INTRODUCTION TO CENTRAL FORCE FIELDS AND CONIC SECTIONS

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OBJECTIVE

- Historical Sketch
  - Ancient Time to 1900*s*
- Central Force Field
  - Kepler's Laws $\Longrightarrow$ Newton's Law of Gravitation
  - Determination of the Orbit from the Law of Force
- Conic Sections

Celestial Mechanics: Celestial mechanics is concerned with the motions and gravitational effects of celestial objects. The field applies principles of physics, historically Newtonian mechanics, to objects in astronomical space.

- Celestial Mechanics is divided into three parts.
- *Mathematical Celestial Mechanics*: aimed primarily at demonstrating existence of solutions.
- *Physical Celestial Mechanics*: aimed primarily at determining astronomical and geodetic constants.

  Result is important; the formulation used to obtain it is not.
- *Astrodynamics*: aimed at developing effective ways to determine orbits, improve them or physically change them.
  - Mathematical Celestial Mechanics + Physical Celestial Mechanics

- Although modern analytic celestial mechanics starts 400 years ago with Isaac Newton, however, the authentic history of Astronomy actually begins with the Greeks, perhaps 3000 years ago.
- Thales (635-543 B.C.): Thales was the first prominent Greek astronomer and best known for predicting a solar eclipse that occurred in the year 585 B.C.
- Pythagoras (569-475 B.C.): Pythagoreans believed that the earth itself was in motion and that the laws of nature could be derived from pure mathematics.
- Aristarchus (310-230 B.C.): Among the one of the earlier astronomers to propose the heliocentric model of the solar system.
- Ptolemy (100-170 A.D.): Ptolemy, carried forward the work of Hipparchus and published a book called the *Almagest*. He is mainly famous for the system of eccentrics and epicycles which he developed to explain the apparent motions of the planets.

- Copernicus (1473-1543): Copernicus is considered as the founder of modern astronomy. He published his masterpiece *De Revolutionibus*, in 1543, in which he presented the *heliocentric theory* of the solar system.
- Kepler (1571-1630)): Assistant of *Tycho Brahe* and announce the *three laws of planetary motion* based upon observations made by Tycho Brahe.
- Galileo (1564-1642): a contemporary of Kepler and like Kepler, was an ardent supporter of the heliocentric theory. Galileo's observations with his new telescope convinced him of the truth of Copernicus's heliocentric theory.
- Aryabhata (476 - 550): an Indian astronomers is believed to propound the Heliocentric theory of gravitation, thus predating *Copernicus by almost 1000 years*. In his book, the *Aryabhatiya*, he provided the value for the length of the year to be *365 days 6 hours 12 minutes 30 seconds* which is remarkably close to the true value: *365 days 6 hours*.

Pre – 1700s **1700s**

$$\int \left\{ \begin{array}{c} Kepler \\ Copernicus \\ Galileo \\ et\ al \end{array} \right\} dt = \left\{ \begin{array}{l} Newton \left\{ \begin{array}{c} calculus\ \ \&\ differential\ equations \\ law\ of\ universal\ gravitation \\ laws\ of\ dynamical\ motion \\ celestial\ mechanics \end{array} \right\} \\ Euler \left\{ \begin{array}{c} variational\ calculus\ \ \&\ PDEs \\ rigid\ body\ dynamics \\ fluid\ mechanics \\ celestial\ mechanics \end{array} \right\} \\ Gauss \left\{ \begin{array}{c} probability\ theory \\ rigid\ body\ fluid\ \&\ dyn \\ systems\ \ of\ \ eqns \\ celestial\ mechanics \end{array} \right\} \end{array} \right.$$

Historical Sketch: 1800s

$$Jacobi\begin{cases} variational\ calculus \\ rigid\ body\ dynamics \\ special\ fcts\ \&\ PDEs \\ celestial\ mechanics \end{cases}$$

$$Laplace\begin{cases} special\ fcts\ \&\ PDEs \\ Laplace\ transform \\ potential\ theory \\ celestial\ mechanics \end{cases}$$

$$Lagrange\begin{cases} variational\ calculus \\ variational\ mechanics \\ generalized\ mechanics \\ celestial\ mechanics \end{cases}$$

$$Hamilton\begin{cases} variational\ calculus \\ canonical\ \ eqs\ of\ mechanics \\ quaternions\,\&\,rotational\ dyn \\ celestial\ mechanics \end{cases}$$

$$\text{Gibbs}\begin{Bmatrix} \text{vector analysis} \\ \text{matrix analysis} \\ \text{fluid mech \& thermodynamics} \\ \text{celestial mechanics} \end{Bmatrix}$$

$$\text{Cayley}\begin{Bmatrix} \text{matrix analysis} \\ \text{differential equations} \\ \text{linear algebra} \\ \text{celestial mechanics} \end{Bmatrix}$$

$$\text{Heaviside}\begin{Bmatrix} \text{Laplace transforms} \\ \text{vector analysis} \\ \text{differential equations} \\ \text{circuit analysis} \end{Bmatrix}$$

$$\text{Einstein}\begin{Bmatrix} \text{quantum mechanics} \\ \text{general relativity} \\ \text{special relativity} \\ \text{modern physics} \end{Bmatrix}$$

