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RECURSIVE KINEMATICS AND INVERSE DYNAMICS FOR PARALLEL MANIPULATORS

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ABSTRACT

We examine here the modular and recursive formulation of the inverse dynamics of parallel architecture manipulators. The concept of the decoupled natural orthogonal complement (DeNOC) is combined with the spatial parallelism of the robots of interest to develop an inverse dynamics algorithm which is both recursive and modular.

INTRODUCTION

The *modular* and *recursive* inverse dynamics formulation for parallel architecture manipulators is the subject of this paper. The *modular* formulation of mathematical models is attractive because existing sub-models may be assembled to create different topologies, e.g. cooperative robotic systems. *Recursive* algorithms are desirable from the viewpoint of simplicity and uniformity of computation, despite the ever-increasing complexity of mechanisms. Additionally we note that, prior to the dynamics computation stage, a forward or inverse kinematics stage is often required. Hence, the development of efficient recursive dynamics algorithms also necessitates the careful investigation of *recursive kinematics algorithms*. However, the development of *modular* and *recursive* kinematics and dynamics algorithms for parallel manipulator architectures remains a challenging research problem.

The literature on mathematical modelling of manipulators has a rich history spanning several decades. We will summarize some critical aspects presently while referring the interested reader to any number of books on the subject [1–5], for details. Methods for formulation of equations of motion (EOM) fall into two main categories: a) Euler-Lagrange and b) Newton-Euler formulations. *Euler-Lagrange* methods, some times referred to as *analytic*, are commonly used in the robotics community to obtain the equations of motion (EOM) of robotic manipulators. Typically, such approaches use the joint-based relative coordinates as the configuration space. For serial chain manipulators, these form a minimal coordinate description and permit a direct mapping to actuator coordinates. *Newton-Euler* (NE) methods, also referred to as *synthetic*, on the other hand, typically favor

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the use of Cartesian variables as configuration-space variables, and develop recursive formulations from the free-body diagram (FBD) of each single body. The *uncoupled* governing equations are then assembled to obtain the model of the entire system.

While efficient formulations exist for serial-chain and tree-structured multi-body systems, the adaptation of these methods to the simulation of closed-chain linkages and parallel manipulators is relatively more difficult. Such systems possess one or more kinematic loops, requiring the introduction of algebraic constraints, typically nonlinear, into the formulation. Considerable work has been reported in the literature on the specialization of the above methods to formulate the EOM of constrained mechanical systems, while including both holonomic and non-holonomic constraints. Parallel mechanisms and manipulators form a special class of constrained mechanical systems where the multiple kinematic loops give rise to systems of nonlinear holonomic constraints; in the ensuing discussion we will focus on the development of EOM of such systems.

Non-Recursive Newton-Lagrange Formulations

The dynamics of constrained mechanical systems with closed loops using the Newton-Lagrange approach is traditionally obtained by cutting the closed loops to obtain various open loops and tree-structured systems, and then writing a system of ordinary differential equations (ODEs) for the corresponding chains in their corresponding generalized coordinates [6]. The solution to these equations are required to satisfy additional algebraic equations guaranteing the closure of the cut-open loops. A Lagrange multiplier term is introduced to represent the forces in the direction of the constraint violation. The resulting formulation, often referred to as a *descriptor form*, yields a usually simpler, albeit larger, system of index-3 differential algebraic equations (DAEs).

Typical methods used to solve the forward and inverse dynamics problems for such constrained systems cover a broad spectrum, namely,

- Direct elimination where the surplus variables are eliminated directly, using the position-level algebraic constraints to explicitly reduce index-3 DAE to an ODE in a minimal set of generalized coordinates (conversion into Lagrange's equations of the second kind) [7];
- Explicit Lagrange-multiplier computation together with the unknown accelerations computed from the augmented index-1 differential algebraic equation (DAE) formed by appending the differentiated acceleration level constraints to the system equations [1, 8];
- Lagrange-multiplier approximation/Penalty formulation, where the Lagrange multipliers are estimated using a compliance-based force-law, based on the extent of constraint violation and assumed penalty factors [2, 9];
- Projection of dynamics onto the tangent space of the constraint manifold, where the constraint-reaction dynamics equations are taken into the orthogonal and tangent subspaces of the vector space of the system generalized velocities. A family of choices exist for the projection, as surveyed by García de Jalón and Bayo [2] and Shabana [5].

Recursive Newton-Euler Formulations

Many variants of fast and readily-implementable recursive algorithms have been formulated within the last two decades, principally within the robotics community to overcome the limitations posed by the complexity of the dynamics equations based on classic Lagrange approaches [6].

The first researchers to develop O(N) algorithms for inverse dynamics for robotics used a Newton-Euler formulation of the problem. Stepanenko and Vukobratovic [10] developed a recursive NE method for human limb dynamics, while Orin et al. [11] improved the efficiency of the recursive method by referring forces and moments to local link-coordinates for the real-time control of a leg of a walking machine. Luh et al. [12] developed a very efficient Recursive Newton-Euler algorithm (RNEA) by referencing most quantities to link-coordinates; this is the most often cited. Further gains have been made in the efficiency over the years, as reported, for example, by Balafoutis et al. [13] and Goldenberg and He [14].

In multi-loop mechanisms, the generalized coordinates (joint variables) are no longer independent, since they are subject to the typically non-linear loop-closure constraints. The most common method for dealing with kinematics is to cut the loop, introduce Lagrange multipliers to substitute for the cut joints and use a recursive scheme for the open-chain system to obtain a recursive algorithm.

Decoupled Natural Orthogonal Complement

The *natural orthogonal complement* (NOC), first introduced in Angeles and Lee [15], belongs to the class of projection methods for dynamics evaluation. Saha [16] showed a method for splitting the NOC of a serial chain into two matrices, one diagonal and one lower block-diagonal, thus introducing the *decoupled natural orthogonal complement* (DeNOC). By doing so, Saha was able to exploit the recursive nature of the DeNOC and apply the concept to model simple serial manipulators. Further, although recursive kinematics algorithms for serial chains have had a long history [10–12], a recursive algorithm for the forward kinematics of *closed-chain systems* first appeared in Saha and Schiehlen [17]. In this work, Saha and Schiehlen [17] showed that the NOC of a parallel manipulator may be split into three parts–one full, one block-diagonal and one lower-triangular, and proposed a recursive minimal-order forward dynamics algorithm for parallel manipulators. Examples of up to two degree-of-freedom planar manipulators were included and various physical

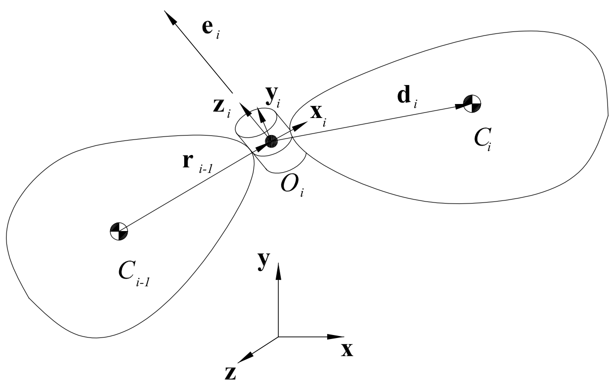

Figure 1. TWO BODIES CONNECTED BY A KINEMATIC PAIR

interpretations were reported.

BACKGROUND

Twists, Wrenches and Equations of Motion

In this section, some definitions and concepts associated with the formulation of the kinematics and dynamics of articulated rigid body systems coupled by lower kinematic pairs will be briefly reviewed. See [18] and [19] for further details.

Figure 1 shows two rigid links connected by a kinematic pair. The mass center of ith link is at C_i while that of link $i-1$ is at C_{i-1}. The axis of the ith pair is represented by a unit vector $\mathbf{e}_i$. We attach a frame $\mathcal{F}_i$ with origin O_i and axes $\mathbf{x}_i$, $\mathbf{y}_i$ and $\mathbf{z}_i$, to link $i-1$, such that $\mathbf{z}_i$ is along $\mathbf{e}_i$. The global inertial reference frame $\mathcal{F}_0$ with axes $\mathbf{x}$, $\mathbf{y}$ and $\mathbf{z}$ is attached to the base of the manipulator, and unless otherwise specified, all quantities will be represented in this global frame in the balance of the paper. Further, we define, the three-dimensional position vectors $\mathbf{d}_i$ from the O_i to the mass center of link i and $\mathbf{r}_{i-1}$ from the mass center of link $i-1$ to O_i.

The six dimensional twist and wrench vector associated with link i, at its mass center C_i, are now defined as

$$\mathbf{t}_i = \begin{bmatrix} \boldsymbol{\omega}_i \\ \mathbf{v}_i \end{bmatrix};\ \mathbf{w}_i = \begin{bmatrix} \mathbf{n}_i \\ \mathbf{f}_i \end{bmatrix} \tag{1}$$

where $\boldsymbol{\omega}$, $\mathbf{v}$, $\mathbf{n}$, and $\mathbf{f}$ are three dimensional angular velocity, linear velocity, moment and force vectors, respectively, associated with link i and represented about C_i.

