Locating a 1-Center on a Manhattan Plane with "Arbitrarily" Shaped Barriers

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Abstract

This paper addresses the 1-center location problem on a 2-dimensional plane having arbitrarily shaped barriers and using Manhattan metric distances. Barriers commonly occur in practical location and layout problems and are regions where neither travel through nor location of the new facility is permitted. Along the lines of Larson and Sadiq (1983) we divide the feasible location region into cells. To overcome the additional complications introduced by the center objective, we develop new analysis and classify cells based on number of cell corners. A procedure to determine the optimal location is established for each class of cells. The overall complexity of the approach is shown to be polynomially bounded. Also, an analogy is drawn to the center problem on a network and generalizations of the model are discussed.

Keywords : barrier, center problem, location.

1 Introduction

Location problems in which regions are excluded from siting new facilities, but travelling through is allowed are called *restricted location problems*. These problems have been solved for median and center objectives, for instance, in Hamacher [10], Nickel [20], Hamacher and Nickel [11], and Hamacher and Schöbel [14]. One often encounters situations in which regions are allowed neither for siting new facilities nor for trespassing. Such regions are called *barriers*. In an urban or larger context, one could just use the street network and

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solve the relevant location problem. Such a street network has obviously been constructed to avoid barriers. A layout context (see Chapters 1 through 4 of Francis, McGinnis and White [21]) in which departments act as barriers to travel is an excellent application area for this model, since material handling needs to occur in a plant wherein goods are moved from one department to another.

A relatively small amount of literature is available on location problems with barriers. Katz and Cooper [15] considered the median problem using Euclidean distance and a barrier consisting of one circle. Aneja and Parlar [1], Butt and Cavalier [4] developed heuristics for the 1-median problem with l_p distance and barriers that are closed polyhedra. Larson and Sadiq [17] solved the p-median problem on a plane with the Manhattan metric in the presence of barriers. Batta, Ghose and Palekar [2] obtained discretization results for the 1-median problem with the Manhattan metric and barriers, by transforming the problem into an equivalent network location problem. While the center problem in the plane without barriers is extensively discussed in the literature (see, e.g., the text books of Drezner [6], Francis *et al.* [21], Hamacher [9] and Love *et al.* [18]), very few references can be found on the corresponding barrier problem. The distinction between the former and the latter problem is that travelling through a barrier is not allowed. A Finite Dominating Set(FDS) for the 1-center problem on a Manhattan plane in the presence of polyhedral, convex barriers is provided by Dearing, Hamacher and Klamroth [5]. A more detailed version of this work is presented in Chapter 9 of the very recent habilitation thesis of Klamroth [16]. A cell decomposition of the feasible region is performed. Building upon this, a bisector algorithm for this problem is developed. This algorithm finds bisectors by adapting an algorithm of Mitchell [19]. The algorithm's output is the set of dominated points. The overall complexity of the algorithm is polynomial in the number of demand points and the number of extreme points of the polygonal barriers.

In this paper, we consider the same problem on a more general class of barriers. This paper differs from [5] and [16] in the following respects:

- It solves the problem optimally for a large class of barriers, that need not be polyhedral or convex. The restrictions we impose are: (i) barriers should have a finite number of tangency lines parallel to the x- and y-axes, and (ii) barriers should be closed and bounded sets.

- It solves problems with constant addends to distance functions.

- It can be readily extended to solve a general class of objective function, which is convex and nondecreasing in the distance functions.

2 Definitions and Notations

In this section, we define the weighted 1-center problem with barriers. Then we consider the properties of the problem. Let $B_1, \ldots, B_N$ be closed, bounded and piecewise disjoint sets in $\Re^2$. Each set $B_i, i = 1, \ldots, N$, is called a barrier. An additional restriction that we place on a barrier is that it has a finite number of horizontal and vertical tangential lines; for instance, the barrier $B_i = \{(x, y) : 0 \leq x \leq 1, 0 \leq y \leq Sin(1/x)\}$ has an infinite number of horizontal tangential lines and would not be permitted. Let $\mathcal{B} = \bigcup_{i=1}^{N} B_i$. The location of the new facility in the interior of $\mathcal{B}, int(\mathcal{B})$, and travel through it is forbidden.

1. The feasible region F is the maximal subset of $\Re^2 \backslash int(\mathcal{B})$ with the property that if $x, y \in F$ then there exists a path between x and y that is wholly contained in F. See Figure 1.

2. A point $P \in F$ has coordinates (x_P, y_P).

3. V is the set of demand nodes labeled $G_1, \ldots, G_n$ with coordinates (x_1, y_1),. . .,(x_n, y_n). $|V| = n$. We assume that each $G_i \in F$, for $i = 1, \ldots, n$.

4. The positive weight and the constant addendum of demand node G_i are w_i and γ_i respectively, for $i = 1, \ldots, n$.

We note here that despite some restrictions on the barriers, the class that we consider is much more general than convex polyhedra.

