Finite-Size Facility Placement in the Presence of Barriers to Rectilinear Travel

by

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Second Revision: August 2001
Revised: April 2001
Submitted: January 2000

Abstract

We consider the placement (location and orientation) of a single finite-size (finite-area, arbitrary shape) facility in the plane under the assumption that all travel occurs according to the rectilinear (or Manhattan) metric in the presence of impenetrable barriers to travel. Facility users are distributed over a finite set of demand points. The facility serves the users via a service point (server) located on the boundary of the facility. We consider an *interactive* model in the sense that there is interaction between not only the facility and the users, but also among the users themselves. We identify the candidates for optimal placement(s) for a facility with a fixed orientation and then for a facility with a fixed server location. Finally, we present a heuristic for the solution of the general problem, when the location and orientation are both unknown.

Keywords: Facility Location, Restricted (Constrained) Location Problems, Barriers to Travel.

1 Introduction

Larson and Sadiq (1983) study the problem of optimally locating p facilities in the plane under the assumption that all travel occurs according to the rectilinear metric in the presence of impenetrable barriers to travel. In their study, Larson and Sadiq assume that all p facilities are infinitesimal. This is a general assumption in location theory. The infinitesimal facility assumption is valid when the physical aspects (i.e., area and dimensions) of the facilities to be located are negligible with respect to the physical aspects of the users of these facilities and the planar area in which the location problem is analyzed. For example, consider the problem of locating an emergency facility in a town. The area required for such a facility will be negligible with respect to the area of the town. However, consider the location of a new department in a shop floor where many other departments are laid out. The area and dimensions of this new department may not be negligible with respect to those of the existing departments. Therefore, in such a problem, the application of a location model with infinitesimal facility assumption will result in false representations of travel distance, since the new department itself creates a barrier to travel. With this motivation, we examine the optimal location of a *finite-size* facility in two-dimensional Euclidean space having fixed barriers to travel, under the assumption that all travel occurs according to the rectilinear (Manhattan, right-angle) metric.

Facility location problems in the plane have been among the most widely studied and used tools in modeling real world problems. Many *restricted* location problems have also been studied. In these types of problems, restrictions correspond to regions in which placement of new facilities is forbidden. Hamacher and Nickel (1995) provide a recent overview of restricted planar location problems and applications. For specific work in this area the reader is referred to books and papers by Hamacher (1995), Nickel (1995), and Hamacher and Schobel (1997). A related but different problem is the location problem with obstacles in which travel barriers are considered. Larson and Sadiq (1993) consider the optimal location of p facilities in the plane under the rectilinear metric assumption in the presence of impenetrable barriers to travel. They show that an optimal set of facility locations can be drawn from a finite set of candidate points. Their analysis is facilitated by the results of Larson and Li (1981) who present an algorithm for finding minimal distance feasible paths between points with polygonal barriers to travel. Dearing, Hamacher, and Klamroth (1998) present the concept of a finite dominating set for the 1-center problem in the presence of polyhedral, convex barriers. Nandikonda, Batta and Nagi (2001) consider the weighted 1-center problem with "arbitrary" shaped barriers. Katz and Cooper (1981) consider the median problem using Euclidean distance and a barrier consisting of one circle. Batta *et al.* (1989) consider the p-median problem in the presence of barriers and convex forbidden regions. In their study, a forbidden region is a region where travel is permitted but facility location is prohibited. They establish that the search for optimal solution can be restricted to a finite set of points. Aneja and Parlar (1994) describe algorithms for optimal single facility location problems with forbidden regions (in this case, travel permitted, location forbidden) and barriers to travel. Their algorithms are valid for the l_p distance metric where $1 < p \leq 2$. A sample analysis of facility location in the presence of forbidden regions under the Euclidean-distance metric is by Butt and Cavalier (1996). In their analysis, a forbidden region is an area where neither travel nor facility location is permitted. They assume convex forbidden regions and provide a solution procedure.

In all the studies referenced above, the new facilities to be located are infinitesimal. In this paper, we study the *finite-size* facility placement problem in two-dimensional Euclidean space having fixed barriers to travel, under the rectilinear metric. A barrier is an area where neither travel nor placement is permitted. Since we are no longer dealing with a single point we use the term "facility placement" rather than "facility location". The rest of this paper is organized as follows. In the next section, we will define our problem. In §3.1 and §3.2, we present preliminaries from [10] and [11] and recast some of their results to help facilitate our analysis. Section 3.3 considers the facility placement problem when the facility orientation is known *a priori*. Then in §3.4, we study the facility placement problem when the server point location is fixed and the facility orientation is the variable. Finally in §3.5 we discuss a heuristic method for the general problem where both the server location and the facility orientation are variables.

2 Problem Definition

There exists a finite number of ***barriers*** where neither travel nor placement is permitted. The existing ***users*** are distributed over a finite set of demand and/or supply points located anywhere in the plane outside the barriers. A new ***facility*** of finite-size is to be placed. The new facility communicates with the users through a single ***server*** located on the facility boundary. The server acts as a demand and/or supply point like all users. The finite-size facility placement problem presents four complications with respect to the infinitesimal facility (point) location problem.

- Since the facility is a finite-size entity, we need to know the *orientation* of the facility in addition to its *location* (coordinates of its server). The location and the orientation determines the *placement* of the facility in the two-dimensional plane.

- As a finite-size entity, the facility may act as a barrier to travel between the users and the server.

- The facility may also increase the travel distances between users. Therefore, we must construct an *interactive* model where there is interaction not only between the users and the new facility (user-server interaction) but also among the users themselves (user-user interaction).

- It is a difficult task to determine the set of feasible placements for the facility (the facility cannot overlap with the existing arbitrary shaped barriers). This is because a feasible location region corresponds to a unique orientation, and infinitely many orientations can be conceived.

Then, the ***finite-size facility placement problem*** is to find the optimal placement(s) for a finite-size facility such that the facility does not overlap with any of the existing barriers, and the sum of user-server and user-user interaction is minimized.

We assume that each barrier is a closed region in $\Re^2$, with finite area and a continuous closed boundary. See Figure 1. We assume that a barrier has a finite number of horizontal and vertical tangential lines. Let B_j (an open set) denote the set of points $(x, y) \in \Re^2$ contained strictly within barrier j. We also define $\overline{B_j} = B_j \cup \{\text{boundary of barrier } j\}$, a closed set. We let $B = \cup_j B_j$ and $\overline{B} = \cup_j \overline{B_j}$. Let H denote the set of points contained strictly within the facility and let $\overline{H} = H \cup \{\text{boundary of the facility}\}$. The distinction between the inside and the boundary of a barrier/facility is necessary to permit travel on the boundary but not on the inside. Let $E(\overline{B})$ define the smallest rectangle (*bounding rectangle*) that encloses all barriers and users and whose sides are parallel to the x and y axes.

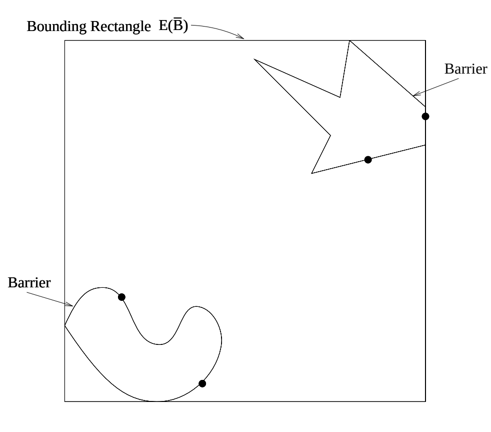

Figure 1: Barriers and the bounding rectangle

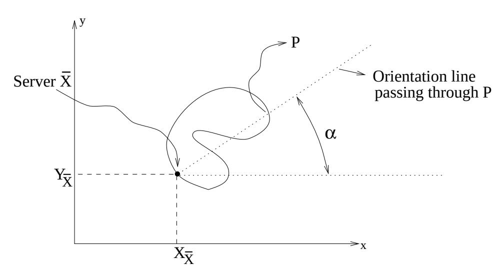

Figure 2: The facility, its location and orientation

In the case of a finite-size facility, the coordinates of a single point cannot convey full information on the placement of the facility on the plane. To this end we let $\ell = [\bar{X}, \alpha]$ denote the *location-orientation vector* for the facility. See Figure 2. Here, $\bar{X} = (X_{\bar{X}}, Y_{\bar{X}})$ represents the location, i.e., point coordinates for the server. The angle $0 \leq \alpha < 2\pi$ between the $+x$-axis and the line joining the server location and a predetermined point P on the boundary of the facility specifies the orientation of the facility. In summary, $\bar{X}$ is the location and α is the orientation of the facility. Together they determine the placement of the facility.

We will now define the feasible region for the finite-size facility placement problem. Let $H(\bar{X}, \alpha)$ (an open set) denote the set of points that correspond to the facility when the server is at $\bar{X}$ and has an orientation α. We also define $\bar{H}(\bar{X}, \alpha) = H(\bar{X}, \alpha) \cup \{\text{boundary of the facility}\}$, a closed set. The feasible region is defined as follows:

$$F = \{[\bar{X}, \alpha] : \bar{H}(\bar{X}, \alpha) \cap B = \emptyset\}.$$

We define $F_{\alpha_0} \subseteq F$ as the feasible region for the facility with a fixed orientation α_0 (see Figure 3). Further, we let $F_{\bar{X}_0} \subseteq F$ denote the feasible region for the facility with a fixed server location $\bar{X}_0$. Clearly, $F = \bigcup_\alpha F_\alpha = \bigcup_{\bar{X}} F_{\bar{X}}$. We note that for an infinitesimal facility, the optimal location is guaranteed to be inside the bounding rectangle. However for a finite-size facility this is no longer true and, depending on the problem specifications, an outside boundary needs to be enforced for the overall problem.