Since 1900

- Substantial Progress along Many Directions:
  - Relativity Theory/Non-Newtonial Effects (Einstein, et al)
  - Integration of Astrodynamics and Control Theory
  - Numerical methods Estimation Theory & Algorithms, e.g., the Kalman Filter
  - Optimization Methods
  - Orbit Transfer Methods/Optimization
  - Orbit Estimation/Navigation
  - Spacecraft Formations/Constellations
- Much of the progress has been associated with exploiting new sensing, computing, and propulsion systems, and generally to accommodate man-made spacecraft, including forces from man-made devices, as opposed to "dealing only with natural forces" & Earth-based measurements, prior to the space age.
- There have been many individuals contributing (un-named here for brevity), rather than the "double handful" of giant contributors who laid the foundations of Celestial Mechanics prior to 1900.

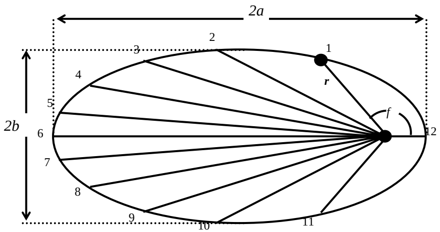
First Law: The orbit of each planet is an ellipse with Sun at one focus.

Second Law: The straight line joining a planet to the Sun sweeps over equal areas in equal interval of time. This is also known as 'Law of Areas'.

Third Law: The squares of the periods (T) of the planets are proportional to the cube of their mean distance (a) from the Sun.

$$T^2 = \left(\frac{2\pi}{k}\right)^2 a^3 \tag{1}$$

The fact remained that Kepler discovered these three laws by *trial and error* without using any *physical law*. More than 60 years later, *Newton* provided theoretical justification for these laws. However, Newton used Kepler's Laws in deriving his Universal Law of Gravitation.



- From Kepler's Second Law:

$$\frac{dA}{dt} = c \;(a\ constant) \tag{2}$$

- Using the fact that $A = r^2 \frac{d\theta}{dt}$, it follows that

$$\frac{d}{dt}\left(r^2 \frac{d\theta}{dt}\right) = 0 \tag{3}$$

Hence, the acceleration perpendicular to the radius vector is zero.$\Rightarrow$ acceleration is radial.

- Equations of Motion:

$$\begin{array}{ll} \textit{Cartesian} & \textit{Polar} \\ \frac{d^2x}{dt^2} = -f\frac{x}{r} & \frac{d^2r}{dt^2} - r\left(\frac{d\theta}{dt}\right)^2 = -f \quad (4) \\ \frac{d^2y}{dt^2} = -f\frac{y}{r} & r\frac{d^2\theta}{dt^2} + 2\frac{dr}{dt}\frac{d\theta}{dt} = 0 \quad (5) \end{array}$$

- From, Eq. (5), $r^2\frac{d\theta}{dt} = h, \ a\ constant \Rightarrow \frac{d^2r}{dt^2} = \frac{h^2}{r^3} - f$
- Let $r = \frac{1}{u}$, therefore,

$$\frac{dr}{dt} = -\frac{1}{u^2}\frac{du}{dt} = -\frac{1}{u^2}\frac{du}{d\theta}\frac{d\theta}{dt} = -h\frac{du}{d\theta} \quad (6)$$

$$\frac{d^2r}{dt^2} = -h^2u^2\frac{d^2u}{d\theta^2} \Rightarrow h^2u^2\left(u + \frac{d^2u}{d\theta^2}\right) = f \quad (7)$$

- From Kepler's first law, we have,
$r = \frac{a(1-e^2)}{1+ecos\theta} \Rightarrow u + \frac{d^2u}{d\theta^2} = \frac{1}{a(1-e^2)} \Rightarrow f = \frac{h^2}{a(1-e^2)}\frac{1}{r^2}$
- Thus, acceleration varies inversely as the square of its distance from the Sun.

