**Problem 4.31** The circular disk of radius $a$ shown in Fig. 4-7 has uniform charge density $\rho_s$ across its surface.

(a) Obtain an expression for the electric potential $V$ at a point $P = (0,0,z)$ on the $z$-axis.

(b) Use your result to find $E$ and then evaluate it for $z = h$. Compare your final expression with (4.24), which was obtained on the basis of Coulomb’s law.

**Solution:**

![Circular disk of charge.](image)

(a) Consider a ring of charge at a radial distance $r$. The charge contained in width $dr$ is

$$dq = \rho_s (2\pi r \, dr) = 2\pi \rho_s r \, dr.$$ 

The potential at $P$ is

$$dV = \frac{dq}{4\pi \varepsilon_0 R} = \frac{2\pi \rho_s r \, dr}{4\pi \varepsilon_0 (r^2 + z^2)^{1/2}}.$$

The potential due to the entire disk is

$$V = \int_0^a dV = \frac{\rho_s}{2\varepsilon_0} \int_0^a \frac{r \, dr}{(r^2 + z^2)^{1/2}} = \left[ \frac{\rho_s}{2\varepsilon_0} \frac{r}{(r^2 + z^2)^{1/2}} \right]_0^a = \frac{\rho_s}{2\varepsilon_0} \left[ (a^2 + z^2)^{1/2} - z \right].$$
\[ E = -\nabla V = -\hat{x} \frac{\partial V}{\partial x} - \hat{y} \frac{\partial V}{\partial y} - \hat{z} \frac{\partial V}{\partial z} = \hat{z} \frac{\rho_s}{2\varepsilon_0} \left[ 1 - \frac{z}{\sqrt{a^2 + z^2}} \right]. \]

The expression for \( E \) reduces to Eq. (4.24) when \( z = h \).
Problem 4.34  Find the electric potential $V$ at a location a distance $b$ from the origin in the $x$–$y$ plane due to a line charge with charge density $\rho_l$ and of length $l$. The line charge is coincident with the $z$-axis and extends from $z = -l/2$ to $z = l/2$.

Solution: From Eq. (4.48c), we can find the voltage at a distance $b$ away from a line of charge [Fig. P4.34]:

$$V(b) = \frac{1}{4\pi\varepsilon} \int \frac{\rho_l}{R'} \, dl' = \frac{\rho_l}{4\pi\varepsilon} \int_{-l/2}^{l/2} \frac{dz}{\sqrt{z^2 + b^2}} = \frac{\rho_l}{4\pi\varepsilon} \ln \left( \frac{l + \sqrt{l^2 + 4b^2}}{-l + \sqrt{l^2 + 4b^2}} \right).$$

Figure P4.34: Line of charge of length $\ell$. 
**Problem 4.38**  Given the electric field

\[ \mathbf{E} = \hat{\mathbf{R}} \frac{18}{R^2} \text{ (V/m)} \]

find the electric potential of point A with respect to point B where A is at +2 m and B at −4 m, both on the \( z \)-axis.

**Solution:**

![Diagram of points A and B with z-axis](image)

Figure P4.38: Potential between B and A.

\[ V_{AB} = V_A - V_B = - \int_B^A \mathbf{E} \cdot d\mathbf{l}. \]

Along \( z \)-direction, \( \hat{\mathbf{R}} = \hat{\mathbf{z}} \) and \( \mathbf{E} = \hat{\mathbf{z}} \frac{18}{z^2} \) for \( z \geq 0 \), and \( \hat{\mathbf{R}} = -\hat{\mathbf{z}} \) and \( \mathbf{E} = -\hat{\mathbf{z}} \frac{18}{z^2} \) for \( z \leq 0 \). Hence,