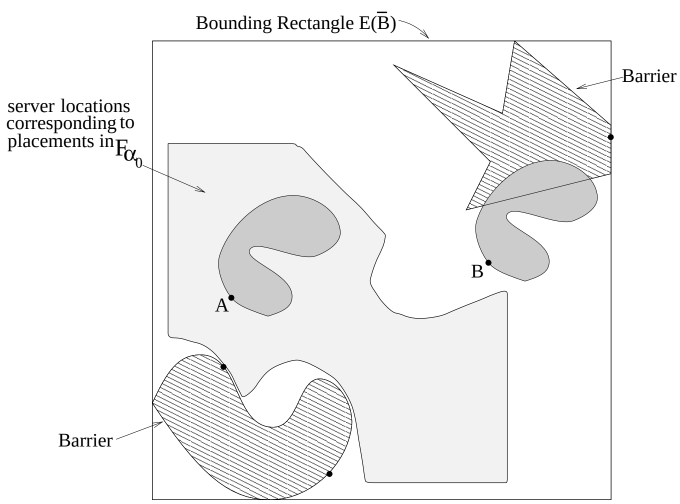

Figure 3: Definition of F_{α_0}

There are two types of interactions in our problem. Firstly, there is interaction between the users and the server of the new facility. The extent of interaction between user i and server $\bar{X}$

is denoted by $u_i \geq 0$. Secondly, there is interaction between all pairs of users. The interaction that takes place from user i to user j is denoted by $w_{ij} \geq 0$ (note: we do not assume $w_{ij} = w_{ji}$). The interaction between any two points takes place through a shortest feasible (penetrating no barriers or facility) rectilinear distance path. Let $d_\ell(i, j)$ represent the length of such a shortest path between two users i and j when the facility placement is ℓ. The subscript ℓ signifies that the distance is a function of the facility placement. Similarly, $d_\ell(i, \bar{X})$ represents the minimum feasible rectilinear distance between user i and the server at $\bar{X}$ when the facility placement is ℓ.

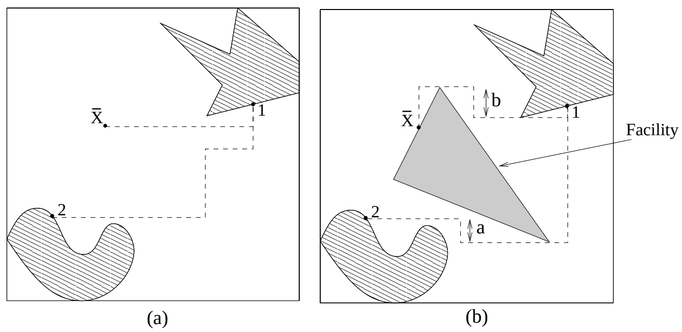

Figure 4: Illustration to show how the facility size and shape affect distances

In Figure 4, we demonstrate how a finite-size facility affects shortest paths. Consider a point facility located at $\bar{X}$ and users 1 and 2 in Figure 4.a. Examples of shortest paths between the facility and user 1, and users 1 and 2 are shown. If the facility has finite-size, then with the given facility placement, the shortest path distance between users 1 and 2 is increased by $2a$. On the other hand, the facility acts as a barrier between its server and user 1 and therefore the shortest path distance between $\bar{X}$ and user 1 is increased by $2b$.

Let D denote the set of all users. For a given facility placement $\ell = [\bar{X}, \alpha]$, the total weighted travel distance between users and the facility (user-server interaction) is:

$$J(\ell) = \sum_{j \in D} u_j \; d_\ell(j, \bar{X})$$

Similarly, the total weighted travel distance between all users (user-user interaction) is:

$$K(\ell) = \sum_{i \in D} \sum_{j \in D} w_{ij} \; d_\ell(i, j)$$

The problem is to find $\ell^* \in F$ such that $J(\ell^*) + K(\ell^*) \leq J(\ell) + K(\ell), \; \forall \ell \in F$.

3 Determining Candidates for Optimal Facility Placement

3.1 Preliminaries

In order to develop our analysis, we need to present a few definitions and results from both Larson and Li [10] and Larson and Sadiq [11]. In §3.2, we develop an adaptation of one of their results which will be critical to our approach and we also discuss our approach in further detail.

Each barrier is characterized by a set of *barrier vertices* (assumed to be finite in number). The barrier vertices are points of tangency, i.e., points in the boundary through which one can pass a horizontal or vertical line segment and for which all boundary points in the neighborhood of this point either lie in or on one side of the horizontal or vertical line. See Figure 6. For segments of the barrier boundary that are horizontal or vertical and whose straight line extensions are fully contained either in the barrier or in $\Re^2 - \bar{B}$, one has a line segment of tangency, both endpoints of which are included as barrier vertices (e.g., points 1 and 2 in Figure 6). The set of *fixed nodes* N for any particular problem is the union of the set of users D and the set of all barrier vertices V. We note here that certain fixed nodes are eliminated from consideration in the grid formation procedure discussed later in this section.

A *stair-case* path between two points (X_i, Y_i) and (X_j, Y_j) is a rectilinear path having length $|X_i - X_j| + |Y_i - Y_j|$. Two points are said to *communicate* if there is at least one feasible stair-case path between them, i.e., a path in $\Re^2 - B$ that is not made longer by the barriers. A nonstaircase path contains at least one *turning step*, i.e., a path step directly connecting two horizontal steps or two vertical steps whose directions of travel are reversed. Hence, the turning step, together with its immediate predecessor and successor steps, form a local "U" shape in the travel path.

A vertex seeking tree (VST), rooted at point $(x, y) \in \Re^2 - B$, is the union of four probes emanating from (x, y), each probe initially following a distinct one of the four feasible travel directions from (x, y). A probe can proceed in a straight line and terminate at a fixed node $i \in N$; or, if it intersects no fixed nodes or barriers, it can proceed to infinity (i.e., become a ray); or, if it intersects a barrier boundary, it then proceeds along the barrier boundary in the direction created by the probe's obtuse angle with the boundary, terminating at the first fixed node encountered. In the last case, if the angle of intersection is a right angle, then the probe splits and follows the barrier boundary along both directions, each segment terminating either at the first fixed node encountered or at the first point in which a straight line extension of the intersected boundary segment enters the barrier, whichever comes first. See Figure 5 for an illustrative example. In this example the probe P intersects the barrier BR at right angles at point c. It then splits following the barrier boundary along both directions. The upward split terminates at barrier vertex a. The downward split terminates at point b, since at this point a straight line extension of the intersected boundary segment enters barrier BR. We note that any point (u, v) on the VST rooted at (x, y) can be reached from (x, y) with travel distance $|x - u| + |y - v|$. We also note that the terminal points of a vertex seeking tree (if one exists) is a point where a probe terminates. Points a and b in Figure 5 are terminal points of the probe P that emanates from point d, and hence become terminal points of the vertex seeking tree that emanates from point d.

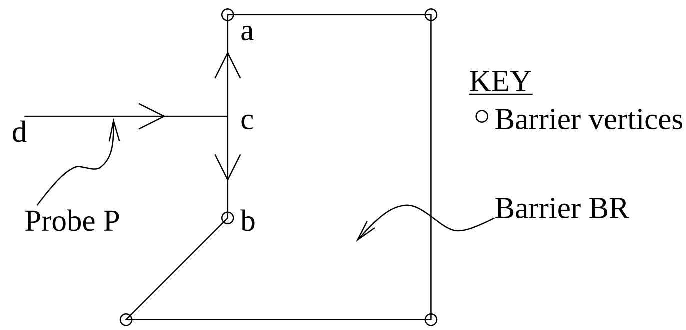

Figure 5: Example of a split probe that follows the barrier boundary

Two points (X_i, Y_i) and (X_j, Y_j) *communicate simply* if they satisfy any of the following three criteria: (i) If (X_i, Y_i) and (X_j, Y_j) are adjacent vertices of a barrier (adjacency determined by clockwise or counter-clockwise ordering of vertices along the boundary); (ii) If (X_i, Y_i) is the root and (X_j, Y_j) is a terminal point of a vertex seeking tree; (iii) If (X_i, Y_i) has an x-directed probe that shares at least one point in common with a y-directed probe from (X_j, Y_j), where the common point must be other than $(X_k, Y_k), k \neq i, j$, i.e., a fixed node.

Path-Push and Amalgamation is a procedure that operates on any staircase path between communicating nodes a and b to obtain a new equal-length staircase path $a - n_1 - n_2 - \cdots - n_k - b$, where $n_1, n_2, \cdots, n_k \in N$ and $(a, n_1), (n_1, n_2), \cdots, (n_k, b)$ are pairs of simply communicating nodes. The new path is called a *nodal path*. Finally, we can state an important result (see Theorem 2 of Larson and Li (1981)) for our purposes as follows:

Result 1: A shortest path in $\Re^2$ between any (X_a, Y_a) and (X_b, Y_b), $a, b \in N$ can be found by restricting travel to nodal paths, i.e., paths containing a sequence of nodes $a - n_1 - n_2 - \cdots - n_k - b$, where $(a, n_1), (n_1, n_2), \cdots, (n_k, b)$ are pairs of simply communicating nodes.