The Newton-Euler equations for link i are

$$\begin{aligned} \mathbf{n}_i &= \mathbf{I}_i\dot{\boldsymbol{\omega}}_i + \dot{\mathbf{I}}_i\boldsymbol{\omega}_i = \mathbf{I}_i\dot{\boldsymbol{\omega}}_i + \boldsymbol{\omega}_i \times \mathbf{I}_i\boldsymbol{\omega}_i \\ \mathbf{f}_i &= m_i\dot{\mathbf{v}}_i \end{aligned}$$

where $\mathbf{I}_i$ is the 3×3 inertia tensor about C_i and m_i is the mass of link i. The above set, in matrix form, may be written as

$$\mathbf{w}_i = \underbrace{\begin{bmatrix} \mathbf{I}_i & \mathbf{O} \\ \mathbf{O} & m_i\mathbf{1} \end{bmatrix}}_{\mathbf{M}_i} \begin{bmatrix} \dot{\boldsymbol{\omega}}_i \\ \dot{\mathbf{v}}_i \end{bmatrix} + \underbrace{\begin{bmatrix} \boldsymbol{\Omega}_i & \mathbf{O} \\ \mathbf{O} & \mathbf{O} \end{bmatrix}}_{\mathbf{W}_i} \begin{bmatrix} \mathbf{I}_i & \mathbf{O} \\ \mathbf{O} & m_i\mathbf{1} \end{bmatrix} \begin{bmatrix} \boldsymbol{\omega}_i \\ \mathbf{v}_i \end{bmatrix} \tag{2}$$

or

$$\mathbf{w}_i = \mathbf{M}_i\dot{\mathbf{t}}_i + \mathbf{W}_i\mathbf{M}_i\mathbf{t}_i \tag{3}$$

For a multi-body system with n rigid links coupled by kinematic pairs, we may write

$$\mathbf{t} = \begin{bmatrix} \mathbf{t}_1 \\ \vdots \\ \mathbf{t}_n \end{bmatrix};\ \mathbf{w} = \begin{bmatrix} \mathbf{w}_1 \\ \vdots \\ \mathbf{w}_n \end{bmatrix} \tag{4}$$

The resulting set of Newton-Euler equations for the entire unconstrained system is then cast in the form

$$\mathbf{w} = \mathbf{M}\dot{\mathbf{t}} + \mathbf{W}\mathbf{M}\mathbf{t} \tag{5}$$

where

$$\mathbf{M} = \begin{bmatrix} \mathbf{M}_1 & \mathbf{O} & \mathbf{O} \\ \mathbf{O} & \ddots & \mathbf{O} \\ \mathbf{O} & \mathbf{O} & \mathbf{M}_n \end{bmatrix};\ \mathbf{W} = \begin{bmatrix} \mathbf{W}_1 & \mathbf{O} & \mathbf{O} \\ \mathbf{O} & \ddots & \mathbf{O} \\ \mathbf{O} & \mathbf{O} & \mathbf{W}_n \end{bmatrix} \tag{6}$$

Kinematic Relations Between Two Bodies Coupled by Lower Kinematic Pair

The twist of link i at O_i can be written recursively in terms of the twist of link $i-1$ at C_{i-1} as

$$\tilde{\mathbf{t}}_i = \tilde{\mathbf{B}}_{i-1}\mathbf{t}_{i-1} + \tilde{\mathbf{p}}_i\dot{\theta}_i \tag{7}$$

where

$$\tilde{\mathbf{B}}_{i-1} = \begin{bmatrix} \mathbf{1} & \mathbf{O} \\ -\mathbf{R}_{i-1} & \mathbf{1} \end{bmatrix} \tag{8}$$

$$\tilde{\mathbf{p}}_i = \begin{bmatrix} \mathbf{e}_i \\ \mathbf{0} \end{bmatrix} \text{ for revolute joint} \tag{9}$$

$$\tilde{\mathbf{p}}_i = \begin{bmatrix} \mathbf{0} \\ \mathbf{e}_i \end{bmatrix} \text{ for prismatic joint} \tag{10}$$

where $\mathbf{R}_{i-1}$ is the cross product matrix of $\mathbf{r}_{i-1}$. Further, the twist of link i about mass center C_i is

$$\mathbf{t}_i = \mathbf{B}_i \tilde{\mathbf{t}}_i; \;\; \mathbf{B}_i = \begin{bmatrix} \mathbf{1} & \mathbf{O} \\ -\mathbf{D}_i & \mathbf{1} \end{bmatrix} \tag{11}$$

where $\mathbf{D}_i$ is the cross product matrix of $\mathbf{d}_i$. Substituting the value of $\tilde{\mathbf{t}}_i$ from Eq.(7) in the above equation we obtain

$$\mathbf{t}_i = \mathbf{B}_i \tilde{\mathbf{B}}_{i-1} \mathbf{t}_{i-1} + \mathbf{B}_i \tilde{\mathbf{p}}_i \dot{\theta}_i \tag{12}$$

We introduce the notation

$$\mathbf{t}_i = \mathbf{B}_{i,i-1} \mathbf{t}_{i-1} + \mathbf{p}_i \dot{\theta}_i \tag{13}$$

where

$$\mathbf{B}_{i,i-1} = \begin{bmatrix} \mathbf{1} & \mathbf{O} \\ -\mathbf{A}_i & \mathbf{1} \end{bmatrix} \tag{14}$$

$$\mathbf{p}_i = \begin{bmatrix} \mathbf{e}_i \\ \mathbf{D}_i \mathbf{e}_i \end{bmatrix} \text{ for revolute joint} \tag{15}$$

$$\mathbf{p}_i = \begin{bmatrix} \mathbf{0} \\ \mathbf{e}_i \end{bmatrix} \text{ for prismatic joint} \tag{16}$$

where $\mathbf{A}_i$ is the cross product matrix of $(\mathbf{r}_{i-1} + \mathbf{d}_i)$. Matrix $\mathbf{B}_{i,i-1}$ may be called the *twist propagation matrix* while $\mathbf{p}_i$, the *twist generator*. The twist $\mathbf{t}_i$ is thus the sum of twist $\mathbf{t}_{i-1}$ and the twist generated at the joint-i, both evaluated at C_i. Equation (13) is recursive in nature and is in fact the forward recursion part of the *recursive Newton-Euler algorithm* proposed by Luh et al. [12].

MODELING OF THE 3R-PLANAR PLATFORM MANIPULATOR

Figure 2 shows the 3R planar platform manipulators with three degrees of freedom [20]. For the sake of simplicity we restrict ourselves to system that has: i) only revolute joints, ii) identical legs and iii) a moving platform in the shape of an equilateral triangle. The three-degree-of-freedom (three-dof) planar manipulator consists of three identical dyads, numbered I, II and III coupling the platform $\mathcal{P}$ with the base, such that their fixed pivots lie on the vertices of an equilateral triangle, as well. The proximal and the distal links of each dyad are numbered 1 and 2 respectively. Joint 1 of each dyad is actuated. The centroidal moment of inertia of each link about the axis normal to the xy-plane is I_i, for $i = 1, 2$. The mass of the platform is given

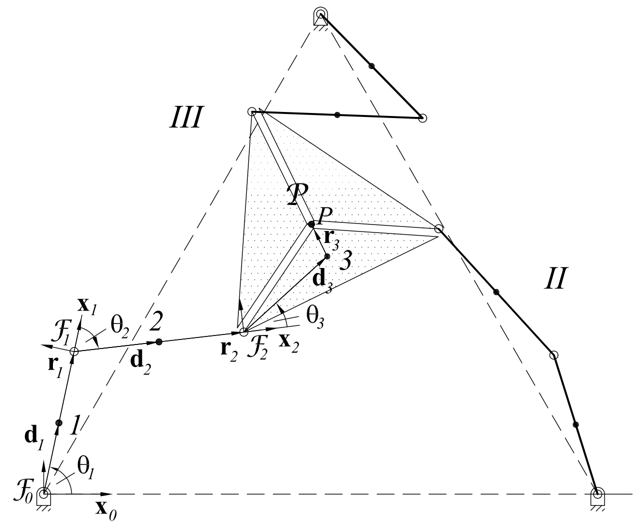

Figure 2. 3-DOF PLANAR PARALLEL MANIPULATOR

by m_P, its mass center located at P, the centroid of the equilateral triangle, and the centroidal moment of inertia about an axis similarly oriented is I_P.

We divide the manipulator into three serial chains, I, II and III, by dividing the rigid platform $\mathcal{P}$ into *three parts* such that the operation point (OP) of the end effector of *each open chain* lies at point P, the mass-center of the platform $\mathcal{P}$. Cutting the platform in this manner is advantageous due to the following reasons:

- Torques may be applied to the joints that otherwise need to be cut to open the chains.
- Joint friction may be accommodated directly for such joints.
- Cutting the links (platform) produces a more streamlined recursive kinematic and dynamic modelling for parallel manipulators.

The first two advantages are discussed in greater detail in Yiu et al. [21]; we will discuss these issues in detail in the ensuing analysis.

Recursive Forward Kinematics

The forward kinematics problem for a parallel manipulator is defined as: Given the actuated-joint angles, velocities and accelerations, find the position, twists and twist-rates of the platform and all the other links.

Position Analysis The displacement analysis is a critical first step and we adopt the approach proposed by Ma and Angeles [22] to this end.

Velocity Analysis Since the manipulator is planar, we use two-dimensional position vectors, three-dimensional twist vector $\mathbf{t} = [\omega, \mathbf{v}]^T$, and three-dimensional wrench vectors $\mathbf{w} = [n, \mathbf{f}]^T$, where ω is the angular velocity, $\mathbf{v}$ is the two-dimensional velocity vector, n is the angular moment and $\mathbf{f}$ is the two-dimensional force vector. For *each chain*, we define position vectors $\mathbf{d}_i$ from the ith joint axis to the mass center of link i, $\mathbf{r}_i$ from mass center of link i to the $(i+1)$st joint axis and $\mathbf{a}_i = \mathbf{d}_i + \mathbf{r}_i$ as shown in Fig. 2.

The twist of the end effector of *any chain* is given by Saha and Schiehlen [17] as

$$\mathbf{t}_P = \mathbf{B}_{P3}\mathbf{t}_3 \tag{17}$$

where

$$\mathbf{B}_{P3} = \begin{bmatrix} 1 & \mathbf{0}^T \\ \mathbf{E}\mathbf{r}_3 & \mathbf{1} \end{bmatrix}; \quad \mathbf{E} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

$\mathbf{t}_3$, the twist of the third link with respect to its mass center, is computed recursively for *each serial chain* from its preceding link as

$$\mathbf{t}_3 = \mathbf{B}_{32}\mathbf{t}_2 + \mathbf{p}_3\dot{\theta}_3 \tag{18}$$
$$\mathbf{B}_{32} = \begin{bmatrix} 1 & \mathbf{0}^T \\ \mathbf{E}(\mathbf{r}_2 + \mathbf{d}_3) & \mathbf{1} \end{bmatrix}; \quad \mathbf{p}_3 = \begin{bmatrix} 1 \\ \mathbf{E}\mathbf{d}_3 \end{bmatrix}$$

where the 3×3 matrix $\mathbf{B}_{32}$ is the *twist-propagation matrix* and $\mathbf{p}_3$ is the *twist generator*; $\mathbf{t}_2$ is the twist of link-2 with respect to its mass center; $\dot{\theta}_3$ is the relative rate of the third joint, while $\mathbf{0}$ is the two-dimensional zero vector and $\mathbf{1}$ is the 2×2 identity matrix.