A Manhattan path P in $\Re^2$ having $2(s+1)$ steps is specified by a sequence of path vertices $V(P) = \{(x_0, y_0), (x_1, y_0), (x_1, y_1), (x_2, y_1), (x_2, y_2), \ldots, (x_s, y_s), (x_{s+1}, y_s), (x_{s+1}, y_{s+1})\}, \; s = 0, 1, \ldots$, such that

$$P = \{(x, y_k) \in \Re^2, x_k \leq x \leq x_{k+1} \; or \; x_{k+1} \leq x \leq x_k, \; k = 0, 1, \ldots, s+1\}$$

$$\cup\{(x_k, y) \in \Re^2; y_{k-1} \leq y \leq y_k \; or \; y_k \leq y \leq y_{k-1}, k = 1, 2, \ldots, s+1\}.$$

That is, a Manhattan path is a connected set of points proceeding in consecutive horizontal and vertical straight line segments. A permitted Manhattan path P penetrates no barriers, hence satisfies the condition $P \cap int(\mathcal{B}) = \phi$. In Fig. 2, paths 1, 2, 3 and 4 are four examples of permitted Manhattan paths between demand nodes G_1 and G_4. A shortest permitted Manhattan path is one with minimum length among all possible permitted Manhattan paths. In Fig. 2, paths 2 and 3 are both shortest permitted Manhattan paths. Let *d(A,B)* be the length of a shortest permitted Manhattan path from A to B. Define the function

$$f_{\mathcal{B}}(X) = \max_{G_i \in V} \; w_i d(X, G_i) + \gamma_i.$$

It is to be noted that the addition of addends can affect the optimization problem (as it essentially amounts to changing the distance function). Then the weighted 1-center problem with barriers is to solve the problem:

$$\min_{X \in F} \; f_{\mathcal{B}}(X).$$

Let R be the smallest rectangle with sides parallel to the coordinate axes, enclosing all the demand nodes and barriers. An optimal solution cannot lie outside the rectangle R, since for any point outside the rectangle R we can find shortest permitted Manhattan paths to all the demand nodes having a common point on the boundary of R. Then the value of the objective function of the center problem can be decreased by pulling the facility location onto the boundary of R. In Fig. 2, the smallest enclosing rectangle is shown – it encloses all the seven demand nodes as well as the two barriers. An optimal solution should lie in the region $F \cap R$. We denote this region as F'.

3 Grid Construction and Cell Formation

A procedure to construct a grid $\mathcal{G}$ that divides the feasible region F' into cells is established in [17]. The same procedure of grid construction is followed here. Let $\mathcal{Q}(B)$ denote the points of tangency for a barrier B, i.e., points on the boundary of B through which one can pass a horizontal or vertical line segment and for which all points in the boundary sufficiently close to this point either lie in or on one side of the horizontal or vertical line. We refer to these as HV-tangency lines. For segments of the barrier boundary that are horizontal or vertical and whose straight line extensions are fully contained either in the barrier or in F', one has a line segment of tangency, both endpoints of which are included in the set $\mathcal{Q}(B)$. We also define $\mathcal{Q}(\mathcal{B}) = \cup_{B \in \mathcal{B}} \mathcal{Q}(B)$. In the region F', bounded by R:

1. Draw lines parallel to coordinate axes from all $X \in V \cup \mathcal{Q}(\mathcal{B})$ with each line terminated in each direction only at the first barrier interior point or at the boundary of R.

2. Exclude from the set of lines in Step 1 any line extending from a point $X \in \mathcal{Q}(\mathcal{B})$ that is not a demand point and for which X is an endpoint of the line. For instance, consider the point $X \in \mathcal{Q}(\mathcal{B})$ for the barrier shown in Fig. 3. The line XY would be constructed in Step 1. Since X is also an endpoint of this line the portion of the line with a common intersection with the barrier XY would be excluded in Step 2.

The resulting lines along with the boundary of barrier set $\mathcal{B}$ form the grid $\mathcal{G}$ which divides the feasible region F' into a finite set of cells $C(\mathcal{G})$. In Fig. 4 the grid is constructed for an example that has two barriers and four demand nodes.

Each cell is bounded by the grid lines. For a given cell C consider the points (x_{min}, y_{min}), (x_{max}, y_{min}), (x_{max}, y_{max}), and (x_{min}, y_{max}) where $x_{min}, x_{max}, y_{min}$, and y_{max} are the respective bounds on X and Y in the cell. Due to our assumption that barriers are closed, bounded and piecewise disjoint, it follows that at least one of the four points (x_{min}, y_{min}), (x_{max}, y_{min}), (x_{min}, y_{max}), and (x_{max}, y_{max}) is contained in C. We refer to all such points contained in C, up to a maximum of four, as the *cell corners* of C. We will later utilize the following result proved in [9]:

Lemma 1 : *Let $G_i \in V$ be a demand point and let C be any cell in $C(\mathcal{G})$ with $X \in C$. Then there exists a shortest permitted Manhattan path connecting G_i and X that passes through a corner point of C.*

Cells that do not have a common boundary with a barrier in the set $\mathcal{B}$ are rectangular and have four cell corners. The cells which have a common boundary with a barrier in the set $\mathcal{B}$ may not be rectangular and may have less than four cell corners. More formally, the set of cells $C(\mathcal{G})$ is divided into four subsets. A subset $C^k(\mathcal{G})$ has all the cells which have k number of cell corners. From the above discussion, we can write

$$C(\mathcal{G}) = C^1(\mathcal{G}) \cup C^2(\mathcal{G}) \cup C^3(\mathcal{G}) \cup C^4(\mathcal{G}).$$

4 Solution Procedure

A grid is constructed and cells are classified as described in Section 3. We now illustrate how to find the local optimum solution for locations restricted to be within a specified cell. Comparison of these local optima yields the global optimum solution. We first examine the case of cells with four corners. Later we examine 1-, 2- and 3-corner cells.

4.1 4-Corner Cells

To facilitate the analysis of the rectangular 4-corner cells we introduce the concepts of Equal Travel Time Lines (ETTLs) and sub-cells.