This is equivalent to restricting travel to a network $G(N, A)$, where N is the set of fixed nodes and the entry (i, j) of the arc set A has length $|X_i - X_j| + |Y_i - Y_j|$ if nodes i and j communicate simply and length $+\infty$ otherwise. A path in G corresponds to a family of equal length paths in $\Re^2$, the only restriction being the sequence of nodes to visit, with travel between each node pair accomplished in a staircase manner. It should be clear that the traversal of any such arc (i, j) corresponds to x and y travel along one of a (usually) noncountably infinite number of staircase paths between the two simply communicating nodes i and j. Thus, specifying a path on the network corresponds to specifying a unique sequence of nodes to visit, but a nonunique path in $\Re^2$.

A *grid* is formed within the bounding rectangle as follows: (i) Pass lines parallel to the x and y axes through all fixed nodes N, with each line terminated at the first barrier interior (i.e., point in B) encountered; (ii) Exclude from the set of lines in (i) any line extending from a barrier vertex that is not also a user, where the vertex is an endpoint of the line. See Figure 6 for an example of the grid formation process.

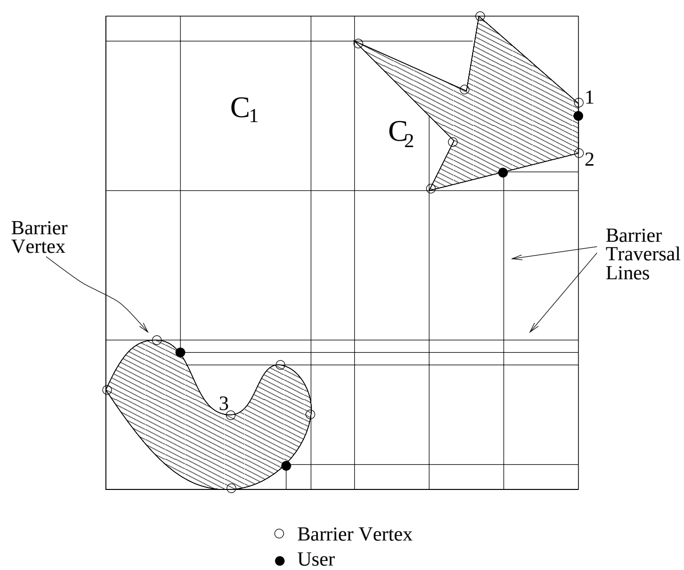

Figure 6: Barrier Traversal Lines, L

An example for an exclusion of a line through a barrier vertex due to rule (ii) is point 3 in Figure 6. We note that these lines are formed due to the barriers and due to users (which are not necessarily related to barriers). Strictly speaking, we should refer to them as barrier-user traversal lines. For simplicity in presentation we refer to the resulting set of lines as simply *barrier traversal lines* (these were called node traversal lines in Larson and Sadiq). We let L_h denote the set of horizontal barrier traversal lines and L_v denote the set of vertical barrier traversal lines. We define $L = L_h \bigcup L_v$ as the set of all barrier traversal lines.

The barriers and L divide $\Re^2 - B$ into a number of *cells*. Each *cell boundary* is composed of segments of barrier boundaries and barrier traversal lines. A cell can either be rectangular or non-rectangular (see C_1 and C_2, respectively in Figure 6). For a given cell C, consider the points $(X_{min}, Y_{min}), (X_{max}, Y_{min}), (X_{max}, Y_{max}), (X_{min}, Y_{max})$, where $X_{min}, Y_{min}, X_{max}, Y_{max}$ are the respective bounds on x and y in the cell. Clearly, at least one of the four points is contained in C; we refer to all such points contained in C, up to a maximum of four, as the *cell corners* of C. Let $E(C)$ denote the smallest enclosing rectangle for any cell C, and also let $E_1(C), E_2(C), E_3(C), E_4(C)$ denote the corners of $E(C)$, starting from the bottom left corner and labeling in the counter-clockwise direction. As a result, the corners of cell C represented earlier in terms of the respective bounds on x and y in the cell can also be represented as $E_1(C), E_2(C), E_3(C), E_4(C)$, respectively.

Consider the nodal path $a - n_1 - n_2 - \cdots - n_k - b$ between two nodes (X_a, Y_a) and (X_b, Y_b), $a, b \in N$. There must exist a shortest path between a and b such that each node in the sequence is a corner of some cell. Due to their formation, cells have useful properties. Let us note three of them which we will refer to later:

Result 2: A shortest feasible rectilinear path from a demand point (user) to an infinitesimal facility located in a cell passes through a corner of the cell (Larson and Sadiq, Lemma 3).

Result 3: It is known that for an infinitesimal point at $\bar{X}$, $J(\bar{X})$ is concave within any cell (Batta *et al.*, Lemma 1).

Result 4: The candidate points for the optimal location of an infinitesimal (point) facility are all user points and all other grid points that represent the intersection of two barrier traversal lines (Larson and Sadiq , p.665).

3.2 The Approach

We know that the shortest path between two users a and b can be found by restricting travel to nodal paths $a - n_1 - n_2 - \cdots - n_k - b$, where each node in the sequence is a corner of some cell. Thus, we have represented the shortest path between a and b in terms of nodes. We can represent the same shortest path in terms of barrier traversal lines. Instead of defining the path with respect to the nodes $a, n_1, \cdots, n_k, b$, we will define the path with respect to a sequence of horizontal barrier traversal lines $h_1, h_2, \cdots, h_m$ and a sequence of vertical barrier traversal lines $v_1, v_2, \cdots, v_n$. In order to accomplish this task, we present the following lemma.

Lemma 1 *A nodal path between two points (X_a, Y_a) and (X_b, Y_b), $a, b \in D$ can also be represented as a* **traversal line path**, *$P(a, b)$, i.e., a path that comes in contact with a sequence of horizontal and vertical barrier traversal lines, $h_1, h_2, \cdots, h_m$ and $v_1, v_2, \cdots, v_n$.*

Proof: Consider the set of nodes $a - n_1 - n_2 - \cdots - n_k - b$ defining a nodal path from a to b. See Figure 7.

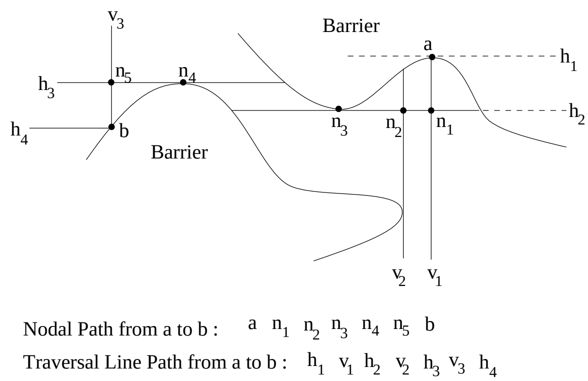

Figure 7: Traversal Line Path

Clearly, each node in the path can be identified by a horizontal line h_i or a vertical line v_j or both lines passing through that node. Hence, each node must correspond to at least one barrier traversal line. In the following sequence of nodes, one node will either be identified by a new horizontal line h_{i+1} or a new vertical line v_{j+1} or both. Also in the sequence, when the path moves from one horizontal line to the next, the y-distance traveled is $|Y_{h_i} - Y_{h_{i+1}}|$ where Y_{h_k} is the ordinate of horizontal line h_k. Similarly, the x-distance traveled between two consecutive vertical lines is $|X_{v_j} - X_{v_{j+1}}|$ where X_{v_k} is the abscissa of vertical line v_k. In this manner, we can identify m horizontal and n vertical barrier traversal lines interspersed in the path. Such a path $P(a, b)$ will be called a traversal line path. $\square$

We note here that the development of nodal paths in Lemma 1 is not necessarily unique. As an example, consider a slightly different barrier shape to that given in Figure 7, where the x-probe emanating from a intersects with the y-probe emanating from n_2. Then the given traversal line path also describes the nodal path $a - n_2 - n_3 - n_4 - n_5 - b$.

From Lemma 1, we can conclude that the length of a traversal line path $P(a, b)$ between two points $a, b \in D$ is $\sum_{i=1}^{m-1} |Y_{h_i} - Y_{h_{i+1}}| + \sum_{j=1}^{n-1} |X_{v_j} - X_{v_{j+1}}|$. We will now set the stage for our analysis of finite-size facility placement and how the existing framework of barrier traversal lines and cells, i.e., the grid structure, is affected once the facility is introduced.