A useful relation is first introduced which will be exploited in the ensuing analysis. Let $\mathbf{a} = \mathbf{b}x + \mathbf{c}$, where $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ are three-dimensional vectors, while x is a scalar; we may determine the value of the scalar as

$$x = \frac{\mathbf{b}^T}{\mathbf{b}^T\mathbf{b}}(\mathbf{a} - \mathbf{c}) \tag{19}$$

Substituting this value for x back in $\mathbf{a} = \mathbf{b}x + \mathbf{c}$ and rearranging terms we may eliminate x from the equation to obtain a relationship between $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ alone as

$$(\mathbf{1} - \frac{\mathbf{b}\mathbf{b}^T}{\mathbf{b}^T\mathbf{b}})\mathbf{a} = (\mathbf{1} - \frac{\mathbf{b}\mathbf{b}^T}{\mathbf{b}^T\mathbf{b}})\mathbf{c} \tag{20}$$

where $\mathbf{1}$ is the 3×3 identity matrix, which is the relation sought.

We employ this process to eliminate the unactuated joint velocities from the recursive kinematic equations. Substituting $\mathbf{t}_3$ into Eq.(17), we obtain

$$\mathbf{t}_P = \mathbf{B}_{P3}(\mathbf{B}_{32}\mathbf{t}_2 + \mathbf{p}_3\dot{\theta}_3) \tag{21}$$

The above equation is solved for $\dot{\theta}_3$:

$$\dot{\theta}_3 = \frac{\tilde{\mathbf{p}}_3^T}{\delta_3}(\mathbf{t}_P - \mathbf{B}_{P2}\mathbf{t}_2) \tag{22}$$

where the three-dimensional vector $\tilde{\mathbf{p}}_3$ is defined as $\tilde{\mathbf{p}}_3 = \mathbf{B}_{P3}\mathbf{p}_3$ and $\delta_3 = \tilde{\mathbf{p}}_3^T\tilde{\mathbf{p}}_3$. Therefore, when we finally substitute $\dot{\theta}_3$ into Eq.(21) we obtain:

$$\mathbf{\Phi}_3\mathbf{t}_P = \mathbf{\Phi}_3\mathbf{B}_{P2}\mathbf{t}_2 \tag{23}$$

where $\mathbf{\Phi}_3 = \mathbf{1} - \tilde{\mathbf{p}}_3\tilde{\mathbf{p}}_3^T/\delta_3$ and the property $\mathbf{B}_{P3}\mathbf{B}_{32} = \mathbf{B}_{P2}$ has been used. Again, the twist of link-2 is then computed recursively from the twist of link-1 as

$$\mathbf{t}_2 = \mathbf{B}_{21}\mathbf{t}_1 + \mathbf{p}_2\dot{\theta}_2$$
$$\mathbf{B}_{21} = \begin{bmatrix} 1 & \mathbf{0}^T \\ \mathbf{E}(\mathbf{r}_1 + \mathbf{d}_2) & \mathbf{1} \end{bmatrix}; \quad \mathbf{p}_2 = \begin{bmatrix} 1 \\ \mathbf{E}\mathbf{d}_2 \end{bmatrix}$$

where $\mathbf{t}_1$ is the twist of link-1 with respect to its mass center, $\dot{\theta}_2$ is the relative angular joint velocity of the second joint, $\mathbf{0}$ is the two-dimensional zero vector and $\mathbf{1}$ is the 2×2 identity matrix. Substituting $\mathbf{t}_2$ into Eq.(23), we obtain

$$\mathbf{\Phi}_3\mathbf{t}_P = \mathbf{\Phi}_3\mathbf{B}_{P2}(\mathbf{B}_{21}\mathbf{t}_1 + \mathbf{p}_2\dot{\theta}_2) \tag{24}$$

Solving for $\dot{\theta}_2$ we obtain

$$\dot{\theta}_2 = \frac{\tilde{\mathbf{p}}_2^T}{\delta_2}(\mathbf{t}_P - \mathbf{B}_{P1}\mathbf{t}_1) \tag{25}$$

where $\tilde{\mathbf{p}}_2 = \mathbf{\Phi}_3\mathbf{B}_{P2}\mathbf{p}_2$ and $\delta_2 = \tilde{\mathbf{p}}_2^T\tilde{\mathbf{p}}_2$. Substituting $\dot{\theta}_2$ into Eq.(24) leads to

$$\mathbf{\Phi}_2\mathbf{t}_P = \mathbf{\Phi}_2\mathbf{B}_{P1}\mathbf{t}_1 \tag{26}$$

where the 3×3 matrix $\mathbf{\Phi}_2$ is defined as $\mathbf{\Phi}_2 = \mathbf{\Phi}_3 - \tilde{\mathbf{p}}_2\tilde{\mathbf{p}}_2^T/\delta_2$ and the properties $\mathbf{B}_{P2}\mathbf{B}_{21} = \mathbf{B}_{P1}$ and $\mathbf{\Phi}_3^T\mathbf{\Phi}_3 = \mathbf{\Phi}_3$ have been used. Noting the similarity between Eq.(23) and Eq.(26), the kinematics relationships may be written in generic form as

$$\mathbf{\Phi}_i \mathbf{t}_P = \mathbf{\Phi}_i \mathbf{B}_{P,i-1} \mathbf{t}_{i-1} \tag{27}$$

where $\mathbf{\Phi}_i$ is evaluated recursively as

$$\mathbf{\Phi}_i = \mathbf{\Phi}_{i+1} - \tilde{\mathbf{p}}_i \tilde{\mathbf{p}}_i^T / \delta_i \tag{28}$$

Finally, since joint 1 is actuated, substituting $\mathbf{t}_1 = \mathbf{p}_1 \dot{\theta}_1$ into Eq.(26), we can express the twist of the platform $\mathcal{P}$ in terms of $\dot{\theta}_1$ as

$$\mathbf{\Phi}_2 \mathbf{t}_P = \mathbf{\Phi}_2 \mathbf{B}_{P1} \mathbf{p}_1 \dot{\theta}_1 \tag{29}$$

This equation illustrates the well known feature for parallel chains. Note that $\mathbf{\Phi}_2$ is a projection matrix and is thus singular.

Next, we write Eq.(29) for *each open chain* to obtain

$$\begin{aligned} \mathbf{K}\mathbf{t}_P &= \mathbf{\Phi}\mathbf{B}\mathbf{P}\dot{\boldsymbol{\theta}}_{ac} \\ \text{where} \quad \mathbf{K} &= \left[\mathbf{\Phi}_2^I + \mathbf{\Phi}_2^{II} + \mathbf{\Phi}_2^{III}\right] \ : \ 3 \times 3 \\ \mathbf{\Phi} &= \left[\mathbf{\Phi}_2^I \ \mathbf{\Phi}_2^{II} \ \mathbf{\Phi}_2^{III}\right] \ : \ 3 \times 9 \\ \mathbf{B} &= \mathrm{diag}(\mathbf{B}_{P1}^I, \mathbf{B}_{P1}^{II}, \mathbf{B}_{P1}^{III}) \ : \ 9 \times 9 \\ \mathbf{P} &= \mathrm{diag}(\mathbf{p}_1^I, \mathbf{p}_1^{II}, \mathbf{p}_1^{III}) \ : \ 9 \times 3 \\ \dot{\boldsymbol{\theta}}_{ac} &= \left[\dot{\theta}_1^I \ \dot{\theta}_1^{II} \ \dot{\theta}_1^{III}\right]^T \end{aligned}$$

where all dimensions have been stated for clarity. Finally when $\mathbf{K}$ is nonsingular[1], we may solve for the end effector velocity in terms of the actuated velocities as

$$\mathbf{t}_P = \mathbf{K}^{-1} \mathbf{\Phi}\mathbf{B}\mathbf{P}\dot{\boldsymbol{\theta}}_{ac} \tag{30}$$

The unactuated joint velocities of *any chain* may be now be computed by substituting $\mathbf{t}_P$ from Eq.(30) into Eq.(25) to yield

$$\dot{\theta}_2 = \frac{\tilde{\mathbf{p}}_2^T}{\delta_2} (\mathbf{K}^{-1}\mathbf{\Phi}\mathbf{B}\mathbf{P}\dot{\boldsymbol{\theta}}_{ac} - \mathbf{B}_{P1}\mathbf{p}_1\dot{\theta}_1) \tag{31}$$

[1] See [23] for a more detailed discussion of the type of singularities.

and substituting $\mathbf{t}_P$, $\mathbf{t}_2$, $\mathbf{t}_1$ and $\dot{\theta}_2$ into Eq.(22),

$$\begin{aligned} \dot{\theta}_3 &= \frac{\tilde{\mathbf{p}}_3^T}{\delta_3}[\mathbf{t}_P - \mathbf{B}_{P2}(\mathbf{B}_{21}\mathbf{t}_1 + \mathbf{p}_2\dot{\theta}_2)] \\ &= \frac{\tilde{\mathbf{p}}_3^T}{\delta_3}(\mathbf{t}_P - \mathbf{B}_{P1}\mathbf{p}_1\dot{\theta}_1 - \mathbf{B}_{P2}\mathbf{p}_2\dot{\theta}_2) \\ &= \frac{\tilde{\mathbf{p}}_3^T}{\delta_3}[(\mathbf{t}_P - \mathbf{B}_{P1}\mathbf{t}_1) - \mathbf{B}_{P2}\mathbf{p}_2\frac{\tilde{\mathbf{p}}_2^T}{\delta_2}(\mathbf{t}_p - \mathbf{B}_{P1}\mathbf{t}_1)] \\ &= \frac{\tilde{\mathbf{p}}_3^T}{\delta_3}(\mathbf{1} - \frac{1}{\delta_2}\mathbf{B}_{P2}\mathbf{p}_2\tilde{\mathbf{p}}_2^T)(\mathbf{t}_P - \mathbf{B}_{P1}\mathbf{t}_1) \end{aligned} \tag{32}$$

which can be written as

$$\dot{\theta}_3 = \frac{\tilde{\mathbf{p}}_3^T}{\delta_3}\mathbf{\Psi}_2^T(\mathbf{K}^{-1}\mathbf{\Phi}\mathbf{B}\mathbf{P}\dot{\boldsymbol{\theta}}_{ac} - \mathbf{B}_{P1}\mathbf{p}_1\dot{\theta}_1) \tag{33}$$

where the 3×3 matrix $\mathbf{\Psi}_2$ is defined as $\mathbf{\Psi}_2 = (\mathbf{1} - \mathbf{B}_{P2}\mathbf{p}_2\tilde{\mathbf{p}}_2^T/\delta_2)^T$ and $\mathbf{1}$ is the 3×3 identity matrix.