4.1.1 Equal Travel Time Lines and Sub-cells

For a fixed location $X \in C$, where C is a cell in $C^4(\mathcal{G})$, we can meaningfully speak of the assignment of demand nodes to cell corners of C. If the assignment of demand nodes to cell corners does not change upon moving the facility location X in cell C, the distance functions $d(G_i, (x, y))$ are linear over cell C. However, the assignment of demand nodes to cell corners can change upon moving the location X in cell C, as illustrated by the example in Fig. 5. When the facility lies in the region 2563 a shortest Manhattan path to the demand node G_1 goes through the cell corner 2, traversing the intervening barrier from its upper apex. The line segment $\overline{5\,6}$ partitions the cell into two regions, 2563 and 5146. If the facility lies in region 2563 but not on line segment $\overline{5\,6}$, a shortest Manhattan path to G_1 from this facility goes through cell corner 2. If it lies in the region 5146 but not on line segment $\overline{5\,6}$, the path goes through cell corner 1. For points on line segment $\overline{5\,6}$ both alternatives (going through 1 or 2) are equally attractive. For this reason we call the line segment $\overline{5\,6}$ an Equal Travel Time Line (ETTL).

We now establish a formal procedure to construct ETTLs associated with a cell $C \in C^4(\mathcal{G})$. Such a cell C is rectangular. Let the coordinates of the four cell corners of C, labeled 1, 2, 3, and 4, be $(0,0), (0,b), (a,b)$, and $(a,0)$, respectively. See Fig. 6 for an illustration. Let d_{ij} denote the length of a shortest Manhattan path from a demand node G_i to cell corner j. If $|d_{i1} - d_{i2}| < b$ then G_i will generate an ETTL, which will be the line segment (parallel to the edges $1-2$ and $3-4$ of the cell) joining $(0, (|d_{i1} - d_{i2}| + b)/2)$ and $(a, (|d_{i1} - d_{i2}| + b)/2)$. This ETTL is parallel to the edges $1-2$ and $3-4$ of the cell. On the other hand, if $|d_{i1} - d_{i3}| < (a+b)$, then G_i will generate an ETTL of a different type, which will be the line segment joining $(a - (|d_{i1} - d_{i3}| + a + b)/2, b)$ and $(a, b - (|d_{i1} - d_{i3}| + a + b)/2)$. This ETTL is at a 45^o angle with sides $2-3$ and $3-4$ of the cell. The first ETTL is generated by corners 1 and 2, whereas the second ETTL is generated by corners 1 and 3. Similarly, we can construct ETTLs generated by corners 3 and 4, 2 and 4, 1 and 4, and 2 and 3. After construction of an ETTL induced on cell C by demand node G_i has been completed, we can repeat the same procedure for all other demand nodes $G_k, k \neq i$. After all ETTLs have been established in cell C, we arrive at a partition of the cell into sub-cells. The number of sub-cells that are generated is polynomial and is further analyzed in Section 5 on solution complexity. This result hinges on a finding that each demand point induces at most one ETTL in a cell.

The importance of a sub-cell SC of C is the following fact: For $(x,y) \in SC$, the functional form of the distance functions $d(G_i, (x,y))$ does not change as long as (x,y) is within SC. This is because for any $(x,y) \in SC$, a shortest Manhattan path always goes through the same cell corner of C.

4.1.2 Linear Programming Formulation

Armed with the concepts of ETTLs and sub-cells we now demonstrate how to check for the local optimal solution to the 1-center problem in a sub-cell by solving an LP.

Let C_1, C_2, C_3 and C_4 be the cell corners of a cell $C \in C^4(\mathcal{G})$. Without loss of generality, we can assume the left lower corner C_1 as the origin. The other corners are $C_2(x_{max}, 0)$, $C_3(x_{max}, y_{max})$, and $C_4(0, y_{max})$, where x_{max} and y_{max} are the respective bounds on x and y in the cell C. Let $d_{ij}(X)$ denote the length of a shortest Manhattan path from a demand node G_i to X that is constrained to pass through cell corner C_j. Thus, $d_{ij}(X) = d(G_i, C_j) + d(C_j, X)$. Here, $d(G_i, C_j)$ is a constant whereas,

$$d(C_j, X) = \begin{cases} x + y & \text{if } \; j = 1, \\ (x_{max} - x) + y & \text{if } \; j = 2, \\ (x_{max} - x) + (y_{max} - y) & \text{if } \; j = 3, \text{and} \\ x + (y_{max} - y) & \text{if } \; j = 4. \end{cases}$$

In a sub-cell SC of the cell C, the assignment of the nearest corners to the demand nodes does not vary. Thus, the set of nodes V can be partitioned into sets of $V_1^{SC}, V_2^{SC}, V_3^{SC}$, and V_4^{SC}, such that $G_i \in V_j^{SC}$ implies that for any $X \in SC$, $d_{ij}(X) \leq d_{ik}(X)$, for $k \neq j$. Thus,

$$d(G_i, X) = \begin{cases} d(G_i, C_1) + x + y & \text{for } \; G_i \in V_1^{SC}, \\ d(G_i, C_2) + (x_{max} - x) + y & \text{for } \; G_i \in V_2^{SC}, \\ d(G_i, C_3) + (x_{max} - x) + (y_{max} - y) & \text{for } \; G_i \in V_3^{SC}, \text{and} \\ d(G_i, C_4) + x + (y_{max} - y) & \text{for } \; G_i \in V_4^{SC}. \end{cases}$$