\[
V_{AB} = - \int_{-4}^{2} \hat{\mathbf{z}} \frac{18}{z^2} \cdot \hat{\mathbf{z}} \, dz = - \left[ \int_{-4}^{0} -\hat{\mathbf{z}} \frac{18}{z^2} \cdot \hat{\mathbf{z}} \, dz + \int_{0}^{2} \hat{\mathbf{z}} \frac{18}{z^2} \cdot \hat{\mathbf{z}} \, dz \right] = 4 \text{ V}.
\]
Problem 4.48  With reference to Fig. 4-19, find $E_1$ if $E_2 = \hat{x}3 - \hat{y}2 + \hat{z}2$ (V/m), $\varepsilon_1 = 2\varepsilon_0$, $\varepsilon_2 = 18\varepsilon_0$, and the boundary has a surface charge density $\rho_s = 3.54 \times 10^{-11}$ (C/m$^2$). What angle does $E_2$ make with the $z$-axis?

Solution: We know that $E_{1t} = E_{2t}$ for any 2 media. Hence, $E_{1t} = E_{2t} = \hat{x}3 - \hat{y}2$. Also, $(D_1 - D_2) \cdot \hat{n} = \rho_s$ (from Table 4.3). Hence, $\varepsilon_1(E_1 \cdot \hat{n}) - \varepsilon_2(E_2 \cdot \hat{n}) = \rho_s$, which gives

$$E_{1z} = \frac{\rho_s + \varepsilon_2E_{2z}}{\varepsilon_1} = \frac{3.54 \times 10^{-11}}{2\varepsilon_0} + \frac{18(2)}{2} = \frac{3.54 \times 10^{-11}}{2 \times 8.85 \times 10^{-12}} + 18 = 20 \text{ (V/m)}.$$  

Hence, $E_1 = \hat{x}3 - \hat{y}2 + \hat{z}20$ (V/m). Finding the angle $E_2$ makes with the $z$-axis:

$$E_2 \cdot \hat{z} = |E_2| \cos \theta, \quad 2 = \sqrt{9+4+4 \cos \theta}, \quad \theta = \cos^{-1}\left(\frac{2}{\sqrt{17}}\right) = 61^\circ.$$
**Problem 4.55** In a dielectric medium with \( \varepsilon_r = 4 \), the electric field is given by

\[
E = \hat{x}(x^2 + 2z) + \hat{y}x^2 - \hat{z}(y + z) \quad (\text{V/m})
\]

Calculate the electrostatic energy stored in the region \(-1 \text{ m} \leq x \leq 1 \text{ m}, 0 \leq y \leq 2 \text{ m}, \) and \(0 \leq z \leq 3 \text{ m} \).

**Solution:** Electrostatic potential energy is given by Eq. (4.124),

\[
W_e = \frac{1}{2} \int \varepsilon |E|^2 \; dV = \frac{\varepsilon}{2} \int_{z=0}^{3} \int_{y=0}^{2} \int_{x=-1}^{1} [(x^2 + 2z)^2 + x^4 + (y + z)^2] \; dx \; dy \; dz
\]

\[
= \frac{4\varepsilon_0}{2} \left( \left( \frac{2}{5}x^5 yz + \frac{2}{3}z^2 x^3 y + \frac{4}{3}z^3 xy + \frac{1}{12}(y+z)^4 x \right) \bigg|^{1}_{x=-1} \right) \bigg|^{2}_{y=0} \bigg|^{3}_{z=0}
\]

\[
= \frac{4\varepsilon_0}{2} \left( \frac{1304}{5} \right) = 4.62 \times 10^{-9} \quad (\text{J}).
\]
**Problem 4.58**  The capacitor shown in Fig. P4.58 consists of two parallel dielectric layers. Use energy considerations to show that the equivalent capacitance of the overall capacitor, $C$, is equal to the series combination of the capacitances of the individual layers, $C_1$ and $C_2$, namely

$$ C = \frac{C_1 C_2}{C_1 + C_2} \quad (22) $$

where

$$ C_1 = \varepsilon_1 \frac{A}{d_1}, \quad C_2 = \varepsilon_2 \frac{A}{d_2} $$

(a) Let $V_1$ and $V_2$ be the electric potentials across the upper and lower dielectrics, respectively. What are the corresponding electric fields $E_1$ and $E_2$? By applying the appropriate boundary condition at the interface between the two dielectrics, obtain explicit expressions for $E_1$ and $E_2$ in terms of $\varepsilon_1$, $\varepsilon_2$, $V$, and the indicated dimensions of the capacitor.