We note that Eqs.(31) and (33) are general and applicable to *each open chain* and that the bracketed term on the right hand side of each equation is the same. This term can be written specifically for *each open chain* as

$$\left[[\mathbf{K}^{-1}\mathbf{\Phi}_2 - \mathbf{1}]^I \ [\mathbf{K}^{-1}\mathbf{\Phi}_2]^{II} \ [\mathbf{K}^{-1}\mathbf{\Phi}_2]^{III}\right]\mathbf{B}\mathbf{P}\dot{\boldsymbol{\theta}}_{ac}$$

$$\left[[\mathbf{K}^{-1}\mathbf{\Phi}_2]^I \ [\mathbf{K}^{-1}\mathbf{\Phi}_2 - \mathbf{1}]^{II} \ [\mathbf{K}^{-1}\mathbf{\Phi}_2]^{III}\right]\mathbf{B}\mathbf{P}\dot{\boldsymbol{\theta}}_{ac}$$

$$\left[[\mathbf{K}^{-1}\mathbf{\Phi}_2]^I \ [\mathbf{K}^{-1}\mathbf{\Phi}_2]^{II} \ [\mathbf{K}^{-1}\mathbf{\Phi}_2 - \mathbf{1}]^{III}\right]\mathbf{B}\mathbf{P}\dot{\boldsymbol{\theta}}_{ac}$$

Thus the final relationship between the joint rates and actuated joint rates is expressed in matrix form as

$$\dot{\boldsymbol{\theta}} = \begin{bmatrix} \bar{\mathbf{P}}^I & \mathbf{O} & \mathbf{O} \\ \mathbf{O} & \bar{\mathbf{P}}^{II} & \mathbf{O} \\ \mathbf{O} & \mathbf{O} & \bar{\mathbf{P}}^{III} \end{bmatrix} \begin{bmatrix} \mathbf{L}^I \\ \mathbf{L}^{II} \\ \mathbf{L}^{III} \end{bmatrix} \mathbf{B}\mathbf{P}\dot{\boldsymbol{\theta}}_{ac} \tag{34}$$

where the 3×9 matrix $\bar{\mathbf{P}}_i$ is defined as $\bar{\mathbf{P}}_i = [\mathrm{diag}(\tilde{\mathbf{p}}_1^T/\delta_1, \tilde{\mathbf{p}}_2^T/\delta_2, \tilde{\mathbf{p}}_3^T\mathbf{\Psi}_2^T/\delta_3)]^i$, while $\tilde{\mathbf{p}}_1^i$ as explicitly $\tilde{\mathbf{p}}_1^i = (\mathbf{B}_{P1}\mathbf{p}_1)^i$ for $i = I, II$ and III, and the 9×9 matrices $\mathbf{L}$ are defined for *each open chain* as

$$\mathbf{L}^I = \begin{bmatrix} \mathbf{1} & \mathbf{O} & \mathbf{O} \\ [\mathbf{K}^{-1}\mathbf{\Phi}_2 - \mathbf{1}]^I & [\mathbf{K}^{-1}\mathbf{\Phi}_2]^{II} & [\mathbf{K}^{-1}\mathbf{\Phi}_2]^{III} \\ [\mathbf{K}^{-1}\mathbf{\Phi}_2 - \mathbf{1}]^I & [\mathbf{K}^{-1}\mathbf{\Phi}_2]^{II} & [\mathbf{K}^{-1}\mathbf{\Phi}_2]^{III} \end{bmatrix}$$

$$\mathbf{L}^{II} = \begin{bmatrix} \mathbf{O} & \mathbf{1} & \mathbf{O} \\ [\mathbf{K}^{-1}\mathbf{\Phi}_2]^I & [\mathbf{K}^{-1}\mathbf{\Phi}_2 - \mathbf{1}]^{II} & [\mathbf{K}^{-1}\mathbf{\Phi}_2]^{III} \\ [\mathbf{K}^{-1}\mathbf{\Phi}_2]^I & [\mathbf{K}^{-1}\mathbf{\Phi}_2 - \mathbf{1}]^{II} & [\mathbf{K}^{-1}\mathbf{\Phi}_2]^{III} \end{bmatrix}$$

$$\mathbf{L}^{III} = \begin{bmatrix} \mathbf{O} & \mathbf{O} & \mathbf{1} \\ [\mathbf{K}^{-1}\mathbf{\Phi}_2]^I & [\mathbf{K}^{-1}\mathbf{\Phi}_2]^{II} & [\mathbf{K}^{-1}\mathbf{\Phi}_2 - \mathbf{1}]^{III} \\ [\mathbf{K}^{-1}\mathbf{\Phi}_2]^I & [\mathbf{K}^{-1}\mathbf{\Phi}_2]^{II} & [\mathbf{K}^{-1}\mathbf{\Phi}_2 - \mathbf{1}]^{III} \end{bmatrix}$$

Equation (34) can be written in compact form as

$$\dot{\boldsymbol{\theta}} = \bar{\mathbf{P}}\mathbf{L}\mathbf{B}\mathbf{P}\dot{\boldsymbol{\theta}}_{ac} \tag{35}$$

where the 9×27 matrix $\bar{\mathbf{P}}$ is defined as $\bar{\mathbf{P}} = \mathrm{diag}(\bar{\mathbf{P}}^I, \bar{\mathbf{P}}^{II}, \bar{\mathbf{P}}^{III})$ and the 27×9 matrix $\mathbf{L}$ is defined as $\mathbf{L} = \left[(\mathbf{L}^I)^T\ (\mathbf{L}^{II})^T\ (\mathbf{L}^{III})^T\right]^T$. Note that, except for $\mathbf{L}$, which is full but still retains a special form, all other matrices are block-diagonal.

Acceleration Analysis The acceleration terms, for *any chain*, by differentiating Eq.(21) as

$$\dot{\mathbf{t}}_P = \dot{\mathbf{B}}_{P3}\mathbf{t}_3 + \mathbf{B}_{P3}(\dot{\mathbf{B}}_{32}\mathbf{t}_2 + \mathbf{B}_{32}\dot{\mathbf{t}}_2 + \dot{\mathbf{p}}_3\dot{\theta}_3) + \mathbf{B}_{P3}\mathbf{p}_3\ddot{\theta}_3 \tag{36}$$

Adopting a process similar to the one discussed for the velocity analysis, we may solve for the unactuated joint acceleration for $\ddot{\theta}_3$ as

$$\ddot{\theta}_3 = \frac{\tilde{\mathbf{p}}_3^T}{\delta_3}[\dot{\mathbf{t}}_P - \dot{\mathbf{B}}_{P3}\mathbf{t}_3 - \mathbf{B}_{P3}(\dot{\mathbf{B}}_{32}\mathbf{t}_2 + \mathbf{B}_{32}\dot{\mathbf{t}}_2 + \dot{\mathbf{p}}_3\dot{\theta}_3)] \tag{37}$$

Substituting $\ddot{\theta}_3$ back into Eq.(36),

$$\mathbf{\Phi}_3\dot{\mathbf{t}}_P = \mathbf{\Phi}_3[\dot{\mathbf{B}}_{P3}\mathbf{t}_3 + \mathbf{B}_{P3}(\dot{\mathbf{B}}_{32}\mathbf{t}_2 + \mathbf{B}_{32}\dot{\mathbf{t}}_2 + \dot{\mathbf{p}}_3\dot{\theta}_3)] \tag{38}$$

Now we obtain the expression for $\dot{\mathbf{\Phi}}_3$. Substituting $\mathbf{t}_3$ and $\dot{\theta}_3$ into Eq.(38) and rearranging leads to,

$$\begin{aligned}\mathbf{\Phi}_3\dot{\mathbf{t}}_P &= \mathbf{\Phi_3}\mathbf{B}_{P2}\dot{\mathbf{t}}_2 + \mathbf{\Phi_3}\dot{\mathbf{B}}_{P2}\mathbf{t}_2 \\ &\quad - [\mathbf{\Phi}_3(\dot{\mathbf{B}}_{P3}\mathbf{p}_3 + \mathbf{B}_{P3}\dot{\mathbf{p}}_3)\frac{\tilde{\mathbf{p}}_3^T}{\delta_3}]\mathbf{B}_{P2}\mathbf{t}_2 \\ &\quad + [\mathbf{\Phi}_3(\dot{\mathbf{B}}_{P3}\mathbf{p}_3 + \mathbf{B}_{P3}\dot{\mathbf{p}}_3)\frac{\tilde{\mathbf{p}}_3^T}{\delta_3}]\mathbf{t}_P\end{aligned} \tag{39}$$

where the relation $(\dot{\mathbf{B}}_{P3}\mathbf{B}_{32} + \mathbf{B}_{P3}\dot{\mathbf{B}}_{32}) = \dot{\mathbf{B}}_{P2}$ has been used. Time-differentiating Eq.(23) we obtain:

$$\mathbf{\Phi}_3\dot{\mathbf{t}}_P = \mathbf{\Phi_3}\mathbf{B}_{P2}\dot{\mathbf{t}}_2 + \mathbf{\Phi_3}\dot{\mathbf{B}}_{P2}\mathbf{t}_2 + \dot{\mathbf{\Phi}}_3\mathbf{B}_{P2}\mathbf{t}_2 - \dot{\mathbf{\Phi}}_3\mathbf{t}_P \tag{40}$$