We conclude that for any $X \in SC$, the distances $d(G_i, X)$ for any demand point $G_i \in V$ takes on a unique and linear (in x and y) functional form. Each sub-cell SC is defined by a set of linear line segments (ETTLs) and the boundary of the rectangular cell C to which it belongs. Therefore, each sub-cell SC is a convex polyhedra defined by linear segments and can be represented by a set of inequalities

$$A^{SC} \begin{pmatrix} x \\ y \end{pmatrix} \leq b^{SC}.$$

We can now set up the problem of finding the local weighted 1-center in sub-cell SC as follows:

$$\text{Minimize: } \; Z$$

Subject to:

$$w_i[d(G_i, C_1) + x + y] + \gamma_i \leq Z \quad \text{for} \quad G_i \in V_1^{SC},$$

$$w_i[d(G_i, C_2) + (x_{max} - x) + y] + \gamma_i \leq Z \quad \text{for} \quad G_i \in V_2^{SC},$$

$$w_i[d(G_i, C_3) + (x_{max} - x) + (y_{max} - y)] + \gamma_i \leq Z \quad \text{for} \quad G_i \in V_3^{SC},$$

$$w_i[d(G_i, C_4) + x + (y_{max} - y)] + \gamma_i \leq Z \quad \text{for} \quad G_i \in V_4^{SC}.$$

$$A^{SC} \begin{pmatrix} x \\ y \end{pmatrix} \leq b^{SC}.$$

4.2 1-, 2- and 3-Corner Cells

To facilitate the analysis of these types of cells, we prove the following result in Theorem 1.

Theorem 1: *Consider a cell $C \in C(\mathcal{G})$ that has less than four cell corners. If an optimal solution X^* lies in this cell C then an alternate optimal solution can be found*

- *on the cell corner itself, if the cell C has one cell corner.*
- *on a shortest permitted Manhattan path joining the two cell corners of the cell C, if the cell C has two cell corners.*
- *on the two cell edges which are not common to any barrier and joining the three cell corners, if the cell C has three cell corners.*

Proof: Let $X^* \in C$ be an optimal solution to our problem. We separately treat the cases of one, two or three cell corners. For each situation we start by assuming that the optimal solution violates the premise in the theorem. We then either establish a contradiction or construct an alternate optimum solution that satisfies the premise. The fundamental result we use is Lemma 1, which tells us that a shortest permitted Manhattan path from X^* to a demand point must pass through a cell corner of C.

Case 1: One cell corner. Let X^* not be a cell corner. From Lemma 1, by moving the solution from X^* to this cell corner we necessarily decrease the distance of the facility to all demand points. This establishes a contradiction.

Case 2: Two cell corners. Let X^* not be on a shortest Manhattan path joining the two cell corners. From its definition, a cell corner must be either (x_{min}, y_{min}), (x_{max}, y_{min}), (x_{max}, y_{max}), or (x_{min}, y_{max}). Thus six cases are generated by choosing two of these four possibilities for cell corners. Fig. 8 shows a case which has (x_{min}, y_{min}), (x_{max}, y_{min}) combination. For this particular case, if X^* does not lie on the line connecting (x_{min}, y_{min}) and (x_{max}, y_{min}) we can move X^* to the point X^{**}, point X^{**} being the closest point on this connecting line to X^*. Since X^{**} is closer in Manhattan distance to both cell corners it follows from Lemma 1 that we necessarily decrease the distance of the facility to all demand points. This establishes a contradiction.

Another case is that of Fig. 9 which shows a $a = (x_{min}, y_{min})$, $b = (x_{max}, y_{max})$ combination. Here any chosen X^* lies on a shortest permitted Manhattan path connecting a and b, so the conditions of the theorem are met. Other cases for 2-corner cells can be treated in a similar manner.

Case 3: Three cell corners. Following the discussion in Case 2 there are four cases generated by choosing three of the four possible cell corners. Fig. 7 shows the (x_{min}, y_{min}), (x_{max}, y_{min}), (x_{min}, y_{max}) case. Suppose that X^* does not satisfy the assertion of the theorem, i.e., it does not lie on the edge C_1C_2 or the edge C_2C_3. Then the situation is as shown in Fig. 7. We can find a point X^1 (which can either be on edge C_1C_2 or C_2C_3 depending on the position of X^* and the dimensions of the cell) by drawing a line from X^* at a 45^o angle and terminating it when it meets either of these edges. By moving the facility from X^* to X^1 the distances to demand points "serviced" through C_1 and C_3 remain the same, whereas the distances to those serviced through C_2 strictly decrease. (Note that the use of the Manhattan metric implies, for example, that the increase in y distance to cell corner C_3 by moving the facility from X^* to X^1 is compensated X^1 is at least as good a solution as X^*.) The other cases for three corner cells can be treated in a similar manner. ■

From theorem 1, in a cell $C \in C^1$, all other locations are dominated by its cell corner. So it is sufficient to evaluate the objective function on the cell corner of C.

Also from theorem 1, for a 2-corner cell the local optimal solution exists on a shortest permitted Manhattan path joining these two cell corners. Consider such as cell C with cell corners C_1 and C_2, and let P_C be such a path. Since we can choose any path P_C we restrict our attention to those paths that have a finite number of horizontal and vertical segments. Consider the example situation in Fig. 8. We can define corner point C_1 as the origin (0,0) and let the coordinates of corner point C_2 be $(x_{max}, 0)$. For any point $X = (x, 0) \in P_C$, with $0 \leq x \leq x_{max}$, let $d_{ij}(X)$ denote the length of a shortest Manhattan path from a demand node G_i to X that is constrained to pass through cell corner C_j. Then, $d_{ij}(X) = d(G_i, C_j) + d(C_j, X)$. Here, $d(G_i, C_j)$ is a constant whereas,

$$d(C_j, X) = \begin{cases} x & \text{if } \; j = 1, \\ x_{max} - x & \text{if } \; j = 2. \end{cases}$$

We now show how to partition the path P_C into segments. To accomplish this we need to find the Equal Travel Time Points (ETTPs) on path P_C. This happens when $d(G_i, C_1) + x_{max} > d(G_i, C_2)$. When this condition is met the ETTP induced by demand node G_i, labeled ETTP$_i$, has a y-coordinate of 0 and an x-coordinate of $(d(G_i, C_2) - d(G_i, C_1) + x_{max})/2$. Segments occur between successive ETTPs on the path P_C.