(b) Calculate the energy stored in each of the dielectric layers and then use the sum to obtain an expression for $C$.

(c) Show that $C$ is given by Eq. (22).

*Figure P4.58:* (a) Capacitor with parallel dielectric layers, and (b) equivalent circuit (Problem 4.58).
Solution:

Figure P4.58: (c) Electric fields inside of capacitor.

(a) If $V_1$ is the voltage across the top layer and $V_2$ across the bottom layer, then

$$V = V_1 + V_2,$$

and

$$E_1 = \frac{V_1}{d_1}, \quad E_2 = \frac{V_2}{d_2}.$$

According to boundary conditions, the normal component of $D$ is continuous across the boundary (in the absence of surface charge). This means that at the interface between the two dielectric layers,

$$D_{1n} = D_{2n}$$

or

$$\varepsilon_1 E_1 = \varepsilon_2 E_2.$$

Hence,

$$V = E_1 d_1 + E_2 d_2 = E_1 d_1 + \frac{\varepsilon_1 E_1}{\varepsilon_2} d_2,$$

which can be solved for $E_1$:

$$E_1 = \frac{V}{d_1 + \frac{\varepsilon_1}{\varepsilon_2} d_2}.$$

Similarly,

$$E_2 = \frac{V}{d_2 + \frac{\varepsilon_2}{\varepsilon_1} d_1}.$$
We \begin{align*}
W_e &= \frac{1}{2} \varepsilon_1 E_1^2 \cdot \gamma_1 = \frac{1}{2} \varepsilon_1 \left( \frac{V}{d_1 + \frac{\varepsilon_1}{\varepsilon_2} d_2} \right)^2 \cdot A d_1 = \frac{1}{2} V^2 \left[ \frac{\varepsilon_1 \varepsilon_2^2 A d_1}{(\varepsilon_2 d_1 + \varepsilon_1 d_2)^2} \right], \\
W_e &= \frac{1}{2} \varepsilon_2 E_2^2 \cdot \gamma_2 = \frac{1}{2} \varepsilon_2 \left( \frac{V}{d_2 + \frac{\varepsilon_2}{\varepsilon_1} d_1} \right)^2 \cdot A d_2 = \frac{1}{2} V^2 \left[ \frac{\varepsilon_2 \varepsilon_1^2 A d_2}{(\varepsilon_1 d_2 + \varepsilon_2 d_1)^2} \right], \\
W_e &= W_e + W_e = \frac{1}{2} V^2 \left[ \frac{\varepsilon_1 \varepsilon_2^2 A d_1 + \varepsilon_2 \varepsilon_1^2 A d_2}{(\varepsilon_1 d_2 + \varepsilon_2 d_1)^2} \right].
\end{align*}

But \( W_e = \frac{1}{2} CV^2 \), hence,
\begin{align*}
C &= \frac{\varepsilon_1 \varepsilon_2^2 A d_1 + \varepsilon_2 \varepsilon_1^2 A d_2}{(\varepsilon_2 d_1 + \varepsilon_1 d_2)^2} = \frac{\varepsilon_1 \varepsilon_2 A}{(\varepsilon_2 d_1 + \varepsilon_1 d_2)^2} = \frac{\varepsilon_1 \varepsilon_2 A}{\varepsilon_2 d_1 + \varepsilon_1 d_2}.
\end{align*}