Comparing Eq.(39) with Eq.(40), we obtain:

$$\dot{\mathbf{\Phi}}_3 = -\mathbf{\Phi}_3(\dot{\mathbf{B}}_{P3}\mathbf{p}_3 + \mathbf{B}_{P3}\dot{\mathbf{p}}_3)\frac{\tilde{\mathbf{p}}_3^T}{\delta_3} \tag{41}$$

or

$$\dot{\mathbf{\Phi}}_3 = -\mathbf{\Phi}_3\frac{\dot{\tilde{\mathbf{p}}}_3\tilde{\mathbf{p}}_3^T}{\delta_3} \tag{42}$$

Now $\mathbf{t}_2 = \mathbf{B}_{21}\mathbf{t}_1 + \mathbf{p}_2\dot{\theta}_2$, and hence, $\dot{\mathbf{t}}_2 = \dot{\mathbf{B}}_{21}\mathbf{t}_1 + \mathbf{B}_{21}\dot{\mathbf{t}}_1 + \mathbf{p}_2\ddot{\theta}_2 + \dot{\mathbf{p}}_2\dot{\theta}_2$. Substituting $\mathbf{t}_2$ and $\dot{\mathbf{t}}_2$ in Eq.(40) we obtain:

$$\begin{aligned}(\dot{\mathbf{\Phi}}_3\mathbf{t}_P + \mathbf{\Phi}_3\dot{\mathbf{t}}_P) &= \mathbf{\Phi}_3\mathbf{B}_{P2}\mathbf{p}_2\ddot{\theta}_2 \\ &\quad + [\mathbf{\Phi}_3\mathbf{B}_{P2}(\dot{\mathbf{B}}_{21}\mathbf{t}_1 + \mathbf{B}_{21}\dot{\mathbf{t}}_1 + \dot{\mathbf{p}}_2\dot{\theta}_2) \\ &\quad + \mathbf{\Phi}_3\dot{\mathbf{B}}_{P2}\mathbf{t}_2 + \dot{\mathbf{\Phi}}_3\mathbf{B}_{P2}\mathbf{t}_2]\end{aligned} \tag{43}$$

Solving the above equation for $\ddot{\theta}_2$

$$\begin{aligned}\ddot{\theta}_2 = \frac{\tilde{\mathbf{p}}_2^T}{\delta_2}&[(\dot{\mathbf{\Phi}}_3\mathbf{t}_P + \mathbf{\Phi}_3\dot{\mathbf{t}}_P) - [\mathbf{\Phi}_3\mathbf{B}_{P2}(\dot{\mathbf{B}}_{21}\mathbf{t}_1 + \mathbf{B}_{21}\dot{\mathbf{t}}_1 + \dot{\mathbf{p}}_2\dot{\theta}_2) \\ &+ \mathbf{\Phi}_3\dot{\mathbf{B}}_{P2}\mathbf{t}_2 + \dot{\mathbf{\Phi}}_3\mathbf{B}_{P2}\mathbf{t}_2]]\end{aligned} \tag{44}$$

substituting $\ddot{\theta}_2$ back into Eq.(43),

$$\begin{aligned}\bar{\mathbf{\Phi}}_2(\dot{\mathbf{\Phi}}_3\mathbf{t}_P + \mathbf{\Phi}_3\dot{\mathbf{t}}_p) &= \bar{\mathbf{\Phi}}_2[\mathbf{\Phi}_3\mathbf{B}_{P2}(\dot{\mathbf{B}}_{21}\mathbf{t}_1 + \mathbf{B}_{21}\dot{\mathbf{t}}_1 + \dot{\mathbf{p}}_2\dot{\theta}_2) \\ &\quad + \mathbf{\Phi}_3\dot{\mathbf{B}}_{P2}\mathbf{t}_2 + \dot{\mathbf{\Phi}}_3\mathbf{B}_{P2}\mathbf{t}_2]\end{aligned} \tag{45}$$

where $\bar{\mathbf{\Phi}}_2 = \mathbf{1} - \tilde{\mathbf{p}}_2\tilde{\mathbf{p}}_2^T/\delta_2$. However, we note that

$$\bar{\mathbf{\Phi}}_2\mathbf{\Phi}_3 = (\mathbf{\Phi}_3 - \frac{\tilde{\mathbf{p}}_2\tilde{\mathbf{p}}_2^T}{\delta_2}) = \mathbf{\Phi}_2$$

where the relation $\mathbf{\Phi}_3^T\mathbf{\Phi}_3 = \mathbf{\Phi}_3$ was used. Substituting $\bar{\mathbf{\Phi}}_2\mathbf{\Phi}_3 = \mathbf{\Phi}_2$ and rearranging eqs.(44 & 45) leads to

$$\ddot{\theta}_2 = \frac{\tilde{\mathbf{p}}_2^T}{\delta_2}[\mathbf{\Phi}_3(\dot{\mathbf{t}}_P - \mathbf{a}_1) - \mathbf{a}_2] \tag{46}$$

$$\mathbf{\Phi}_2\dot{\mathbf{t}}_P = \mathbf{\Phi}_2\mathbf{a}_1 + \bar{\mathbf{\Phi}}_2\mathbf{a}_2 \tag{47}$$

where $\mathbf{a}_1 = \mathbf{B}_{P2}(\dot{\mathbf{B}}_{21}\mathbf{t}_1 + \mathbf{B}_{21}\dot{\mathbf{t}}_1 + \dot{\mathbf{p}}_2\dot{\theta}_2) + \dot{\mathbf{B}}_{P2}\mathbf{t}_2$ and $\mathbf{a}_2 = \dot{\mathbf{\Phi}}_3(\mathbf{B}_{P2}\mathbf{t}_2 - \mathbf{t}_P)$.

Finally, adding Eq.(47) for *each open chain* and solving for $\dot{\mathbf{t}}_p$,

$$\begin{aligned}\dot{\mathbf{t}}_P = \ &\mathbf{K}^{-1}([\mathbf{\Phi}_2\mathbf{a}_1 + \bar{\mathbf{\Phi}}_2\mathbf{a}_2]^I + [\mathbf{\Phi}_2\mathbf{a}_1 + \bar{\mathbf{\Phi}}_2\mathbf{a}_2]^{II} \\ &+ [\mathbf{\Phi}_2\mathbf{a}_1 + \bar{\mathbf{\Phi}}_2\mathbf{a}_2]^{III})\end{aligned} \tag{48}$$

Summary of Forward Kinematics The overall process of computing the forward kinematics can be summarized as:

1. With the data, calculate $\mathbf{B}_{21}$, $\mathbf{B}_{32}$, $\mathbf{B}_{31}$, $\mathbf{B}_{P3}$,$\mathbf{B}_{P2}$, $\mathbf{B}_{P1}$, $\mathbf{p}_1$, $\mathbf{p}_2$ and $\mathbf{p}_3$. For *each chain*, we calculate:

$$\begin{aligned}\tilde{\mathbf{p}}_3 &= \mathbf{B}_{P3}\mathbf{p}_3\\ \delta_3 &= \tilde{\mathbf{p}}_3^T\tilde{\mathbf{p}}_3\\ \mathbf{\Phi}_3 &= [\mathbf{1} - \frac{\tilde{\mathbf{p}}_3\tilde{\mathbf{p}}_3^T}{\delta_3}]\\ \tilde{\mathbf{p}}_2 &= \mathbf{B}_{P2}\mathbf{p}_2\\ \delta_2 &= \tilde{\mathbf{p}}_2^T\tilde{\mathbf{p}}_2\\ \mathbf{\Phi}_2 &= \mathbf{\Phi}_3 - \frac{\tilde{\mathbf{p}}_2\tilde{\mathbf{p}}_2^T}{\delta_2}\end{aligned}$$

2. Form matrices $\mathbf{K}$, $\mathbf{\Phi}$, $\mathbf{B}$ and $\mathbf{P}$ with values received from *each chain* and calculate the platform twist $\mathbf{t}_P$ from:

$$\mathbf{K}\mathbf{t}_p = \mathbf{\Phi}\mathbf{B}\mathbf{P}\dot{\boldsymbol{\theta}}_{ac}$$

3. Obtain the twists and joint rates recursively for *each chain*, using $\mathbf{t}_P$.

$$\begin{aligned}\mathbf{t}_1 &= \mathbf{p}_1\dot{\theta}_1\\ \dot{\theta}_2 &= \frac{\tilde{\mathbf{p}}_2^T}{\delta_2}(\mathbf{t}_P - \mathbf{B}_{P1}\mathbf{t}_1)\\ \mathbf{t}_2 &= \mathbf{B}_{21}\mathbf{t}_1 + \mathbf{p}_2\dot{\theta}_2\\ \dot{\theta}_3 &= \frac{\tilde{\mathbf{p}}_3^T}{\delta_3}(\mathbf{t}_P - \mathbf{B}_{P2}\mathbf{t}_2)\\ \mathbf{t}_3 &= \mathbf{B}_{32}\mathbf{t}_2\mathbf{p}_3\dot{\theta}_3\end{aligned}$$