In a segment SP of the path P_C, the assignment of the nearest corners to the demand nodes does not vary. The set of nodes can be partitioned into sets V_1^{SP}, and V_2^{SP}, such that $G_i \in V_j^{SP}$ implies that for any $X \in SP$, $d_{ij}(X) \leq d_{ik}(X)$, for $k \neq j$. Thus,

$$d(G_i, X) = \begin{cases} d(G_i, C_1) + x & \text{for } \; G_i \in V_1^{SP}, \text{and} \\ d(G_i, C_2) + x_{max} - x & \text{for } \; G_i \in V_2^{SP}. \end{cases}$$

It is now clear that we can set up a linear program (as for the 4-cell corner case) to solve for the local weighted 1-center on a segment of the path P_C.

We now consider the case of 3-corner cells. From theorem 1 we know that a local optimal solution can be found on the two cell edges which are not common to any barrier and join the three cell corners. The situation is shown in Fig. 7. An optimal solution lies on either $\overline{C_1C_2}$ or $\overline{C_2C_3}$. We can partition both line segments into segments by identifying ETTPs, following the logic used in the 2-corner cell case. A linear program can be solved for identifying the local weighted 1-center on each segment.

5 Solution Complexity

Most 1-, 2- and 3-corner cells have a common boundary with a four-corner cell. From the discussion in the previous section, we can eliminate all of these cells from consideration, since the local 1-center will lie on a common point with an adjacent four-corner cell. For instance, in Figure 4 the only cells (that have 1, 2 or 3 corners) that need further consideration are cells C_1 and C_2. Now, the dominating edge of cell C_1 is its common edge with cell C_3. However, since cell C_3 is a 3-corner cell we know that its dominating edges are those shared with two of its adjacent four-corner cells. Therefore, even cell C_1 does not need further consideration.

There are some unusual circumstances (involving 2-corner cells) that would require consideration of these cells. The cell shown in Fig. 9 has two corners a and b. The dominating set of points lie on a permitted Manhattan path P_C joining a and b. It is clear that a local 1-center for this cell may not be part of an adjacent cell.

We now turn our attention to four-corner cells. The number of linear programs that we would need to solve depends upon the number of sub-cells in the four-corner cell. Since sub-cells are formed by ETTLs, it is critical to know how many ETTLs can be generated by a given demand point. We focus on this issue in Theorem 2.

Theorem 2: *A demand point G_i generates at most one ETTL in a four-corner cell.*

Proof: Consider a four-corner cell as shown in Fig. 10. Suppose that the demand point generates an ETTL using cell corners a and b. Then a shortest Manhattan path (SMP) between a and G_i must have a $top-left-bottom$ turning step and a SMP between b and G_i must have a $bottom-left-top$ turning step. Now suppose that there also exists an ETTL due to cell corners c and b. Then a SMP between c and G_i must have a $left-bottom-right$ turning step. However, this contradicts an earlier statement. By a similar argument other ETTL combinations can be eliminated. ■

The precise number of sub-cells generated in a cell will depend upon which type of ETTLs are generated by the demand points. Refer to Figure 10. ETTLs generated using cell corners a and b are parallel to the line segment $\overline{a\,c}$, whereas those generated using cell corners b and c are at 45^o to the line segment $\overline{a\,c}$; see discussion in Section 4.1.1. In a worst-case situation we could have n ETTLs that cut a cell in some order of angles 0^o, 90^o, 45^o and -45^o, as shown in Fig. 11.

An upper bound on the number of sub-cells could be derived by lifting the restriction on the angle of the intersecting ETTLs. n lines intersect in at most $C_2^n = n(n-1)/2$ points. Furthermore, every line is cut into at most n pieces (or edges). We can now use Euler's relationship for the number of sub-cells in a plane as follows (see e.g., Edelsbrunner [7]):

$$F = E - V + 1 = n \times n - n(n-1)/2 + 1 = O(n^2)$$

where V is the number of vertices or points, E is the number of edges, and F is the number of sub-cells. In typical problems which we have tried, the number of sub-cells is much smaller since the vast majority of demand nodes do not generate an ETTL in a cell.

Another factor that governs complexity of the proposed solution procedure is the number of cells formed. This is a function of the number of demand points and their locations. It is also a function of the number, shape and placement of barriers, and, in particular, in the number of HV-tangency lines. To formalize this we note that the N barriers generate at most CN horizontal and CN vertical lines, where C is a constant that signifies an upper bound on the number of HV-tangency lines. Similarly, the demand nodes can generate at most n horizontal and n vertical lines. Hence the maximal number of 4-corner cells is $O(N^2 + n^2)$. We do not typically generate this many 4-corner cells since (as discussed in Section 3) grid lines that intersect with a barrier get terminated.

In summary, based on the number of sub-cells per cell, $O(n^2)$, and the number of cells, $O(N^2 + n^2)$, the proposed solution procedure is $O(N^2 n^2 + n^4)$. The number of linear programming problems is polynomially bounded.