(c) Multiplying numerator and denominator of the expression for \( C \) by \( A/d_1 d_2 \), we have
\begin{align*}
C &= \frac{\frac{\varepsilon_1 A}{d_1} \frac{\varepsilon_2 A}{d_2}}{\frac{\varepsilon_1 A}{d_1} + \frac{\varepsilon_2 A}{d_2}} = \frac{C_1 C_2}{C_1 + C_2},
\end{align*}
where
\begin{align*}
C_1 &= \frac{\varepsilon_1 A}{d_1}, \quad C_2 = \frac{\varepsilon_2 A}{d_2}.
\end{align*}
Problem 4.62  Conducting wires above a conducting plane carry currents $I_1$ and $I_2$ in the directions shown in Fig. P4.62. Keeping in mind that the direction of a current is defined in terms of the movement of positive charges, what are the directions of the image currents corresponding to $I_1$ and $I_2$?

![Figure P4.62: Currents above a conducting plane (Problem 4.62).](image)

Solution:

(a) In the image current, movement of negative charges downward = movement of positive charges upward. Hence, image of $I_1$ is same as $I_1$.

\[ I_1 \uparrow \quad + q @ t=t_1 \]
\[ \quad \cdot \quad + q @ t=0 \]

\[ \cdots \cdots \cdots \cdots \cdots \cdots \cdots \]

\[ I_1 \downarrow \quad - q @ t=0 \]
\[ \quad \cdot \quad - q @ t=t_1 \]

![Figure P4.62(a): Solution for part (a).](image)

(b) In the image current, movement of negative charges to right = movement of positive charges to left.
Figure P4.62(b): Solution for part (b).
Problem 5.5  In a cylindrical coordinate system, a 2-m-long straight wire carrying a current of 5 A in the positive $z$-direction is located at $r = 4$ cm, $\phi = \pi/2$, and $-1 \text{ m} \leq z \leq 1$ m.

(a) If $\mathbf{B} = \hat{r}0.2 \cos \phi$ (T), what is the magnetic force acting on the wire?

(b) How much work is required to rotate the wire once about the $z$-axis in the negative $\phi$-direction (while maintaining $r = 4$ cm)?

(c) At what angle $\phi$ is the force a maximum?

Solution:

(a) 

$$\mathbf{F} = I \ell \times \mathbf{B} = 5 \hat{z} \times [\hat{r}0.2 \cos \phi] = \hat{\phi}2 \cos \phi.$$ 

At $\phi = \pi/2$, $\hat{\phi} = -\hat{x}$. Hence, 

$$\mathbf{F} = -\hat{x}2 \cos(\pi/2) = 0.$$ 

(b) 

$$W = \int_{\phi=0}^{2\pi} \mathbf{F} \cdot d\mathbf{l} = \int_{0}^{2\pi} \hat{\phi} [2 \cos \phi] \cdot (-\hat{\phi}) r \, d\phi \bigg|_{r=4 \text{ cm}}$$

$$= -2r \left[ \cos \phi \right]_{\phi=0}^{2\pi} \bigg|_{r=4 \text{ cm}} = -8 \times 10^{-2} \left[ \sin \phi \right]_{0}^{2\pi} = 0.$$
The force is in the \( +\hat{\phi} \)-direction, which means that rotating it in the \( -\hat{\phi} \)-direction would require work. However, the force varies as \( \cos \phi \), which means it is positive when \( -\pi/2 \leq \phi \leq \pi/2 \) and negative over the second half of the circle. Thus, work is provided by the force between \( \phi = \pi/2 \) and \( \phi = -\pi/2 \) (when rotated in the \( +\hat{\phi} \)-direction), and work is supplied for the second half of the rotation, resulting in a net work of zero.

