int \frac{\sqrt{R}}{z} dz$$
where \( R = a_0 + b_0 z + c_0 z^2 \) and

\[
\begin{align*}
    a_0 &= a^2 - d^2 \\
    b_0 &= 2d \\
    c_0 &= -1 \\
    \Delta &= 4a_0c_0 - b_0^2 = -4a^2 < 0
\end{align*}
\]

From Gradshteyn and Ryzhik, *Table of Integrals, Series, and Products* (Academic Press, 1980, p. 84), we have

\[
\int \frac{\sqrt{R}}{z} \, dz = \sqrt{R} + a_0 \int \frac{dz}{z/\sqrt{R}} + b_0 \int \frac{dz}{\sqrt{R}}.
\]

For

\[
\sqrt{R} \bigg|_{z=d-a}^{d+a} = \sqrt{a^2 - d^2 + 2dz - z^2} = 0 - 0 = 0.
\]

For \( \int \frac{dz}{z\sqrt{R}} \), several solutions exist depending on the sign of \( a_0 \) and \( \Delta \).

For this problem, \( \Delta < 0 \), also let \( a_0 < 0 \) (i.e., \( d > a \)). Using the table of integrals,

\[
a_0 \int \frac{dz}{z\sqrt{R}} = a_0 \left[ \frac{1}{\sqrt{-a_0}} \sin^{-1} \left( \frac{2a_0 + b_0 z}{\sqrt{b_0^2 - 4a_0c_0}} \right) \right]_{z=d-a}^{d+a}
\]

\[
= -\sqrt{d^2 - a^2} \left[ \sin^{-1} \left( \frac{a^2 - d^2 + dz}{az} \right) \right]_{z=d-a}^{d+a}
\]

\[
= -\pi \sqrt{d^2 - a^2}.
\]

For \( \int \frac{dz}{\sqrt{R}} \), different solutions exist depending on the sign of \( c_0 \) and \( \Delta \).

In this problem, \( \Delta < 0 \) and \( c_0 < 0 \). From the table of integrals,

\[
b_0 \int \frac{dz}{z\sqrt{R}} = \frac{b_0}{2} \left[ \frac{-1}{\sqrt{-c_0}} \sin^{-1} \left( \frac{2c_0 z + b_0}{\sqrt{-\Delta}} \right) \right]_{z=d-a}^{d+a}
\]

\[
= -d \left[ \sin^{-1} \left( \frac{d - z}{a} \right) \right]_{z=d-a}^{d+a} = \pi d.
\]

Thus

\[
L_{12} = \frac{\mu_0}{\pi} \left[ \pi d - \pi \sqrt{d^2 - a^2} \right]
\]

\[
= \mu_0 \left[ d - \sqrt{d^2 - a^2} \right].
\]