1- A WSS random process $X(t)$ with autocorrelation $R_{XX}(\tau) = Ae^{-a|\tau|}$, where $A$ and $a$ are real positive constants, is applied to the input of a linear time-invariant (LTI) system with impulse response $h(t) = e^{-bt}u(t)$, where $b$ is a real positive constant. Find the power spectral density of the output $Y(t)$ of the system.

Solution:

\[ H(\omega) = \mathcal{F}[h(t)] = \frac{1}{j\omega + b} \Rightarrow |H(\omega)|^2 = \frac{1}{\omega^2 + b^2}. \]

Also, $S_{XX}(\omega) = \mathcal{F}[R_{XX}(\tau)] = A\frac{2a}{\omega^2 + a^2}$.

\[ S_{YY}(\omega) = |H(\omega)|^2 S_{XX}(\omega) = \left(\frac{1}{\omega^2 + b^2}\right)\left(\frac{2aA}{\omega^2 + a^2}\right) = \frac{aA}{(a^2 - b^2)b} - \frac{A}{a^2 - b^2}\left(\frac{2a}{\omega^2 + a^2}\right). \]

2- Consider a random process given by $X(t) = A\cos(\omega t + \theta)$, where $A$ and $\omega$ are constants, and $\Theta$ is a uniform random variable over $[-\pi, \pi]$. Show that $X(t)$ is WSS.

Solution:

\[ f_\Theta(\theta) = \begin{cases} 
\frac{1}{2\pi}, & \text{for } -\pi \leq x \leq \pi \\
0, & \text{otherwise}
\end{cases} \]

For WSS we need to show

1- $E[X(t)] = \bar{X} = \text{constant},$

2- Autocorrelation depends only on the time difference $\tau$. $E[X(t)X(t + \tau)] = R_{XX}(\tau)$.

\[ E[X(t)] = \int_{-\infty}^{\infty} A\cos(\omega t + \theta) f_\Theta(\theta) d\theta = \frac{A}{2\pi} \int_{-\pi}^{\pi} \cos(\omega t + \theta) d\theta = 0, \text{ which satisfies the first condition.} \]
\[ R_{XX}(\tau) = E[X(t)X(t+\tau)] = \frac{A^2}{2\pi} \int_{-\pi}^{\pi} \cos(\omega t + \theta) \cos(\omega(t + \tau) + \theta) d\theta = \frac{A^2}{2} \cos \omega \tau , \]

which satisfies the second condition.

**Bonus:** In problem 2 above, show that the random process \( X(t) \) is ergodic in both the mean and the autocorrelation.

*For the bonus:* up to two extra points.

**Solution:**

A random process \( X(t) \) is **ergodic** if time-averages are the same for all sample functions, and are equal to the corresponding ensemble averages.

In an ergodic process, all its statistics can be obtained from a single sample function.

A stationary process \( X(t) \) is called **ergodic in the mean** if \( \langle x(t) \rangle = \bar{X} \),

and **ergodic in the autocorrelation** if \( \langle x(t)x(t+\tau) \rangle = R_{XX}(\tau) \).

\[
\langle x(t) \rangle = \lim_{T \to \infty} \frac{1}{T} \int_{-T/2}^{T/2} x(t) dt = \lim_{T \to \infty} \frac{1}{T} \int_{-T/2}^{T/2} A \cos(\omega t + \theta) dt = \lim_{T \to \infty} \frac{1}{T} \int_{-\omega T/2}^{\omega T/2} A \cos(\omega t + \theta) d\omega t = \frac{A^2}{2\pi} \int_{-\pi}^{\pi} \cos(\omega t + \theta) d\omega t = 0 \]

which is equal to \( E[X(t)] \).

Similarly, \( \langle x(t)x(t+\tau) \rangle = \frac{A^2}{2} \cos \omega \tau = R_{XX}(\tau) \).