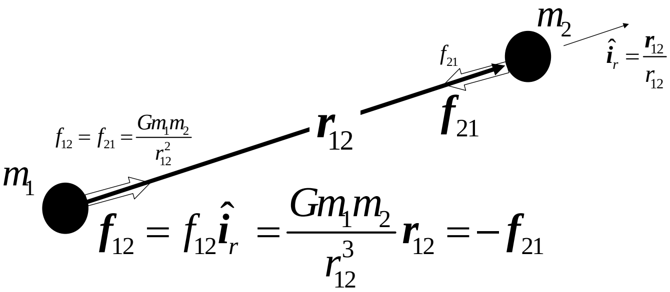
- From Newton's 3^{rd} Law of motion:

$$M_S f_P = M_P f_S \Rightarrow M_S \frac{k_P}{r^2} = M_P \frac{k_S}{r^2} \tag{8}$$

$$\Rightarrow \frac{k_P}{k_S} = \frac{M_P}{M_S} \Rightarrow k_P = G M_P \tag{9}$$

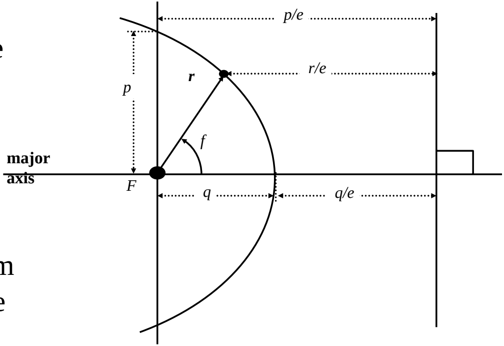
Therefore, $F_{PS} = M_S G \frac{M_P}{r^2}$

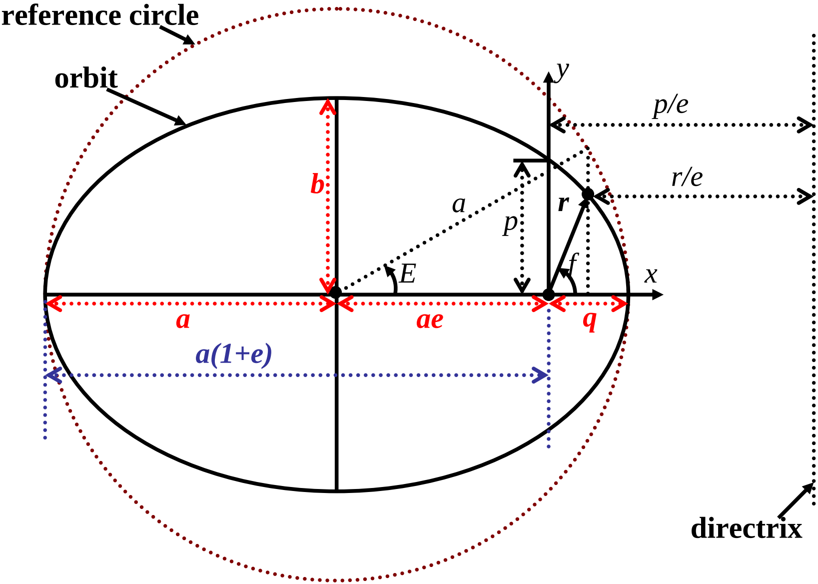
- Which is *Newton's Universal Law of Gravitation.*



Newton conjectured this force law to be consistent with Kepler's laws, his calculus, differential equations, and to make the Earth-Moon dynamics ($m_1 = M_{Earth} \approx 80M_{Moon} = 80m_2$) become consistent with Newton's corrected version of Kepler's Laws.

- The directrix definition of a conic section: The most general conic section is the locus of all points whose distance (r) from a fixed point (the occupied focus F) have a constant ratio (the eccentricity e) to the perpendicular distance from a typical point on the curve to a fixed line (*the directrix*).




reference circle
orbit
y
p/e
r/e
b
a
p
r
f
E
x
a
ae
q
a(1+e)
directrix

Universal	Elliptic Orbit Special Case
$r = \frac{p}{1+e\cos f}$	$r = a(1-e\cos E)$, $p = a(1-e^2)$
$x = r\cos f$	$x = a(\cos E - e)$, $b = a\sqrt{1-e^2}$
$y = r\sin f$	$y = b\sin E$, $\tan\frac{f}{2} = \sqrt{\frac{1+e}{1-e}}\tan\frac{E}{2}$

This is a small but important subset of an incredible number of elegant geometrical relationships. You will require skill in doing geometry of conic sections. These skills themselves are rather "universal", so the investment is worth the effort.

References

1. S. Herrick, "Astrodynamics," Vol. 1, *Van Norstrand Reinhold Co.*, 1972.
2. *http://en.wikipedia.org/wiki/*