Before the facility is placed, we know that the barrier traversal lines L that were drawn due to the existing barriers and users were adequate to provide the framework that is necessary to obtain the minimum feasible rectilinear distance between any two users. When the facility is placed, the existing framework is altered. More specifically, two things happen. First, the facility itself provides a barrier to travel and therefore a new set of traversal lines parallel to the x and y axes must be passed through the facility vertices (similar to the barrier vertices) and the server on the facility boundary. This new set of lines will be referred to as *facility traversal lines*, L'. See Figure 8. When

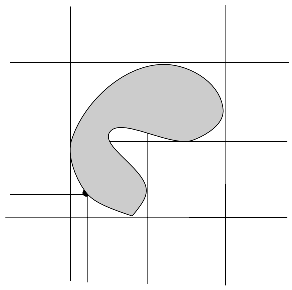

Figure 8: L'

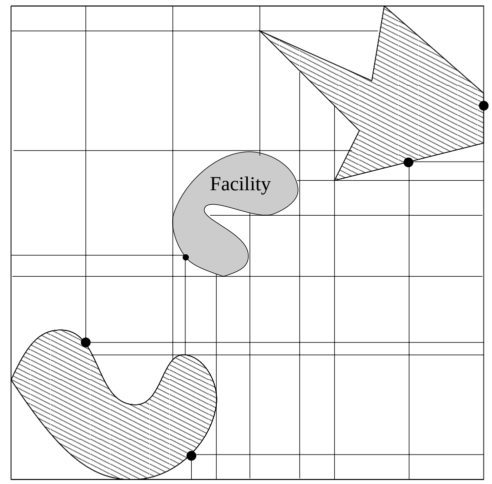

Figure 9: L''

the facility is placed, the facility traversal lines will be drawn with the line termination rule as in (i) and the exclusion rule as in (ii) of the two-step grid formation process given earlier. The second thing that happens due to the placement of the facility is that some existing barrier traversal lines will be cut off since these lines will encounter the facility interior. As a result of these changes, a new set of lines, L'' are formed for the overall problem. See Figure 9. Clearly, the grid structure before and after the facility placement will be different. In fact, for any $\ell \in F$, we may have a different set of lines L'' and a different grid structure. We can say that the existing set of lines L before the facility placement is transformed into a new set of lines L'' after the facility placement:

$$L \xrightarrow{\ell} L''$$

The solution to the overall problem lies in understanding how the distance functions behave as the facility is moved from one placement to a different placement. We know that we can find shortest paths by simply looking at the grid structure. Lemma 1 provides this tool. It is also clear that the results we have summarized and adapted are valid for any given grid structure, i.e., both before and after the facility placement. Therefore our approach is to study the grid structure as a function of the placement of the facility.

We analyze the relationship between the original barrier traversal lines L (Figure 6) and the facility traversal lines L' (Figure 8) as the facility is moved from one placement to another. This is because we know that shortest paths between interacting points in the problem can be identified as a sequence of these lines.

Any facility movement can be a result of two things: Changing the server coordinates and/or changing the facility orientation. Consider the slightest movement of the facility from one feasible placement to another. In the background, the barrier traversal lines will always be fixed, i.e., the coordinate of such a line will not change. However, the coordinates of the facility traversal lines may change as a result of the facility movement. By exploiting the relationship between the set of *fixed* lines and the set of *moving* lines, we will observe how the distances change as a function of the facility movement.

3.3 Optimal Placement Candidates for the Facility with Fixed Orientation

In this section, we are interested in placement of a facility with known orientation α_0. We first study aspects of the problem when the facility does not interfere with any barrier traversal lines in L (§3.3.1). Then in §3.3.2, we study the problem when the facility interferes with at least one barrier traversal line in L. We must note that since the facility orientation is fixed, we are only interested in the server coordinates, hence the feasible region F_{α_0} (which consists of two-tuple vector information consisting of location and orientation) can be represented by only location information (i.e., server coordinates). In that case, we are simply concerned with a two-dimensional area which we will denote by $F_{\alpha_0}^{\bar{X}}$. That is, $F_{\alpha_0}^{\bar{X}} = \{(X_{\bar{X}}, Y_{\bar{X}}) : \ell = [\bar{X}, \alpha_0] \in F_{\alpha_0}\}$. In other words, F_{α_0} is the set of feasible placements and $F_{\alpha_0}^{\bar{X}}$ is the corresponding set of feasible locations.

3.3.1 The facility placement does not cut off any barrier traversal lines

For a feasible placement $[\bar{X}, \alpha_0]$, it is possible that the area occupied by the interior of the facility will not interfere with any barrier traversal lines. That is, for all barrier traversal lines $l_i \in L$, $l_i \cap H = \emptyset$. This also means that the area occupied by the facility is a subset of a cell C, i.e., $\overline{H} \subseteq C$. See Figure 10.

We will refer to such a case as the *containment* of the facility within cell C. The set of such placements within cell C will be denoted by $F_{\alpha_0}(C)$ and the corresponding locations by $F_{\alpha_0}^{\bar{X}}(C)$. We will show that the objective function is concave when $\ell \in F_{\alpha_0}(C)$. Before we present a lemma leading to Theorem 1, let us denote the smallest enclosing rectangle for the facility by $E(\overline{H})$. Similar to what we have defined for a cell C, let us denote the corners of $E(\overline{H})$ by $E_k(\overline{H})$, $k = 1, 2, 3, 4$.

Lemma 2 *A shortest path that passes through $E_k(C)$ and $E_k(\overline{H})$, $k \in \{1, 2, 3, 4\}$, exists from any user $i \in D$ to the server at $\bar{X}$ for all $\ell \in F_{\alpha_0}(C)$.*

Proof: For a non-rectangular cell C, it is clear that $E_k(C) \notin C$ for some k (e.g. $k = 1$ in

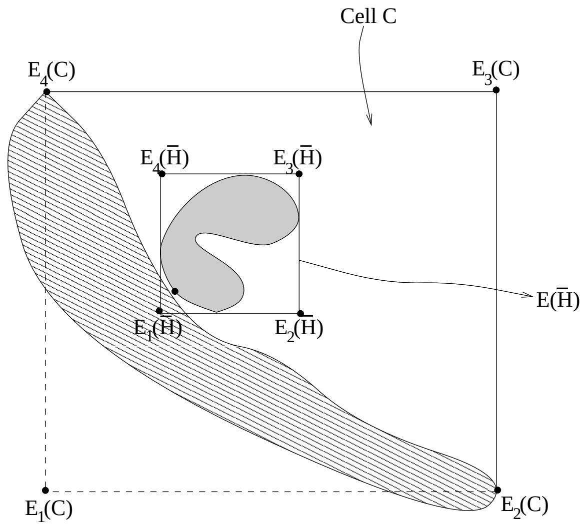

Figure 10: Facility contained within cell C

Figure 10). We will first prove that if a particular corner of the cell is contained in the cell, then the corresponding corner for the facility bounding rectangle $E(\overline{H})$ is also contained in that cell. That is, if $E_k(C) \in C$, then $E_k(\overline{H}) \in C$ for any $k = 1, 2, 3, 4$.

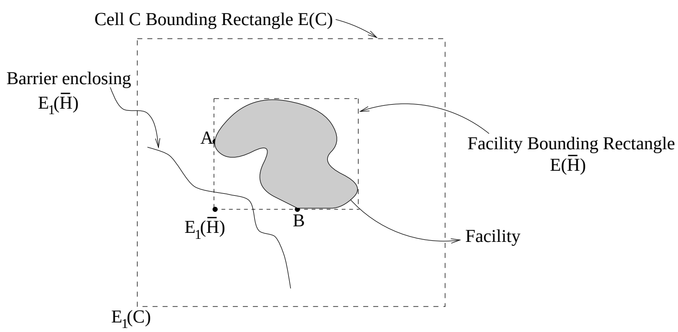

Figure 11: Lemma 2

See Figure 11. Using contradiction, we will illustrate the result for $k = 1$. A similar approach can be used for all corners. By definition, the boundary of the facility bounding rectangle $E(\overline{H})$ intersects with a set of points on the facility boundary. Let A and B denote such points that are closest to $E_1(\overline{H})$. Since the facility is contained in cell C, the points A and B must be in the cell. Now suppose $E_1(C) \in C$, but $E_1(\overline{H}) \notin C$. Any barrier that would create $E_1(\overline{H}) \notin C$ would have to enclose $E_1(\overline{H})$ in a way that destroys the current cell structure and $E_1(C) \in C$ would be a contradiction. Therefore, if $E_k(C) \in C$, then $E_k(\overline{H}) \in C$ for all k.

Now consider a shortest rectilinear path from a user outside cell C to a user placed within cell C. From Result 2 we can conclude that a shortest rectilinear path from a user outside cell C to a corner $E_i(\overline{H})$ in cell C passes through a corner of $E_j(C)$ of C. Also from (repeated applications of) Result 2 we can conclude that a shortest rectilinear path from a user outside $E(\overline{H}) \cap C$ to a point inside the region $E(\overline{H}) \cap C$ passes through cell corner $E_i(\overline{H})$ of $\overline{H}$ that is contained in cell C (e.g., $i = 2, 3, 4$ for the situation shown in Figure 10). More than one application of the result is needed for the situation shown in Figure 12, because the server at a is inside a cell $C_1 \subset E(\overline{H})$.

Combining these observations we conclude that a shortest path to the server from a user outside cell C passes through both a corner $E_k(C)$ and a corner $E_j(\overline{H})$.