To improve the computational time we can attempt to prune some cells based on a bound. For instance, it is possible that a four-corner cell is dominated by a current upper bound (i.e., the best known solution). Consider the cell C in Figure 10, with "length" l and "width" w, and with the objective function value at the four cell corners a, b, c and d being $f_{\mathcal{B}}(a), f_{\mathcal{B}}(b), f_{\mathcal{B}}(c)$ and $f_{\mathcal{B}}(d)$, respectively. Let $w_{max} = \underset{i=1,...,n}{max} \quad w_i$. Then the objective $f_{\mathcal{B}}(x)$ for $X \in C$ is bounded by

$$LB_C = min\{f_{\mathcal{B}}(a), f_{\mathcal{B}}(b), f_{\mathcal{B}}(c), f_{\mathcal{B}}(d)\} - w_{max}(l + w).$$

It is possible to develop stronger lower bounds following the analysis of the weighted 1-center problem on a network. This will obviously be of relevance when developing a computer code to solve the problem.

6 Analogies with Network Problems and Discussion

There is a close relationship between the problem studied in this paper and the network counterparts. On a network, cycles introduce alternate paths to reach from one node to another. Thus one would expect a planar problem with no barriers to be "equivalent" to a tree network (which has no cycles). A similar correspondence would be expected between a planar problem with barriers and a cyclic network problem.

The notion of "bottleneck" points on a link of a network is introduced in Garfinkel, Neebe and Rao [8]. Berman, Larson and Chiu [3] refer to these as "breakpoints". Hooker [12] solves single facility location problems on a network by breaking it up into a collection of subproblems, each of which is tractable. He uses the bottleneck points' concept from Garkinkel *et al.* [8] to do this. A similar approach is used by Berman *et al.* [3] for solving the Stochastic Queue Median problem on a network.

Our approach is similar. We divide each cell (which corresponds to link on the network) into sub-cells using ETTLs. The ETTLs correspond to breakpoints or bottleneck points on a network. Each sub-cell generates its own subproblem, which is tractable. The best solution among all subproblems is chosen as the solution to the original problem.

It is worthwhile noting that essentially the same approach as in this paper could solve $\underset{X \in F}{min} \quad g_{\mathcal{B}}(X)$, where

$$g_{\mathcal{B}}(X) = h(d(X, G_1), \ldots, d(X, G_{|V|})),$$

with h being a convex, nondecreasing function. This problem can be transformed into:

$$\text{(Q)} \quad \text{Minimize:} \quad h(Z_1, \ldots, Z_n),$$

Subject to:

$$d(X, G_i) \leq Z_i \qquad \forall i \in V \quad \text{and} \quad X \in F.$$

The problem (Q) can be set up as a convex program in each sub-cell and solved using a suitable solver.

We further note that though there is a close analogy between the problem studied here and the 1-center problem on a network, it does not appear that the problem reduces to that on a network. This is because of the 2-dimensional space available for location within 4-corner cells (as opposed to a one-dimensional link on a network). Exploration of cases where such a reduction is possible would be a fruitful future research area since this would enable the use of localization results from Hooker, Garfinkel and Chen [13]. In a special case of our problem where there are only 1-, 2- and 3-corner cells, the problem becomes equivalent to the 1-center problem on a network. The following can be reasoned about the network from the discussion of Section 4.2. For a 1-corner cell, the network consists of a single node corresponding to this corner and no edges. For a 2-corner cell, the network has two nodes corresponding to the corners, and a single edge which is either an edge of the cell (Figure 8) or a Manhattan path connecting the two corners (Figure 9). For a 3-corner cell, the network has three nodes corresponding to the corners, and two edges that are the edges of the cell connecting the corners.

An extension of the paper's results to the more general Block norm case does not appear to be possible. This is because the fundamental result of Lemma 1 breaks down, i.e., the shortest permissible paths from the interior of a cell to a demand point do not have to go through a cell corner. The figure on page 125 of Klamroth [16] illustrates this point. Once the result of Lemma 1 does not hold we cannot make the assertion in Section 4.

Acknowledgments

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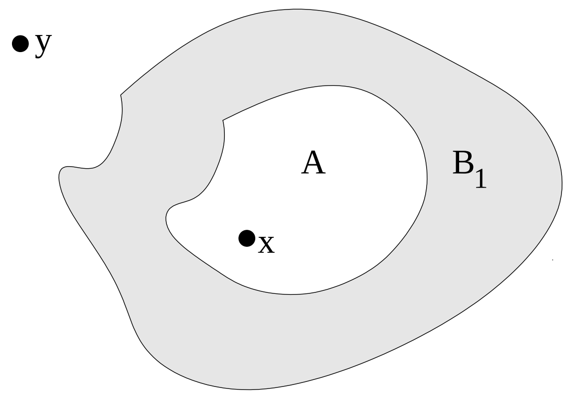


Key:

- $F = \Re^2 \backslash int(\mathcal{B}) \backslash A$
- Choosing $x \in A$ and $y \in F$, no path joining x and y is wholly contained in F. Thus $x \notin F$.