Now calculate twist rates and joint accelerations for *each chain*. First, calculate i.e. $\dot{\mathbf{B}}_{21}$, $\dot{\mathbf{B}}_{32}$, $\dot{\mathbf{B}}_{31}$, $\dot{\mathbf{B}}_{P3}$,$\dot{\mathbf{B}}_{P2}$, $\dot{\mathbf{B}}_{P1}$, $\dot{\mathbf{p}}_1$, $\dot{\mathbf{p}}_2$ and $\dot{\mathbf{p}}_3$ using joint velocities calculated above:

$$\begin{aligned}\dot{\mathbf{t}}_1 &= \mathbf{p}_1\ddot{\theta}_1 + \dot{\mathbf{p}}_1\dot{\theta}_1\\ \mathbf{a}_1 &= \mathbf{B}_{P2}(\dot{\mathbf{B}}_{21}\mathbf{t}_1 + \mathbf{B}_{21}\dot{\mathbf{t}}_1 + \dot{\mathbf{p}}_2\dot{\theta}_2) + \dot{\mathbf{B}}_{P2}\mathbf{t}_2\\ \dot{\mathbf{\Phi}}_3 &= -\mathbf{\Phi}_3(\dot{\mathbf{B}}_{P3}\mathbf{p}_3 + \mathbf{B}_{P3}\dot{\mathbf{p}}_3)\frac{\tilde{\mathbf{p}}_3^T}{\delta_3}\\ \mathbf{a}_2 &= \dot{\mathbf{\Phi}}_3(\mathbf{B}_{P2}\mathbf{t}_2 - \mathbf{t}_P)\\ \bar{\mathbf{\Phi}}_2 &= \mathbf{1} - \frac{\tilde{\mathbf{p}}_2\tilde{\mathbf{p}}^T}{\delta_2}\end{aligned}$$

4. Using $\mathbf{\Phi}_2$, $\bar{\mathbf{\Phi}}_2$, $\mathbf{a}_1$ and $\mathbf{a}_2$ from each chain calculate $\dot{\mathbf{t}}_P$ from:

$$\mathbf{K}\dot{\mathbf{t}}_P = [\mathbf{\Phi}_2\mathbf{a}_1 + \bar{\mathbf{\Phi}}_2\mathbf{a}_2]^I + [\mathbf{\Phi}_2\mathbf{a}_1 + \bar{\mathbf{\Phi}}_2\mathbf{a}_2]^{II} + [\mathbf{\Phi}_2\mathbf{a}_1 + \bar{\mathbf{\Phi}}_2\mathbf{a}_2]^{III}$$

5. Calculate joint accelerations and twist rates for *each chain*:

$$\begin{aligned}\ddot{\theta}_2 &= \frac{\tilde{\mathbf{p}}_2^T}{\delta_2}[\mathbf{\Phi}_3(\dot{\mathbf{t}}_P - \mathbf{a}_1) - \mathbf{a}_2]\\ \dot{\mathbf{t}}_2 &= \dot{\mathbf{B}}_{21}\mathbf{t}_1 + \mathbf{B}_{21}\dot{\mathbf{t}}_1\dot{\mathbf{p}}_2\dot{\theta}_2 + \mathbf{p}_2\ddot{\theta}_2\\ \ddot{\theta}_3 &= \frac{\tilde{\mathbf{p}}_3^T}{\delta_3}[\dot{\mathbf{t}}_P - \dot{\mathbf{B}}_{P3}\mathbf{t}_3 - \mathbf{B}_{P3}(\dot{\mathbf{B}}_{32}\mathbf{t}_2 + \mathbf{B}_{32}\dot{\mathbf{t}}_2 + \dot{\mathbf{p}}_3\dot{\theta}_3)]\\ \dot{\mathbf{t}}_3 &= \mathbf{B}_{32}\dot{\mathbf{t}}_2 + \dot{\mathbf{B}}_{32}\mathbf{t}_2 + \mathbf{p}_3\dot{\theta}_3\dot{\mathbf{p}}_3\ddot{\theta}_3\end{aligned}$$

Inverse Dynamics

The inverse-dynamics problem is defined as: Given the time-histories of all the system degrees-of-freedom, compute the time-histories of the controlling actuated joint torques and forces. As in the case of the kinematics calculations, we again divide the platform into three parts and assign cut sections of platform $\mathcal{P}$ to each open chain. Each cut section thus becomes the "end-effector" of the corresponding serial chain. Further, we divide the mass of the platform (including any tool carried by the platform) and assign its corresponding moment of inertia, with respect to the mass center of the platform, to the "end-effector" of each chain. Any working wrench applied to the platform has to be appropriately divided in a similar fashion. The Newton-Euler equations for *each open chain* is, thus,

$$\mathbf{M}\dot{\mathbf{t}} + \dot{\mathbf{M}}\mathbf{t} = \mathbf{w}^A + \underbrace{\overline{\mathbf{w}}^W + \mathbf{w}^g}_{\mathbf{w}^G} + \mathbf{w}^C \tag{49}$$

where $\mathbf{M}$ is the 9×9 mass matrix, $\mathbf{t}$ is the 9-dimensional twist vector of the whole chain, $\mathbf{w}^A$ is the wrench applied by the actuators, $\overline{\mathbf{w}}^W = [\mathbf{0}^T, \mathbf{0}^T, (\mathbf{w}^W)^T]^T$, where $\mathbf{w}^W$ is the corresponding working wrench applied at the "end-effector" of the corresponding chain, $\mathbf{w}^g$ is the gravity wrench and $\mathbf{w}^C$ are the constraint wrenches all these being 9-dimensional vectors. The friction forces have been left out for the sake of simplicity but they can be readily taken into account by means of dissipation function [18].

In particular, the twist vector $\mathbf{t}$ may now be written as [16]

$$\mathbf{t} = \mathbf{N}_l\mathbf{N}_d\dot{\boldsymbol{\theta}} \tag{50}$$

where $\mathbf{N}_l\mathbf{N}_d$ is the decoupled orthogonal complement and $\dot{\boldsymbol{\theta}}$ is the joint-rate vector of the chain. For our manipulator, for *each open chain* Eq.(50) is

$$\mathbf{t} = \begin{bmatrix}\mathbf{t}_1\\ \mathbf{t}_2\\ \mathbf{t}_3\end{bmatrix} = \underbrace{\begin{bmatrix}\mathbf{1} & \mathbf{O} & \mathbf{O}\\ \mathbf{B}_{21} & \mathbf{1} & \mathbf{O}\\ \mathbf{B}_{31} & \mathbf{B}_{32} & \mathbf{1}\end{bmatrix}}_{\mathbf{N}_l} \underbrace{\begin{bmatrix}\mathbf{p}_1 & \mathbf{0} & \mathbf{0}\\ \mathbf{0} & \mathbf{p}_2 & \mathbf{0}\\ \mathbf{0} & \mathbf{0} & \mathbf{p}_3\end{bmatrix}}_{\mathbf{N}_d} \begin{bmatrix}\dot{\theta}_1\\ \dot{\theta}_2\\ \dot{\theta}_3\end{bmatrix} \tag{51}$$

where all terms have been previously defined. Now, the constraint wrenches $\mathbf{w}^C$ do not develop any power, and hence, $\mathbf{t}^T\mathbf{w}^C$ is 0; by virtue of Eq.(50), $\mathbf{w}^C$ lies in the nullspace of $\mathbf{N}_d^T\mathbf{N}_l^T$. To eliminate joint constraint wrenches, we pre-multiply both sides of Eq.(49) by $\mathbf{N}_d^T\mathbf{N}_l^T$, and noting that, for planar manipulators $\dot{\mathbf{M}} = \mathbf{O}$,

$$\mathbf{N}_d^T\mathbf{N}_l^T\mathbf{M}\dot{\mathbf{t}} = \tilde{\boldsymbol{\tau}} + \mathbf{N}_d^T\mathbf{N}_l^T\mathbf{w}^G \tag{52}$$

Time differentiating Eq.(50) and substituting the expression for $\dot{\mathbf{t}}$ in Eq.(52),

$$\mathbf{N}_d^T\mathbf{N}_l^T[\mathbf{M}\mathbf{N}_l\mathbf{N}_d\ddot{\boldsymbol{\theta}}+(\mathbf{M}\dot{\mathbf{N}}_l\mathbf{N}_d+\mathbf{M}\mathbf{N}_l\dot{\mathbf{N}}_d)\dot{\boldsymbol{\theta}}] = \tilde{\boldsymbol{\tau}}+\mathbf{N}_d^T\mathbf{N}_l^T\mathbf{w}^G \tag{53}$$

where $\tilde{\boldsymbol{\tau}} = \mathbf{N}_d^T\mathbf{N}_l^T\mathbf{w}^A$ is the joint torque vector for the chain and is given by $\tilde{\boldsymbol{\tau}} = \left[\tilde{\tau}_1\ \tilde{\tau}_2\ \tilde{\tau}_3\right]^T$ for *each open chain*. We can write the above equation in compact form as

$$\mathbf{I}\ddot{\boldsymbol{\theta}} + \mathbf{C}\dot{\boldsymbol{\theta}} = \tilde{\boldsymbol{\tau}} + \boldsymbol{\tau}^G \tag{54}$$

where $\mathbf{I} = \mathbf{N}_d^T\mathbf{N}_l^T\mathbf{M}\mathbf{N}_l\mathbf{N}_d$ and $\mathbf{C} = \mathbf{N}_d^T\mathbf{N}_l^T(\mathbf{M}\dot{\mathbf{N}}_l\mathbf{N}_d + \mathbf{M}\mathbf{N}_l\dot{\mathbf{N}}_d)$, $\mathbf{I}$ being the *generalized inertia matrix of the chain* and $\mathbf{C}$ the matrix of coriolis and centrifugal forces.

Notice that the distribution of the working wrench between chains is not important, as the torques evaluated at this stage are projected onto the minimum actuated joint space in the step that follows. We will discuss the case where the working wrench is assumed to have been distributed evenly among the subsystems.