Since $E_k(\overline{H})$ is the closest corner to $E_k(C)$ when $\overline{H} \subseteq C$ it follows that such a shortest path to the server must pass through both $E_k(C)$ and $E_k(\overline{H})$. □

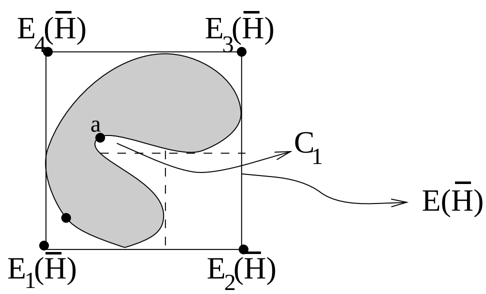

Figure 12: Example for repeated application of Result 2

Theorem 1 *The function $J(\ell) + K(\ell)$ is concave over the set $F_{\alpha_0}(C)$.*

Proof: Consider a location A such that $[A, \alpha_0] \in F_{\alpha_0}(C)$. Further consider some feasible direction of movement from location A, as shown in Figure 13, to a location B, where $[B, \alpha_0] \in F_{\alpha_0}(C)$. What we mean here is that location B is such that every location Δ on the line segment joining A and B has the property $[\Delta, \alpha_0] \in F_{\alpha_0}(C)$. (If no such point B exists then $F_{\alpha_0}(C)$ is a singleton set and the result follows.)

Let the variable δ parametrize the movement along $\overline{AB}$ with $\delta = 0$ signifying point A. Further, let Δ be a point on $\overline{AB}$ which is δ Euclidean distance units from A.

From Lemma 2, we know that the shortest rectilinear path between any user and the server passes through a corner $E_k(C)$ of cell C and corner $E_k(\overline{H})$ of $E(\overline{H})$ ($k = 1, 2, 3, 4$). For the kth corner, the distance between user i and the facility server at Δ is as follows:

$$d_\ell(i, E_k(C)) + d_\ell(E_k(C), E_k(\overline{H})) + d_\ell(E_k(\overline{H}), \Delta)$$

Here, $d_\ell(i, E_k(C))$ and $d_\ell(E_k(\overline{H}), \Delta)$ are constants with respect to δ and it is easy to show that $d_\ell(E_k(C), E_k(\overline{H}))$ is linear in δ. Let $d_\ell(E_k(C), E_k(\overline{H}))|[A, \alpha_0]$ represent the shortest distance be-

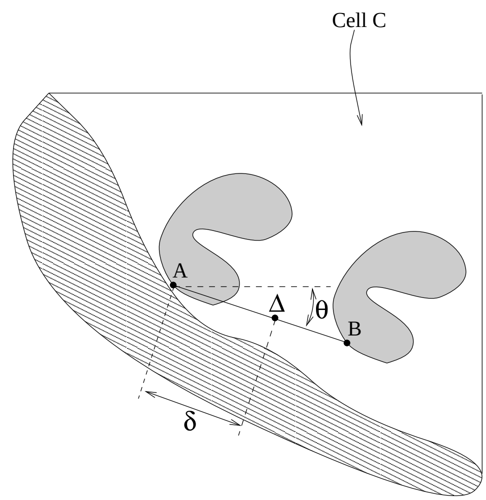

Figure 13: Movement of the facility within cell C

tween cell corner $E_k(C)$ and facility bounding rectangle corner $E_k(\overline{H}))$ when facility location is $\ell = [A, \alpha_0]$. We can then write:

$$d_\ell(E_k(C), E_k(\overline{H}))|[\Delta, \alpha_0] = \begin{cases} d_\ell(E_k(C), E_k(\overline{H}))|[A, \alpha_0] + \delta(+\cos\theta - \sin\theta) & \text{if } k = 1 \\ d_\ell(E_k(C), E_k(\overline{H}))|[A, \alpha_0] + \delta(-\cos\theta - \sin\theta) & \text{if } k = 2 \\ d_\ell(E_k(C), E_k(\overline{H}))|[A, \alpha_0] + \delta(-\cos\theta + \sin\theta) & \text{if } k = 3 \\ d_\ell(E_k(C), E_k(\overline{H}))|[A, \alpha_0] + \delta(+\cos\theta + \sin\theta) & \text{if } k = 4 \end{cases}$$

where θ is as shown in Figure 13.

We know that cell C has at most four corners. Then, $d_\ell(i, \Delta)$ is the minimum of up to four linear functions and is therefore concave. Since $J(\ell) = \sum_i u_i \; d_\ell(i, \Delta)$ is the sum of positively weighted concave functions, it is also concave. Clearly, $K(\ell)$ remains unchanged as long as the facility remains inside cell C. Therefore, $J(\ell) + K(\ell)$ is concave. □

From Theorem 1, it follows that the optimal location lies on the boundary of $F_{\alpha_0}^{\bar{X}}(C)$. For certain situations it is possible to further narrow down the candidates for optimal placement. Consider the case where both the facility $\bar{H}$ and the cell C are rectangular. Then the set $F_{\alpha_0}^{\bar{X}}(C)$ is a rectangle and from Theorem 1 it follows that the optimal placement candidates can be restricted to be at the four corners of this set. See Figure 14 for an example. In this figure, points a, b, c and d are candidate locations when the facility $\bar{X}$ is contained in cell C. Another case might result when $F_{\alpha_0}^{\bar{X}}(C)$ is a convex polygon. The polygon vertices are the candidates for the optimal placement.

3.3.2 The facility placement cuts off at least one barrier traversal line

When the new facility is placed over a number of barrier traversal lines, one of the things that may happen is that the facility may *cut off* (interfere with) the shortest paths between existing users.

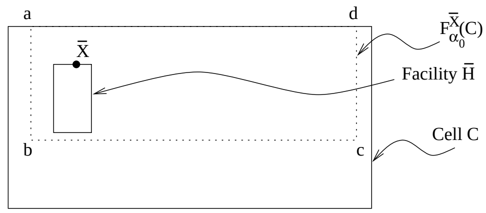

Figure 14: Illustration for identifying candidate locations in the case of a rectangular cell and facility

The interaction between users is represented by weights (w_{ij}'s) associated to each user pair. Higher interaction between users will affect the optimal placement of the facility, causing the placement to avoid cutting off the flow between highly interacting users. Another consequence of the facility cutting off a number of barrier traversal lines is that the shortest feasible rectilinear path from a user to the server $\bar{X}$ may have to travel around the facility. That is, the facility itself may act as a barrier to travel between its server and some user.

Consider a feasible initial placement $\ell_{ini} = [\bar{X}_{ini}, \alpha_0]$ of the facility, such that

- The facility interferes with at least one barrier traversal line, that is, given $\ell_{ini} \in F_{\alpha_0}$, there exists r ($r \geq 1$) barrier traversal lines $l_1, \cdots, l_r \in L$ such that $l_i \cap H \neq \emptyset$, $\forall i \in \{1, \cdots, r\}$, and
- No barrier traversal line coincides with a facility traversal line, that is, $X_{v_j} \neq X_{v'_j}$, $\forall v_j, v'_j$ and $Y_{h_i} \neq Y_{h'_i}$, $\forall h_i, h'_i$.

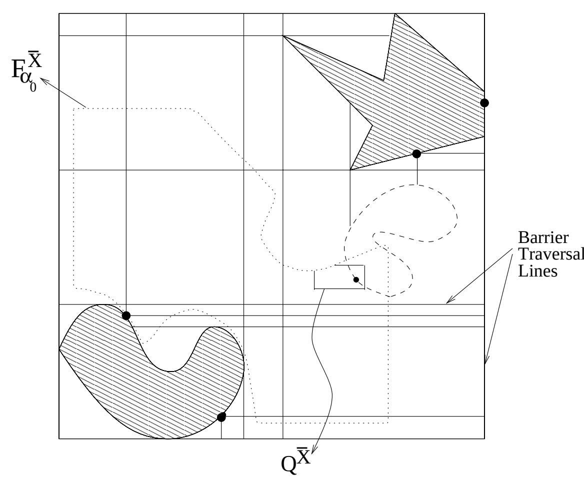

Figure 15: $\mathcal{Q}^{\bar{X}}$

See Figure 15. Now we will sketch a set of placements $\mathcal{Q}$ such that when $\ell \in \mathcal{Q}$, the facility will always interfere with the same set of barrier traversal lines $l_1, \cdots, l_r$. We again note that since the facility orientation is fixed, we are only interested in the server coordinates, hence we define

$\mathcal{Q}^{\bar{X}} = \{(X_{\bar{X}}, Y_{\bar{X}}) : \ell = [\bar{X}, \alpha_0] \in \mathcal{Q}\}$. The area $\mathcal{Q}^{\bar{X}}$ represents the area of facility location and can be constructed by moving the facility in all directions from the initial location $\bar{X}_{ini}$. The area $\mathcal{Q}^{\bar{X}}$ is the set of all locations which can be obtained by moving the facility from an initial location without intersecting any barrier traversal lines other than $l_1, \cdots, l_r$. The boundary of such an area originating from $\bar{X}_{ini}$ can consist of two segments: (a) locations such that facility boundary intersects with some barrier boundary, and (b) locations such that some barrier traversal line(s) coincide with some facility traversal line(s), i.e., there exists some v_j, v_j' such that $X_{v_j} = X_{v_j'}$ and/or there exists some h_i, h_i' such that $Y_{h_i} = Y_{h_i'}$ We will now demonstrate that the objective function is concave when the facility location is in $\mathcal{Q}^{\bar{X}}$ and this will provide clues for determining candidates for optimal facility placement. We state the following theorem:

Theorem 2 *The function $J(\ell) + K(\ell)$ is concave over the set $\mathcal{Q}$.*

Proof: Following the reasoning in Theorem 1, let the variable δ parametrize the movement along a line segment $\overline{AB}$ within $\mathcal{Q}^{\bar{X}}$ where any point on the line represents the facility location and with $\delta = 0$ signifying point A. Further, let Δ be a point on $\overline{AB}$ which is δ Euclidean distance units from A (Figure 16).