Figure 1: Illustration for Barriers and the Feasible Region

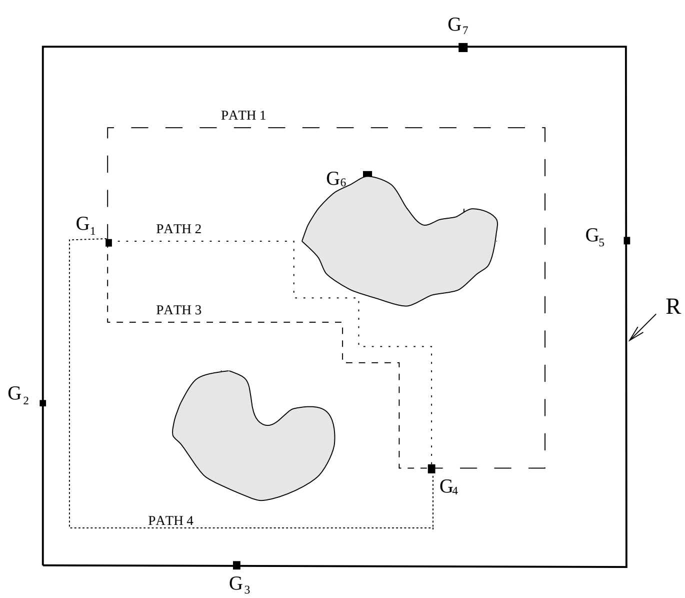


Figure 2: Manhattan paths joining G_1 and G_4, and enclosing rectangle R

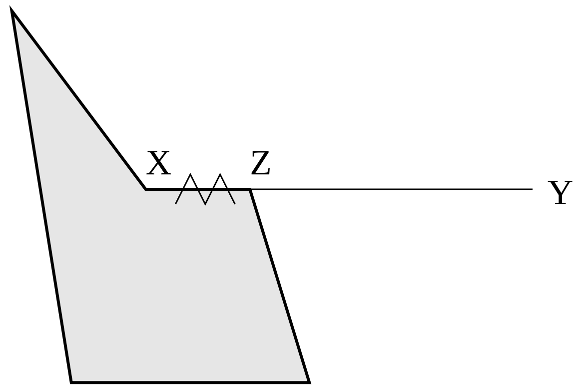


Figure 3: Example of a portion of a barrier tangency line to be excluded in Step 2 of the grid construction procedure