Projecting Joint Torques onto Minimal-Coordinate Space As a second step, we write the dynamics equation for *each chain* and couple them with Lagrange multipliers, thereby obtaining the dynamics equation of the whole manipulator as

$$\begin{bmatrix} [\mathbf{I}\ddot{\boldsymbol{\theta}} + \mathbf{C}\dot{\boldsymbol{\theta}}]^I \\ [\mathbf{I}\ddot{\boldsymbol{\theta}} + \mathbf{C}\dot{\boldsymbol{\theta}}]^{II} \\ [\mathbf{I}\ddot{\boldsymbol{\theta}} + \mathbf{C}\dot{\boldsymbol{\theta}}]^{III} \end{bmatrix} = \begin{bmatrix} \boldsymbol{\tau}^{I} \\ \boldsymbol{\tau}^{II} \\ \boldsymbol{\tau}^{III} \end{bmatrix} + \begin{bmatrix} \boldsymbol{\tau}_G^{I} \\ \boldsymbol{\tau}_G^{II} \\ \boldsymbol{\tau}_G^{III} \end{bmatrix} - \mathbf{A}^T\boldsymbol{\lambda} \tag{55}$$

where $\mathbf{A}$ is the *loop-closure constraint Jacobian*, which appears in the constraints in the form

$$\mathbf{A}\dot{\boldsymbol{\theta}} = \mathbf{0}$$

Now, by defining $\dot{\boldsymbol{\theta}} = \mathbf{J}\dot{\boldsymbol{\theta}}_{AC}$, it is clear that $\mathbf{J}$ lies in the nullspace of $\mathbf{A}$ and may be called the *loop-closure orthogonal complement*. Pre-multiplying both sides of Eq.(55) by $\mathbf{J}^T$ we obtain:

$$\mathbf{J}^T \begin{bmatrix} [\mathbf{I}\ddot{\boldsymbol{\theta}} + \mathbf{C}\dot{\boldsymbol{\theta}} - \boldsymbol{\tau}_{\boldsymbol{G}}]^I \\ [\mathbf{I}\ddot{\boldsymbol{\theta}} + \mathbf{C}\dot{\boldsymbol{\theta}} - \boldsymbol{\tau}_{\boldsymbol{G}}]^{II} \\ [\mathbf{I}\ddot{\boldsymbol{\theta}} + \mathbf{C}\dot{\boldsymbol{\theta}} - \boldsymbol{\tau}_{\boldsymbol{G}}]^{III} \end{bmatrix} = \boldsymbol{\tau}_{AC} \tag{56}$$

where $\boldsymbol{\tau}_{AC}$ is the vector of actuator torques. Notice that the bracketed terms are nothing else than $\tilde{\boldsymbol{\tau}}^j$, which can be found for *each open chain*, for $j = I, II$ and III, recursively [16]. We may therefore write Eq.(56) as

$$\mathbf{J}^T \begin{bmatrix} \tilde{\boldsymbol{\tau}}^I \\ \tilde{\boldsymbol{\tau}}^{II} \\ \tilde{\boldsymbol{\tau}}^{III} \end{bmatrix} = \boldsymbol{\tau}_{AC} \tag{57}$$

which is the relation sought, where $\mathbf{J}$, given by Eq.(35), may be written in a slightly different form

$$\mathbf{J} = \bar{\mathbf{P}}\mathbf{L}\tilde{\mathbf{P}} \tag{58}$$

$$\text{where} \quad \mathbf{L} = \begin{bmatrix} \mathbf{1} & \mathbf{O} & \mathbf{O} \\ (\mathbf{S}^I - \mathbf{1}) & \mathbf{S}^{II} & \mathbf{S}^{III} \\ (\mathbf{S}^I - \mathbf{1}) & \mathbf{S}^{II} & \mathbf{S}^{III} \\ \mathbf{O} & \mathbf{1} & \mathbf{O} \\ \mathbf{S}^I & (\mathbf{S}^{II} - \mathbf{1}) & \mathbf{S}^{III} \\ \mathbf{S}^I & (\mathbf{S}^{II} - \mathbf{1}) & \mathbf{S}^{III} \\ \mathbf{O} & \mathbf{O} & \mathbf{1} \\ \mathbf{S}^I & \mathbf{S}^{II} & (\mathbf{S}^{III} - \mathbf{1}) \\ \mathbf{S}^I & \mathbf{S}^{II} & (\mathbf{S}^{III} - \mathbf{1}) \end{bmatrix},$$

$$\tilde{\mathbf{P}} = \begin{bmatrix} \tilde{\mathbf{p}}_1^I & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \tilde{\mathbf{p}}_1^{II} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \tilde{\mathbf{p}}_1^{III} \end{bmatrix}$$

and $\mathbf{S}^j = \mathbf{K}^{-1}\boldsymbol{\Phi}_2^j$ for $j = I, II$ and III. Substituting $\mathbf{J}$ into Eq.(57) and rearranging we obtain the actuated torques as

$$\begin{aligned}
\tau_1^I &= (\tilde{\mathbf{p}}_1^I)^T[(\mathbf{S}^I)^T(\hat{\mathbf{p}}_2^I + \hat{\mathbf{p}}_3^I + \hat{\mathbf{p}}_2^{II} + \hat{\mathbf{p}}_3^{II} + \hat{\mathbf{p}}_2^{III} + \hat{\mathbf{p}}_3^{III}) \\
&\quad + \hat{\mathbf{p}}_1^I - \hat{\mathbf{p}}_2^I - \hat{\mathbf{p}}_3^I] \\
\tau_1^{II} &= (\tilde{\mathbf{p}}_1^{II})^T[(\mathbf{S}^{II})^T(\hat{\mathbf{p}}_2^I + \hat{\mathbf{p}}_3^I + \hat{\mathbf{p}}_2^{II} + \hat{\mathbf{p}}_3^{II} + \hat{\mathbf{p}}_2^{III} + \hat{\mathbf{p}}_3^{III}) \\
&\quad + \hat{\mathbf{p}}_1^{II} - \hat{\mathbf{p}}_2^{II} - \hat{\mathbf{p}}_3^{II}] \\
\tau_1^{III} &= (\tilde{\mathbf{p}}_1^{III})^T[(\mathbf{S}^{III})^T(\hat{\mathbf{p}}_2^I + \hat{\mathbf{p}}_3^I + \hat{\mathbf{p}}_2^{II} + \hat{\mathbf{p}}_3^{II} + \hat{\mathbf{p}}_2^{III} + \hat{\mathbf{p}}_3^{III}) \\
&\quad + \hat{\mathbf{p}}_1^{III} - \hat{\mathbf{p}}_2^{III} - \hat{\mathbf{p}}_3^{III}]
\end{aligned}$$

where $\hat{\mathbf{p}}_k^j = [\tilde{\tau}_k\boldsymbol{\Psi}_{k-1}\tilde{\mathbf{p}}_k/\delta_k]^j$ for $k = 1,\ 2$ and 3 and $j = I,\ II$ and III, where $\boldsymbol{\Psi}_0$ and $\boldsymbol{\Psi}_1$ are the three-dimensional identity matrices. But $\tilde{\mathbf{p}}_1^T\hat{\mathbf{p}}_1 = \tilde{\tau}_1$ and hence the above equation set is written finally as

$$\boldsymbol{\tau}_{AC} = \begin{bmatrix} \tilde{\tau}_1^I + (\tilde{\mathbf{p}}_1^I)^T[(\mathbf{S}^I)^T(\bar{\mathbf{p}}^I + \bar{\mathbf{p}}^{II} + \bar{\mathbf{p}}^{III}) - \bar{\mathbf{p}}^I] \\ \tilde{\tau}_1^{II} + (\tilde{\mathbf{p}}_1^{II})^T[(\mathbf{S}^{II})^T(\bar{\mathbf{p}}^I + \bar{\mathbf{p}}^{II} + \bar{\mathbf{p}}^{III}) - \bar{\mathbf{p}}^{II}] \\ \tilde{\tau}_1^{III} + (\tilde{\mathbf{p}}_1^{III})^T[(\mathbf{S}^{III})^T(\bar{\mathbf{p}}^I + \bar{\mathbf{p}}^{II} + \bar{\mathbf{p}}^{III}) - \bar{\mathbf{p}}^{III}] \end{bmatrix}$$

where $\bar{\mathbf{p}} = \hat{\mathbf{p}}_2 + \hat{\mathbf{p}}_3$ for corresponding chains.

Summary of Inverse Dynamics To summarize the process of computation of the inverse dynamics of PSGP

1. From Eq.(52) it is clear that

$$\tilde{\boldsymbol{\tau}} = \mathbf{N}_d^T\mathbf{N}_l^T(\mathbf{M}\dot{\mathbf{t}} + \mathbf{w}^G)$$

which can be calculated recursively for each chain as follows:

$$\begin{aligned} \boldsymbol{\gamma}_3 &= (\mathbf{M}_3\dot{\mathbf{t}}_3 + \mathbf{w}_3^G) \\ \boldsymbol{\gamma}_2 &= (\mathbf{M}_2\dot{\mathbf{t}}_2 + \mathbf{w}_2^G) + \mathbf{B}_{32}^T\boldsymbol{\gamma}_3 \\ \boldsymbol{\gamma}_1 &= (\mathbf{M}_1\dot{\mathbf{t}}_1 + \mathbf{w}_1^G) + \mathbf{B}_{21}^T\boldsymbol{\gamma}_2 \\ \tilde{\tau}_3 &= \mathbf{p}_3^T\boldsymbol{\gamma}_3 \\ \tilde{\tau}_2 &= \mathbf{p}_2^T\boldsymbol{\gamma}_2 \\ \tilde{\tau}_1 &= \mathbf{p}_1^T\boldsymbol{\gamma}_1 \end{aligned}$$

Now we calculate $\bar{\mathbf{p}}$:

$$\bar{\mathbf{p}} = \tilde{\tau}_2\frac{\tilde{\mathbf{p}}_2}{\delta_2} + \tilde{\tau}_3\frac{\boldsymbol{\Psi}_2\tilde{\mathbf{p}}_3}{\delta_3}$$

2. Add all $\bar{\mathbf{p}}$ from each chain
3. Calculate actuated joint torques:
   Chain I:

$$\tau_1^I = \tilde{\tau}_1^I + (\tilde{\mathbf{p}}_1^I)^T[(\mathbf{S}^I)^T(\bar{\mathbf{p}}^I + \bar{\mathbf{p}}^{II} + \bar{\mathbf{p}}^{III}) - \bar{\mathbf{p}}^I]$$

Chain II:

$$\tau_1^{II} = \tilde{\tau}_1^{II} + (\tilde{\mathbf{p}}_1^{II})^T[(\mathbf{S}^{II})^T(\bar{\mathbf{p}}^I + \bar{\mathbf{p}}^{II} + \bar{\mathbf{p}}^{III}) - \bar{\mathbf{p}}^{II}]$$