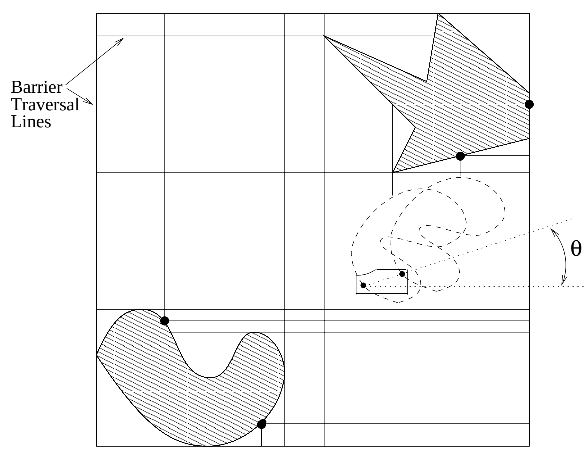

Figure 16: Movement within $\mathcal{Q}^{\bar{X}}$

Consider any traversal line path $P(i,j)$ between two demand/supply points (users or server), $i, j \in D'$, where $D' = D \cup \bar{X}$. As a function of δ, the distance between any two consecutive vertical lines in the path will either decrease or increase linearly or not change at all. This is due to the construction of $\mathcal{Q}^{\bar{X}}$, that is, two parallel lines will not pass over each other, in other words, for any two consecutive vertical lines v_i and v_{i+1} for example, either $X_{v_i} > X_{v_{i+1}}$ or $X_{v_i} < X_{v_{i+1}}$ is always true. More specifically, at Δ, the distance between any two consecutive vertical lines will either decrease or increase by $\delta \cos\theta$ or not change at all. Similarly, if the distance between any two consecutive horizontal lines in the path changes, the change will be linear. More specifically, at Δ, the distance between any two consecutive horizontal lines will either decrease or increase by $\delta \sin\theta$ or not change at all.

Consequently, the total length of $P(i,j)$ will change linearly as a function of δ within $\mathcal{Q}^{\bar{X}}$ or not change at all. The shortest distance path between i and j is the minimum length path among a finite number of such paths. This means that within $\mathcal{Q}^{\bar{X}}$, both $d_\ell(i,\bar{X})$ and $d_\ell(i,j)$ are piecewise linear and concave in δ. Therefore, $J(\ell)$ and $K(\ell)$ are concave (positively weighted sums of $d_\ell(i,\bar{X})$ and $d_\ell(i,j)$, respectively). As a result, the objective $J(\ell)+K(\ell)$ must also be concave for $\ell \in \mathcal{Q}$. $\square$

We will explore the area $\mathcal{Q}^{\bar{X}}$ further. Consider the initial location around which $\mathcal{Q}^{\bar{X}}$ was constructed, i.e., $\ell_{ini} = [\bar{X}_{ini}, \alpha_0]$. Clearly, the facility location must belong to some cell C, i.e., $\bar{X}_{ini} \subseteq C$. In constructing $\mathcal{Q}^{\bar{X}}$, the server cannot pass over any barrier traversal lines. Therefore it is also true that $\mathcal{Q}^{\bar{X}} \subseteq C$. As a result, we can state that $\mathcal{Q}^{\bar{X}}$ is simply a sub-cell and the facility interferes with the same set of barrier traversal lines when the facility is located in $\mathcal{Q}^{\bar{X}}$.

From Theorem 2 it follows that an optimal location lies on the boundary, $\widetilde{\mathcal{Q}^{\bar{X}}}$, of $\mathcal{Q}^{\bar{X}}$. In the case where all barriers and the facility are polygonal structures, $\mathcal{Q}^{\bar{X}}$ will always be polygonal. See example shown in Figure 17. In this example, the facility intersects a barrier traversal line $e-f$. The region $\mathcal{Q}^{\bar{X}}$ shown is such that the facility only intersects the same barrier traversal line (in region *abcd*). However, there is no guarantee that $\mathcal{Q}^{\bar{X}}$ is convex. By partitioning the non-convex polygon $\mathcal{Q}^{\bar{X}}$ into convex polygonal subsets, we can identify the set of candidate locations as the union of the vertices of these polygonal subsets. See the example shown in Figure 18.

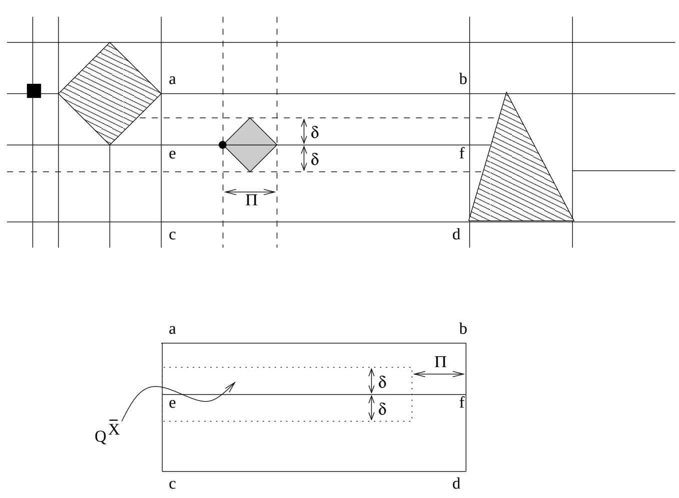

Figure 17: Example of a convex polygonal $\mathcal{Q}^{\bar{X}}$

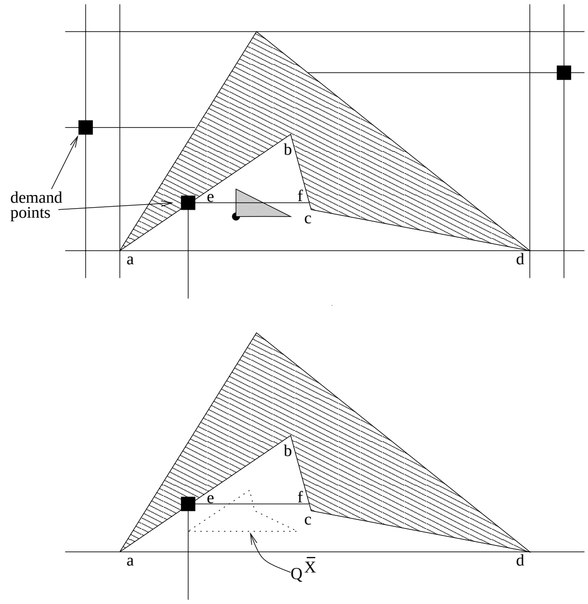

Figure 18: Example of a nonconvex polygonal $\mathcal{Q}^{\bar{X}}$

Up to this point we have discussed the behavior of the individual shortest path distance functions and the overall objective function for a specific region $\mathcal{Q}^{\bar{X}}$. However, $\mathcal{Q}^{\bar{X}}$ is only a subset of the overall set of feasible locations, $F_{\alpha_0}^{\bar{X}}$. Let $\widetilde{F_{\alpha_0}^{\bar{X}}}$ represent the boundary of $F_{\alpha_0}^{\bar{X}}$. By the way $\mathcal{Q}^{\bar{X}}$ is structured, its boundary $\widetilde{\mathcal{Q}^{\bar{X}}}$ will consist of at least one of three segments:

1. straight line segments parallel to the x axis corresponding to the intersection of some horizontal barrier traversal line h_i and some horizontal facility traversal line h'_i;

2. straight line segments parallel to the y axis corresponding to the intersection of some vertical barrier traversal line v_j and some vertical facility traversal line v'_j;

3. segments where the facility boundary intersects with some barrier boundary, which corresponds to $\widetilde{\mathcal{Q}^{\bar{X}}} = \widetilde{F_{\alpha_0}^{\bar{X}}}$.

The overall feasible region for the server, $F_{\alpha_0}^{\bar{X}}$, is made up of a number of regions $\mathcal{Q}^{\bar{X}}$. The number and structure of $\mathcal{Q}^{\bar{X}}$ within $F_{\alpha_0}^{\bar{X}}$ depends on the number of barriers, the shapes of barriers, and the proximity of barriers. However, a lot of $\mathcal{Q}^{\bar{X}}$ will be rectangular. The corners of such a rectangular $\mathcal{Q}^{\bar{X}}$ correspond to a server location $\bar{X} \in \mathcal{Q}^{\bar{X}}$ such that at that location $\ell = [\bar{X}, \alpha_0]$, some vertical barrier traversal line intersects with some vertical facility traversal line, i.e., $X_{v_j} = X_{v'_j}$ for some j and some horizontal barrier traversal line intersects some horizontal facility traversal line, i.e., $Y_{h_i} = Y_{h'_i}$ for some i.