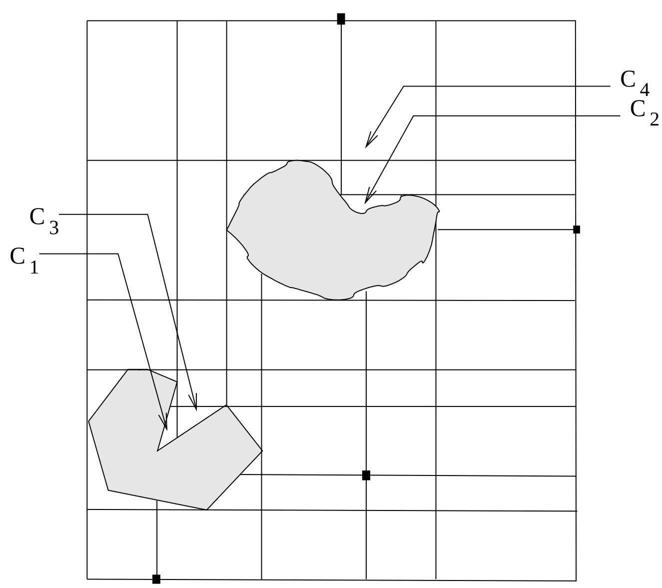


Figure 4: Example to illustrate the grid construction procedure

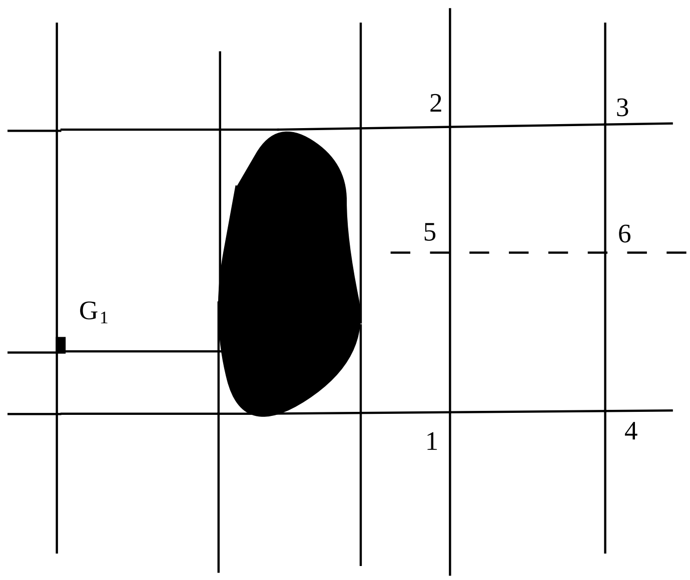


Figure 5: Example for varying cell corner assignment to demand node

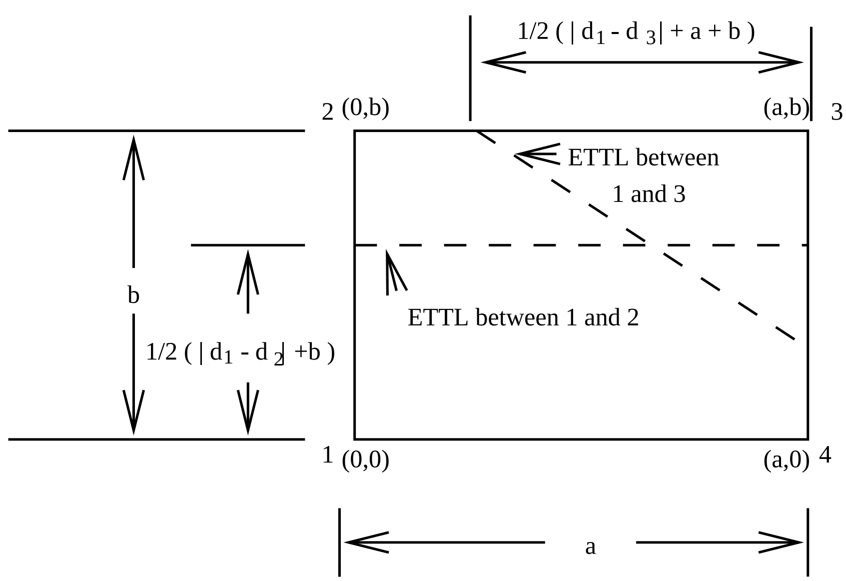


Figure 6: ETTLs in a cell

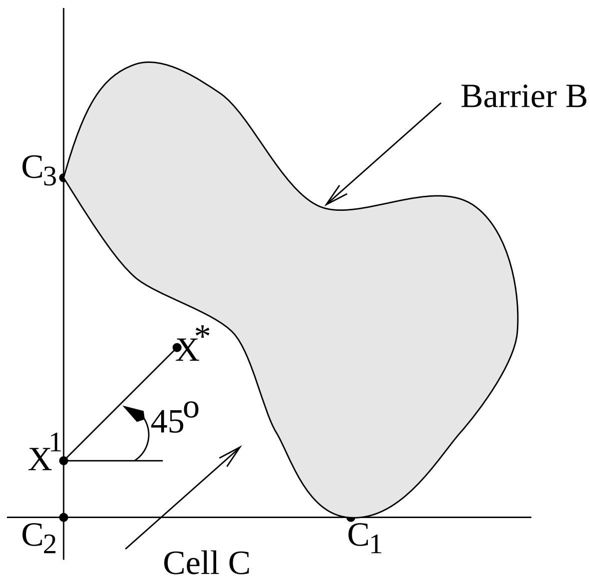


Figure 7: The case of a 3-corner cell (used in the proof of Theorem 1)

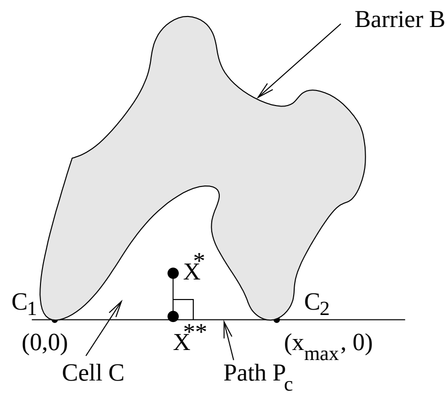


Figure 8: The case of a 2-corner cell

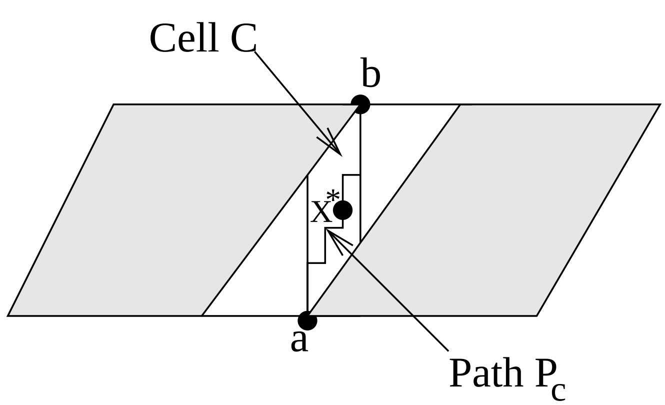


Figure 9: A 2-corner cell that may not be dominated by an adjacent cell

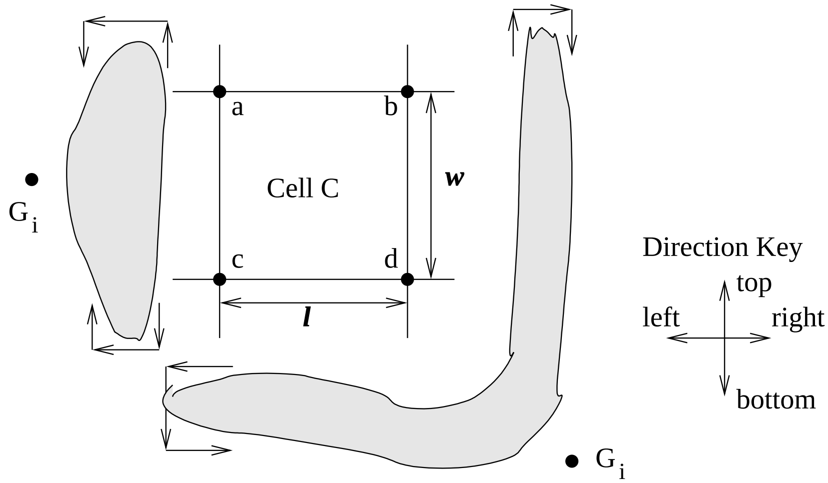
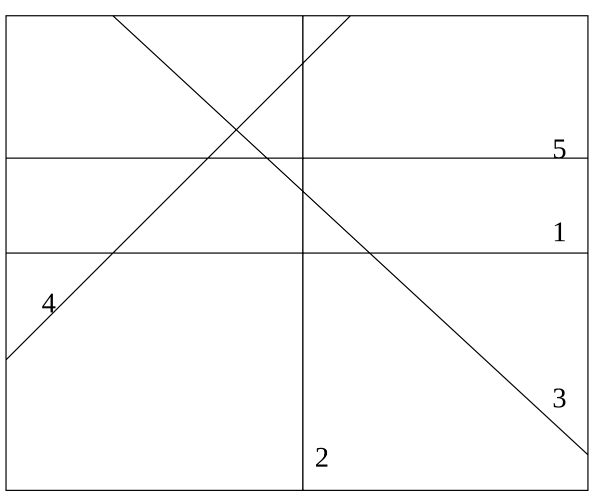


Figure 10: A 4-corner cell used in Theorem 2



n	Number of sub−cells
1	2
2	4
3	7
4	11
5	15
6	20
7	33
8	40, etc.

Figure 11: Maximal number of sub-cells generated by alternating ETTLs