Chain III:

$$\tau_1^{III} = \tilde{\tau}_1^{III} + (\tilde{\mathbf{p}}_1^{III})^T[(\mathbf{S}^{III})^T(\bar{\mathbf{p}}^I + \bar{\mathbf{p}}^{II} + \bar{\mathbf{p}}^{III}) - \bar{\mathbf{p}}^{III}]$$

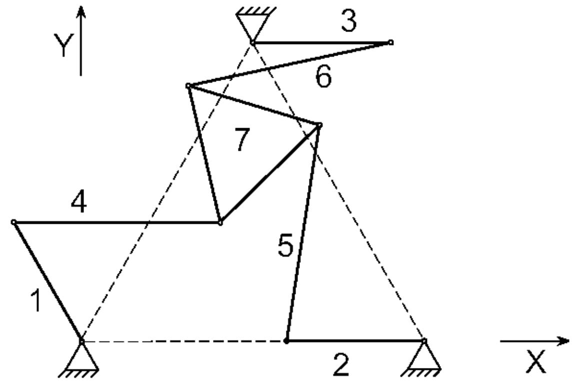

Figure 3. THE THREE-DOF PLANAR PARALLEL MANIPULATOR USED IN THE EXAMPLE

EXAMPLE

Parameters and Initial Conditions

We use the same parameters for the 3R planar Stewart-Gough platform shown in Fig. 3 as in Ma and Angeles [22]. The end effector, labelled 7, has the shape of an equilateral triangle, with sides of length l_7, links 1,2 and 3 have a length l_1, links 4, 5 and 6 have length l_4, and the three fixed revolute joints form an equilateral triangle with sides of length l_0. The prescribed motion drivers given by Ma and Angeles [22] were:

$$\begin{aligned} \theta_1 &= \frac{1}{3}\pi + \frac{1}{6}(\frac{2\pi t}{T} - \sin\frac{2\pi t}{T}) \\ \theta_2 &= \frac{4}{3}\pi + \frac{1}{6}(\frac{2\pi t}{T} - \sin\frac{2\pi t}{T}) \\ \theta_3 &= \frac{11}{6}\pi + \frac{1}{12}(\frac{2\pi t}{T} - \sin\frac{2\pi t}{T}) \end{aligned}$$

where $T = 3$-sec. However, since these initial conditions are not sufficient to define a unique initial posture of the manipulator, we use the initial configuration given by Geike and McPhee [24]:

$\theta_1 = \frac{1}{3}\pi$ rad	$\theta_4 = -0.865$ rad	$x_7 = 0.728$ m
$\theta_2 = \frac{4}{3}\pi$ rad	$\theta_5 = -2.102$ rad	$y_7 = 0.233$ m
$\theta_3 = \frac{11}{6}\pi$ rad	$\theta_6 = -0.976$ rad	$\theta_7 = 3.916$ rad

The parameters of the manipulator are given in Table 1, gravity acts in the $-Y$ direction.

Results

We first perform the inverse dynamics in order to compute a time history of actuation forces that would realize the prescribed motions. Using the above parameters, the resulting torques for the actuated joints 1, 2 and 3 were evaluated using the inverse dynamics model discussed in this paper. The resulting set of

Link i	L_i(m)	m_i(kg)	I_i(Kg m2)
1,2,3	0.4	3.0	0.04
4,5,6	0.6	4.0	0.12
7	0.4	8.0	0.0817

Table 1. DIMENSION AND INERTIA PROPERTIES

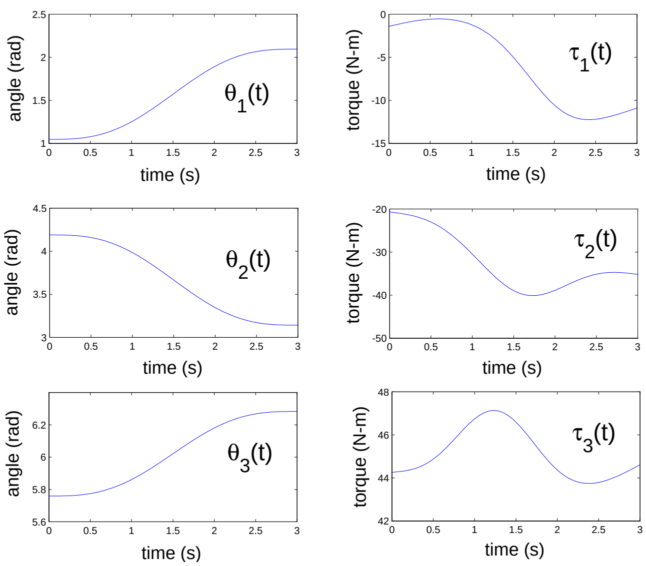

Figure 4. DESIRED TRAJECTORY AND REQUIRED DRIVING TORQUES

torques which realize the prescribed motions is shown in Fig. 4, which tally with those given by Ma and Angeles [22].

CONCLUSION

A modular and recursive inverse dynamics algorithm based on decoupled natural orthogonal complements is presented. The modularity is achieved by projecting the set of sub-model dynamics onto the space of feasible motions. The decoupled natural orthogonal complement is used for this projection. The recursive computation of DeNOC thus leads to the recursive nature of the suggested algorithm.

REFERENCES

[1] Ascher, U. and Petzold, L., 1998. *Computer Methods for Ordinary Differential Equations and Differential-Algebraic Equations*. SIAM, Philadelphia.

[2] García de Jalón, J. and Bayo, E., 1994. *Kinematic and Dynamic Simulation of Multibody Systems: The Real-Time Challenge*. Springer-Verlag, New York.

[3] Haug, E., 1989. *Computer Aided Kinematics and Dynamics of Mechanical Systems*. Allyn and Bacon, Boston.

[4] Schiehlen, W., 1990. *Multibody Systems Handbook*. Springer-Verlag, Berlin.

[5] Shabana, A. A., 2001. *Computational Dynamics*. Wiley, New York.

[6] Featherstone, R., 1987. *Robot Dynamics Algorithms*. Kluwer Academic Publishers, Boston-Dordrecht-Lancaster.

[7] Kecskemethy, A., Krupp, T., and Hiller, M., 1997, "Symbolic Processing of Multi-loop Mechanism Dynamics Using Closed Form Kinematic Solutions," Multibody System Dynamics, **1**(1), pp. 23–45.

[8] Murray, R., Li, Z., and Sastry, S., 1994. *A Mathematical Introduction to Robotic Manipulation*. CRC Press, Boca Raton, FL.

[9] Wang, J., Gosselin, C., and Cheng, L., 2000, "Dynamic Modelling and Simulation of Parallel Mechanisms Using Virtual Spring Approach," *Proc. 2000 ASME Design Engineering Technical Conferences*, Baltimore, Maryland, pp. 1–10.

[10] Stepanenko, Y. and Vukobratovic, M., 1976, "Dynamics of Articulated Open-Chain Active Mechanism," Math. Biosciences, **28**, pp. 137–170.

[11] Orin, D., McGhee, R., Vukobratovic, M., and Hartoch, G., 1979, "Kinematic and Kinetic Analysis of Open-chain Linkages Utilizing Newton-Euler Methods," Mathematical Biosciences, **43**, pp. 107–130.

[12] Luh, J., Walker, M., and Paul, R., 1980, "On-line Computational Schemes for Mechanical Manipulators," ASME Trans. J. Dynamics Systems, Measurement and Control, **102**(2), pp. 69–76.

[13] Balafoutis, C., Patel, R., and Cloutier, B., 1988, "Efficient Modelling and Computation of Manipulator Dynamics Using Orthogonal Cartesian Tensors," IEEE Journal of Robotics and Automation, **4**, pp. 665–676.

[14] Goldenberg, A. and He, X., 1989, "An Algorithm for Efficient Computation of Dynamics of Robotic Manipulators," *Proc. Fourth International Conference on Advanced Robotics*, Columbus, OH, pp. 175–188.

[15] Angeles, J. and Lee, S., 1988, "The Formulation of Dynamical Equations of Holonomic Mechanical Systems Using a Natural Orthogonal Complement," ASME Journal of Applied Mechanics, **55**, pp. 243–244.

[16] Saha, S. K., 1999, "Dynamics of Serial Multibody Systems Using the Decoupled Natural Orthogonal Complement Matrices," ASME Journal of Applied Mechanics, **66**, pp. 986–996.

[17] Saha, S. K. and Schiehlen, W. O., 2001, "Recursive Kinematics and Dynamics for Parallel Structured Closed-loop

Multibody Systems," Mechanics of Structures and Machines, An International Journal, **29**(2), pp. 143–175.

[18] Angeles, J., 2002. *Fundamentals of Robotic Mechanical Systems*. Springer-Verlag, New York, 2 edition.

[19] Saha, S. K., 1997, "A Decomposition of the Manipulator Inertia Matrix," IEEE Trans. Robotics and Automation, **13** (2), pp. 301–304.

[20] Merlet, J.-P., 2000. *Parallel Robots*. Kluwer Academic Publishers, Dordrecht.

[21] Yiu, Y., Cheng, H., Xiong, Z., Liu, G., and Li, Z., May 2001, "On the Dynamics of Parallel Manipulator," *Proc. IEEE international Conference on Robotics and Automation*, Seoul, Korea, pp. 3766–3771.

[22] Ma, O. and Angeles, J., 1989, "Direct kinematics and dynamics of a planar 3-dof parallel manipulator," *Advances in Design Automation*, Montreal, Quebec, Vol. 3, pp. 313–320.

[23] Gosselin, C. and Angeles, J., 1990, "Singularity Analysis of Closed-Loop Kinematic Chains," IEEE Transactions on Robotics and Automation, **6**(3), pp. 281–290.

[24] Geike, T. and McPhee, J., 2002, "On the Automatic Generation of Inverse Dynamic Solutions for Parallel Manipulators," *Proc. Workshop on Fundamental Issues and Future Research Directions for Parallel Mechanisms and Manipulators*, Quebec city, pp. 348–358.