3.4 Optimal Orientation for the Facility with Fixed Server Location

In §3.3, we have assumed that the orientation of the facility was fixed. In this section we study the facility placement problem when the facility orientation is the variable and the server location is fixed at $\bar{X}_0$. Here, when we speak of a feasible placement ℓ, we mean $\ell \in F_{\bar{X}_0}$. Now consider an initial feasible location $\ell_{ini} = [\bar{X}_0, \alpha_{ini}]$, with the initial orientation of α_{ini}. Assume that no facility traversal line coincides with a barrier traversal line, that is, $X_{v_j} \neq X_{v'_j}$; $\forall v_j, v'_j$ and $Y_{h_i} \neq Y_{h'_i}$; $\forall h_i, h'_i$; and also that there is no intersection between the facility boundary and a barrier boundary.

From this initial location, we will construct an orientation interval $[\alpha_a, \alpha_b]$ such that any facility placement $\ell = [\bar{X}_0, \alpha_c]$ with $\alpha_a \leq \alpha_c \leq \alpha_b$ will interfere with the same set of barrier traversal lines only. Let us start rotating the facility from the initial location in both counter-clockwise and clockwise directions, keeping the server location fixed at $\bar{X}_0$. We will continue the rotations until we reach the two orientations, α_a and α_b respectively, where four things can happen:

1. A vertical barrier traversal line v_j coincides with a vertical facility traversal line v'_j and/or a horizontal barrier traversal line h_i coincides with a horizontal facility traversal line h'_i.

2. $\alpha_a + \epsilon$ $(\alpha_b - \epsilon)$ creates infeasibility, where $\epsilon > 0$ is sufficiently small.

3. Some horizontal facility traversal line coincides with some other horizontal facility traversal line, i.e., $\exists\ h'_{i_1}, h'_{i_2}$ such that $Y_{h'_{i_1}} = Y_{h'_{i_2}}$ and/or some vertical facility traversal line coincides with some other vertical facility traversal line, i.e., $\exists\ v'_{j_1}, v'_{j_2}$ such that $X_{v'_{j_1}} = X_{v'_{j_2}}$. As an example, consider the case of a circular facility as shown in Figure 19. When $\alpha_0 = 90^o$ the horizontal lines h'_1 and h'_2 coincide.

4. A new horizontal and/or vertical facility traversal line is created or destroyed.

Theorem 3 *Consider the interval $[\alpha_a, \alpha_b]$. This interval can be partitioned into a finite number of sub-intervals, such that the objective function $J(\ell) + K(\ell)$ on each such sub-interval is either increasing or decreasing.*

Proof: From Lemma 1, we know that any path between i and j, $i, j \in D'$ is a sequence of horizontal and vertical traversal lines. The interval $[\alpha_a, \alpha_b]$ is defined in such a way that the sequence of horizontal and vertical lines in the path will be the same for any facility orientation in that interval. As the facility is moved, the barrier traversal lines remain fixed. In addition,

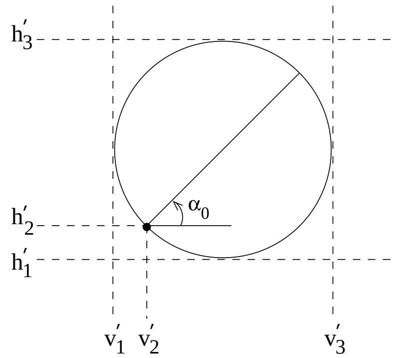

Figure 19: Illustration for the case where horizontal facility traversal lines coincide

since we keep the server location $\bar{X}_0$ fixed, the facility traversal lines passing through the server will also be fixed when such lines can be drawn through the server. However, as the orientation is changed, other facility traversal lines will be altered. Therefore in the following discussion, all barrier traversal lines and the facility traversal lines passing through the server are *fixed* lines (the coordinate of the line does not change as a function of α). On the other hand, a facility traversal line other than the one passing through the server is a *moving* line (the coordinate of the line changes as a function of α). For ease of presentation we drop the prime ($'$) notation for facility traversal lines (shown by dashed lines in Figures 20 and 21). The sequence of horizontal traversal lines $h_1, h_2, \cdots, h_m$ (or vertical traversal lines $v_1, v_2, \cdots, v_n$) that make up the path $P(i, j)$ can be represented as a combination of the following four sequences.

(i) fixed line - fixed line

(ii) fixed line - moving line(s) - fixed line

(iii) moving line - moving line

(iv) moving line - fixed line(s) - moving line

See Figures 20 and 21, where the facility is moved from $[\bar{X}_0, \alpha_1]$ to $[\bar{X}_0, \alpha_2]$, where $\alpha_a \leq \alpha_1 < \alpha_2 \leq \alpha_b$.

The distance between two fixed lines never changes (e.g., horizontal lines $h_1 - h_2$). There are two cases for the fixed-moving-fixed line sequence in (ii). For case 1, the distance between the two fixed lines will not change (e.g., horizontal lines $h_2 - h_3 - h_4$). Even if there are more than one moving lines between two fixed lines, the result remains. For case 2, the distance between the moving line and either fixed line is either increasing or decreasing in α (e.g., horizontal lines $h_2 - h_3 - h_5$). If there are more than one moving lines between the two fixed lines (e.g., $h_5 - h_6 - h_7 - h_1$), consider the moving line that is the furthest away (h_7) from the fixed lines; it is clear that the distance

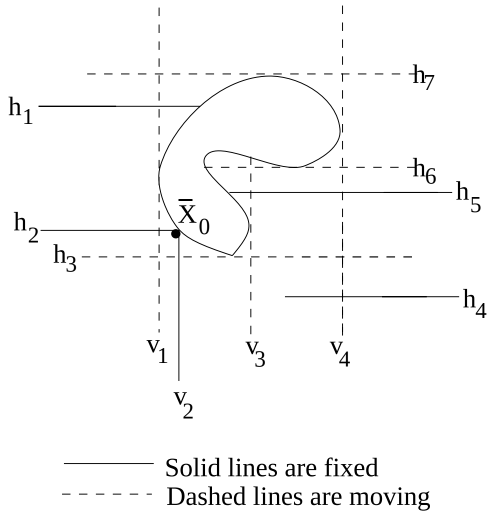

Figure 20: Server location $\bar{X}_0$; orientation α_1

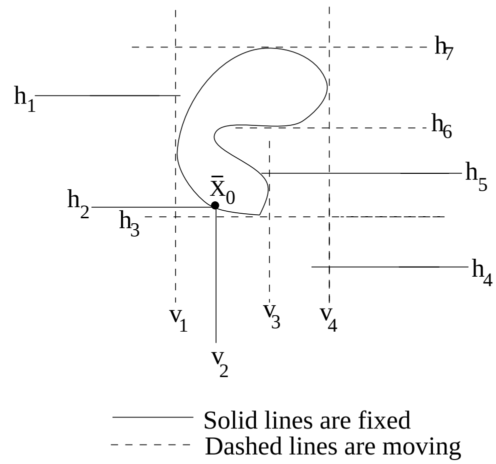

Figure 21: Server location $\bar{X}_0$; orientation α_2

between this moving line and either fixed lines is either increasing or decreasing in α.

We now focus on (iii), e.g., $v_3 - v_4$. In this situation it is possible that the distance between two moving lines first decreases as α is increased and later decreases as α is further increased. To see this consider the example of a square facility rotating about one of its corner points (with no additional barrier traversal line in its vicinity). However, in such a situation a given interval $[\alpha_1, \alpha_2]$, $0 \leq \alpha_1 < \alpha_2 \leq \pi$ will be divided into at most two-subintervals $[\alpha_1, \alpha_3]$ and $[\alpha_3, \alpha_2]$, with $\alpha_1 < \alpha_3 < \alpha_2$, such that the distance between the lines is either increasing or decreasing in these intervals. When a fixed line is between two moving lines as in (iv) (e.g., $v_1 - v_2 - v_3$), the distance between the moving lines is either non-increasing or non-decreasing. This remains true even if there are more than one fixed lines between the two moving lines.

From Lemma 1, we know that the length of any path $P(i,j)$ is $\sum_{i=1}^{m-1} |Y_{h_i} - Y_{h_{i+1}}| + \sum_{j=1}^{n-1} |X_{v_j} - X_{v_{j+1}}|$. Since the path can be constructed from the four possible sequences and the distances in each sequence are either increasing or decreasing, the length of $P(i,j)$ is either increasing or decreasing in α in each sub-interval of $[\alpha_a, \alpha_b]$. The shortest distance between i and j is the minimum among a finite number of such paths and therefore must be either increasing or decreasing in each sub-interval. Then, since $J(\ell)$ and $K(\ell)$ are both positively weighted sums of $d_\ell(i, \bar{X})$ and $d_\ell(i,j)$, respectively, the objective $J(\ell) + K(\ell)$ must be either increasing or decreasing in each sub-interval. By the manner in which sub-intervals are constructed, it follows there are a finite number of such sub-intervals. $\square$

Using the above theorem, we can determine the set of candidate optimal facility locations when the server is fixed at $\bar{X}_0$. Let us start rotating the facility from the initial location ℓ_{ini} in the counter-clockwise direction and keeping the server location fixed at $\bar{X}_0$. In this manner, we look at all orientations $0 \leq \alpha < 2\pi$ and mark an orientation α_m when at least one of the following five things occur:

1. A vertical barrier traversal line v_j coincides with a vertical facility traversal line v'_j and/or a horizontal barrier traversal line h_i coincides with a horizontal facility traversal line h'_i.
2. $\alpha_m + \epsilon$ creates infeasibility, where $\epsilon > 0$ is sufficiently small.
3. $\alpha_m - \epsilon$ creates infeasibility, where $\epsilon > 0$ is sufficiently small.
4. Some horizontal facility traversal line coincides with some other horizontal facility traversal line, i.e., $\exists h'_{i_1}, h'_{i_2}$ such that $Y_{h'_{i_1}} = Y_{h'_{i_2}}$ and/or some vertical facility traversal line coincides with some other vertical facility traversal line, i.e., $\exists v'_{j_1}, v'_{j_2}$ such that $X_{v'_{j_1}} = X_{v'_{j_2}}$.
5. A new horizontal and/or vertical facility traversal line is created.