(c) The force \( \mathbf{F} \) is maximum when \( \cos \phi = 1 \), or \( \phi = 0 \).
Problem 5.12 Two infinitely long, parallel wires are carrying 6-A currents in opposite directions. Determine the magnetic flux density at point $P$ in Fig. P5.12.

\[ B = \hat{\Phi} \frac{\mu_0 I_1}{2\pi(0.5)} + \hat{\Phi} \frac{\mu_0 I_2}{2\pi(1.5)} = \hat{\Phi} \frac{\mu_0}{\pi} (6 + 2) = \hat{\Phi} \frac{8\mu_0}{\pi} \text{ (T)}. \]
**Problem 5.19**  Three long, parallel wires are arranged as shown in Fig. P5.19. Determine the force per unit length acting on the wire carrying $I_3$. 

![Figure P5.19: Three parallel wires of Problem 5.19.](image)

**Solution:** Since $I_1$ and $I_2$ are equal in magnitude and opposite in direction, and 

![Figure P5.19: (a) B fields due to $I_1$ and $I_2$ at location of $I_3$.](image)
equidistant from \( I_3 \), our intuitive answer might be that the net force on \( I_3 \) is zero. As we will see, that’s not the correct answer. The field due to \( I_1 \) (which is along \( \hat{y} \)) at location of \( I_3 \) is

\[
B_1 = \hat{b}_1 \frac{\mu_0 I_1}{2\pi R_1}
\]

where \( \hat{b}_1 \) is the unit vector in the direction of \( B_1 \) shown in the figure, which is perpendicular to \( \hat{R}_1 \). The force per unit length exerted on \( I_3 \) is

\[
F'_{31} = \frac{\mu_0 I_1 I_3}{2\pi R_1} (\hat{y} \times \hat{b}_1) = -\hat{R}_1 \frac{\mu_0 I_1 I_3}{2\pi R_1}.
\]

Similarly, the force per unit length excited on \( I_3 \) by the field due to \( I_2 \) (which is along \( -\hat{y} \)) is

\[
F'_{32} = \hat{b}_2 \frac{\mu_0 I_2 I_3}{2\pi R_2}.
\]

The two forces have opposite components along \( \hat{x} \) and equal components along \( \hat{z} \). Hence, with \( R_1 = R_2 = \sqrt{8} \) m and \( \theta = \sin^{-1}(2/\sqrt{8}) = \sin^{-1}(1/\sqrt{2}) = 45^\circ \),

\[
F_3 = F'_{31} + F'_{32} = 2 \left( \frac{\mu_0 I_1 I_3}{2\pi R_1} + \frac{\mu_0 I_2 I_3}{2\pi R_2} \right) \sin \theta
= 2 \left( \frac{4\pi \times 10^{-7} \times 10 \times 20}{2\pi \times \sqrt{8}} \right) \times \frac{1}{\sqrt{2}} = 2 \times 10^{-5} \text{ N/m}.
\]
**Problem 5.20** A square loop placed as shown in Fig. P5.20 has 2-m sides and carries a current $I_1 = 5 \text{ A}$. If a straight, long conductor carrying a current $I_2 = 10 \text{ A}$ is introduced and placed just above the midpoints of two of the loop’s sides, determine the net force acting on the loop.

**Solution:** Since $I_2$ is just barely above the loop, we can treat it as if it’s in the same plane as the loop. For side 1, $I_1$ and $I_2$ are in the same direction, hence the force on side 1 is attractive. That is,

$$F_1 = \frac{\mu_0 I_1 I_2 a}{2\pi (a/2)} = \frac{4\pi \times 10^{-7} \times 5 \times 10 \times 2}{2\pi \times 1} = 2 \times 10^{-5} \text{ N}.$$ 

$I_1$ and $I_2$ are in opposite directions for side 3. Hence, the force on side 3 is repulsive, which means it is also along $\hat{y}$. That is, $F_3 = F_1$.

The net forces on sides 2 and 4 are zero. Total net force on the loop is

$$F = 2F_1 = 4 \times 10^{-5} \text{ N}.$$