Let M denote the set of all orientations α_m marked by the above procedure. We point out that the set M is finite. The fact that the number of barriers is finite implies that the number of vertical and horizontal barrier traversal lines is finite. Also, the number of vertical and horizontal facility traversal lines is finite, since we have a single facility. It then follows that the number of occurrences of conditions 1 through 5 is finite, implying the finiteness of the set M. Consider two successive orientations $\alpha_i, \alpha_{i+1} \in M$, with $\alpha_i < \alpha_{i+1}$. The interval $[\alpha_i, \alpha_{i+1}]$ can, by Theorem 1, be partitioned into a finite number of sub-intervals such that the objective function is increasing or decreasing over each such sub-interval. We have the following result.

Corollary 1 *The search for the optimal orientation α for a facility placed at $\ell = [\bar{X}_0, \alpha]$ can be restricted to a finite number of choices of α, for all $\ell \in F_{\bar{X}_0}$.*

3.5 Discussion of A Heuristic Method for the Placement of a Finite-Size Facility and Solution Complexity Issues

In §3.3 and §3.4, we dealt with given orientations and locations, respectively. The results in those sections can be used to eliminate placements that are dominated. In fact, there may be practical situations where the orientation of the facility is fixed. For example, a given side of a building may need to face a direction away from the strong winds or facing east for the morning sun or simply face a river or a mountain. Under these situations, §3.3 results are useful. On the other hand, §3.4 results can be used for situations where a structure must be built around an immovable resource.

The major difficulty with identifying a set of candidates for optimal facility placement is the determination of the feasible region. This difficulty exists even for placements with fixed orientation as demonstrated in §3.3. For that special case, we simply dealt with an area on the plane corresponding to feasible server locations. We have shown that the boundary of the feasible area may contain optimal location(s). Therefore the boundary of the feasible area must be investigated. Inside the feasible area, the set of candidates are finite.

For the general problem in which neither the orientation nor the location of the facility is known *a priori*, the determination of the feasible region is an even more difficult task. Using our analysis in §3.3 and §3.4, we will provide the ideas towards a heuristic method for the finite-size facility placement problem. Let us refer to the problem of locating an infinitesimal (point) facility with barriers to travel (as studied in Larson and Sadiq) as the facility location problem. The candidates for optimality in the facility location problem are all user points and all other grid points that represent the intersection of two barrier traversal lines (Result 4). We start by solving the facility location problem, i.e., (i) finding the distance matrix for the network whose node set is the set of candidate points and whose arc set contains only arcs between simply communicating nodes, with node weights being the appropriate rectilinear distances, and (ii) solving the network problem. Moreover, the candidate locations are listed in the ascending order of objective function value.

The optimal facility location is denoted by $\bar{X}_1$. The objective function value for this location is denoted by $f'(\bar{X}_1) = J'(\bar{X}_1) + K'(\bar{X}_1)$, where $J'(\bar{X}_1)$ and $K'(\bar{X}_1)$ correspond to the user-server and user-user interaction for the facility location problem, respectively. Clearly, $K'(\bar{X}_1)$ is constant in the facility location problem since the facility does not change the distances between users. In order to solve the facility placement problem, we will use the candidate locations for the facility location problem and results of §3.3 and §3.4 to create and continuously refine an upper bound until the stopping condition is satisfied.

We start with the facility placement $[\bar{X}_1, \alpha_1]$. We initially select $\alpha_1 = 0$. If the placement is feasible, then the objective function is calculated, $f([\bar{X}_1, \alpha_1]) = J([\bar{X}_1, \alpha_1]) + K([\bar{X}_1, \alpha_1])$. If $f([\bar{X}_1, \alpha_1]) = f'(\bar{X}_1)$ then the placement is clearly optimal. If $f([\bar{X}_1, \alpha_1]) > f'(\bar{X}_1)$, then we have obtained an upper bound UB for the facility placement problem. At this point we can use this upper bound to eliminate some part of the feasible region. Consider any cell C. The facility location objective at the corner $E_i(C)$ is given by $f'(E_i(C)),\ i = 1, 2, 3, 4$. If $f'(E_i(C)) \geq UB,\ \forall i$, then all placements with the server located in cell C are dominated and hence eliminated from consideration. With the server still located at $\bar{X}_1$, the analysis in §3.4 can be used to determine all other candidates for optimal facility placement. For all those candidates, the objective function is determined and UB is updated accordingly. Each time the upper bound is updated, more cells can also be eliminated from consideration.

If no feasible placement is possible with the server located at $\bar{X}_1$, then we look at the second best node, i.e., the grid point with the second best optimal location objective, $\bar{X}_2$. We need to look at this node as long as $f'(\bar{X}_1) < UB$. We try the placement $[\bar{X}_2, \alpha_1]$, again starting with $\alpha_1 = 0$. As we did for $\bar{X}_1$, we investigate placements with server located at $\bar{X}_2$, updating UB and eliminating more cells from future consideration. We continue this domination and elimination process following the ascending sequence of candidate points for optimal location until we reach a candidate point $\bar{X}_i$, where $f'(\bar{X}_i) \geq UB$.

At this point, we have identified a number of server locations, $\bar{X}_1, \cdots, \bar{X}_{i-1}$ along with a number of orientations at each server location and an upper bound for the optimal placement objective function value. One way to finalize the heuristic is to select a finite number of orientations and then identify a set of candidate locations. A number of stopping conditions can be implemented after this point: exhausting a predetermined set of orientations, the difference between the optimal objective value of the facility location problem and the upper bound on the facility placement problem, the rate of improvement on the upper bound or a combination of these criteria. Although the method presented here does not guarantee optimal placement, it overcomes the need to completely identify the feasible region, which is a difficult task. For convex polygonal barriers and facility, certain improvements can be made. These results are being implemented in a layout context in a forthcoming paper.

Another factor that affects solution complexity is the number of cells. The number of cells formed is a function of the number of demand points and their locations. It is also a function of the number, shape and placement of barriers, and, in particular, on the number of tangential lines to barriers.

The procedure would be more efficient if 1-, 2- and 3-corner cells are omitted from consideration. This is "equivalent" to considering all barriers (and the facility) to be rectangular in shape, and to restricting the orientation of the facility so that its sides are parallel to the travel axes. In such a situation, the region shapes are rectangular and easy to identify (see, e.g., Figure 14). The rectangular shape of the regions also imply, from Theorems 1 and 2, that the set of candidate points can be readily identified. A practical application of this paper relates to issues in layout analysis, where one is attempting to place a new department in the presence of existing departments in a rectangular building. In the literature this is modeled using infinitesimal department sizes (both for the existing and new department) - see Chapter 4 of Francis, McGinnis and White (1992). The typical department shapes in a factory are rectangular and placements of departments usually have departmental boundaries running parallel to the sides of the building (which are also generally rectangular). In such a context the concept of rectangular barriers and facility shape are applicable and the analysis can be greatly simplified. The authors are currently in the process of writing a paper that addresses such a layout application. Solution complexity issues relative to this simplified scenario will be addressed in this work.

4 Conclusion

In summary, this paper examines the optimal placement of a single finite-size facility when there are arbitrary barriers to rectilinear travel. The server location and the orientation of the facility define its placement. There are two types of interactions in the problem. Firstly, there is interaction between existing demand/supply points and secondly, the new facility interacts with some existing demand/supply points through the server located on the facility boundary. The interaction between any two points in the problem takes place on the shortest feasible rectilinear path(s) between the points. All interactions are positively weighted and the sum of the two types of interactions must be minimized.

We first study the facility placement problem with a fixed facility orientation. Sets of horizontal and vertical lines are drawn around the barriers and the new facility, referred to as barrier traversal lines (L) and facility traversal lines (L'), respectively. Two sets of candidate placements exist. Firstly, any placement created when a horizontal L coincides with a L' and a vertical L coincides with a vertical L' is a candidate. Secondly, the boundary of the feasible region can include the optimal placement. We then study the facility placement problem with a fixed server location. We obtain domination results and determine the set of candidates for optimal placements. In both problems, if the barriers and facility are polygonal, the set of candidate placements is finite. Finally, we use our results to develop a heuristic for the finite-size facility placement problem. Although no current method exists for a guaranteed optimal placement of an arbitrary facility, optimal methods are being implemented in a layout context for rectangular barriers and facility.

Acknowledgements

This work was supported by the National Science Foundation, via grant no. DMI-9800429. This support is gratefully acknowledged. The authors also wish to acknowledge the help of two anonymous referees whose insightful comments enhanced this paper.

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