The K-Connection Location Problem in a Plane

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Abstract

This paper determines the optimal location of K connections in the plane, where a connection links pairs of existing facilities. Both uncapacitated and capacitated versions of the problem are considered. Discretization results for general polyhedral gauges and other properties are established. Two heuristic algorithms are developed for each case using the concept of a shortest path flow set coupled with a sequential location and allocation approach. Computational results show that the algorithms are efficient and accurate.

Keywords: planar location; connection; capacity.

Introduction

This paper considers a location problem in which flows between pairs of facilities must flow through a connection. Examples of such connections include an input/output (I/O) station of a department in a facility layout (Montreuil and Ratliff (1988)), a hub in hub-and-spoke network (O'Kelly (1986)), and an international land border crossing between two countries (Robenhymer and Estrada (1998)). We seek to locate a given number of connections and allocate flows to them, with the goal of minimizing transportation cost. Both the cases of capacitated and uncapacitated connections are considered.

Three of the co-authors have considered discrete versions of this problem. In a recent paper, Huang, Batta, and Nagi (2003) discuss the discrete version when the capacity of the connection is variable. In another paper, Huang, Batta, and Nagi (2002) consider the discrete case while modeling the connections as $M/G/1$ queues. In these papers the potential locations are from a discrete set but the problem is NP-hard. Therefore if a fine discretization of the continuous version were to be treated, the above approaches would quickly become inapplicable. Therefore, we use a direct approach in this paper to address the planar version of the problem. This requires significant new analysis and computational procedures that constitute the major contribution of this paper.

We note that the planar connection location problem is related to three well-studied planar location problems: the planar K-median problem, the location-allocation problem and the hub location problem. We now summarize each of these problems and will later draw upon these to perform our analysis and solution methods. The planar K-median problem (see, e.g., Plastria (1995) for a review) seeks to locate a set of new facilities with respect to a set of existing facilities in the plane so as to minimize costs. It turns out that the optimal locations for new facilities only depend on their geographical relationship with respect to each single existing facility, since the interactions between new facilities and existing facilities are given. In our situation, the optimal locations for connections depend on each pair of origin-destination facilities. We also have to assign each flow to a specific connection. However, as we shall see in Section 2.2, when the assignments are given, our problem reduces to the planar K-median problem.

Another closely related problem is the location-allocation problem addressed in the papers by Cooper (1963) and Cooper (1964): an overview of applications of location-allocation problems is given in Hodgon, Rosing, and Shmulevitz (1993). Here the focus is to locate a set of K new facilities in a plane and allocate a set of M existing facilities to the new facilities in order to minimize total weighted distances. When we let each pair of origin and destination facilities be the same facility, our problem reduces to the planar location-allocation problem. The principal differences between these problems are:

- The connection location problem has interactions among the existing facilities.
- For M existing facilities and K new facilities or connections, the location-allocation problem must assign M demands to new facilities. However, the connection location problem must assign $M(M-1)$ pairs of flows to connections.
- The aggregation technique is a common approach to reduce the problem size in the location-allocation problem: see Francis, Lowe, and Rayco (1996). This approach is not suitable for the connection location problem because we cannot cluster flows using proximity as a criterion.

Finally, the connection location problem can also be viewed as a one-stop hub location problem, which is a special case of the planar hub location problem by O'Kelly (1986). Discussions of the planar hub location problem can be found in Aykin (1988), O'Kelly (1992) and Aykin and Brown (1992). However, the connection location problem assumes that there are no interactions between connections. This restriction helps to establish some discretization results and to develop more effective algorithms. Further, we propose both uncapacitated and capacitated cases, which are not considered in the planar hub location problem.

This paper is organized as follows: Section 1 provides some preliminaries. Section 2 considers the uncapacitated K-connection location problem and a mathematical formulation of the problem is provided. Some properties and an algorithm for this case are developed under the Manhattan distance. Section 3 discusses the capacitated K-connection location problem and an algorithm under the Manhattan distance is proposed. Section 4 reports computational results. Finally, Section 5 provides a summary and gives directions for future work.

1 Preliminaries

We let $\{Ex_i, i = 1, \ldots, M\}$ be the given set of existing facilities and M be the total number of existing facilities. We shall denote the origin-destination flow between existing facilities Ex_i and Ex_j by (Ex_i, Ex_j) and the flow amount from Ex_i to Ex_j by w_{ij}. For simplicity, we assume that $w_{ii} = 0$ and that the connections allow bi-directional flow. Other variations (e.g., unidirectional flows) can be analyzed in a similar manner. We want to locate a total of K connections, $X_1, ..., X_K$, in the plane $\Re^2$. Let $d(Ex_i, X_k)$ be the distance function between existing facility Ex_i and connection X_k, and w_{ijk} be the fraction of flow amount from Ex_i to Ex_j, w_{ij}, by way of connection $k \in \{1, ..., K\}$.

We use the notation $Z^P(\boldsymbol{x})$ to denote the objective function of a certain problem, where (P) indicates the problem and bold $\boldsymbol{x}$ indicates the vector of decision variables.

2 Uncapacitated K-connection location problem

2.1 Formulation

The uncapacitated K-connection location problem can be formulated as follows:

$$(P_1) \qquad \min_{\boldsymbol{x}, \boldsymbol{w}} Z^{P_1}(\boldsymbol{x}, \boldsymbol{w}) = \sum_{i=1}^{M} \sum_{j=1}^{M} \sum_{k=1}^{K} w_{ijk}(d(Ex_i, X_k) + d(X_k, Ex_j)), \tag{1}$$

$$\text{subject to} \qquad \sum_{k=1}^{K} w_{ijk} = w_{ij}, \qquad \forall\, i, j, \tag{2}$$

$$w_{ijk} \geq 0, \qquad \forall\, i, j, k. \tag{3}$$

where $\boldsymbol{x} = (X_1, \ldots, X_K) \in \Re^{2K}$ and $\boldsymbol{w} = (w_{ijk})_{i,j \in \{1,\ldots,M\}, k \in \{1,\ldots,K\}} \in \Re^{M^2 K}$. The objective function (1) minimizes the total transportation cost. Constraints (2) stipulate that the flows are only transported through connections. Constraints (3) are non-negativity constraints. The major difficulty in solving this problem is due to the form of the objective function, which is neither convex nor concave because we can find a Hessian matrix that is neither positive nor negative semidefinite.

2.2 Relationship to the K-median Problem

We define a related K-median problem as follows: For given w_{ijk}, let $\bar{w}_{ik} = \sum_{j=1}^{M} w_{ijk} + \sum_{j=1}^{M} w_{jik}$, which is the sum of inflow and outflow amounts at the existing facility Ex_i by way of connection k. We refer to a special case of a planar multifacility location problem where for the set of existing facilities Ex_i, $i \in \{1, ..., M\}$, the location of K connections, $\{X_1, ..., X_K\}$, is sought with respect to accumulated demands $\bar{w}_{ik}$ of existing facility Ex_i that is served by connection X_k. Then the related K-median problem in the plane is formulated as follows:

$$(RP_1) \qquad \min_{\boldsymbol{x}} Z^{RP_1}(\boldsymbol{x}) = \sum_{i=1}^{M} \sum_{k=1}^{K} \bar{w}_{ik} d(Ex_i, X_k). \tag{4}$$

From the definition, if we know the optimal assignment variables w^*_{ijk}, the uncapacitated K-connection location problem is equivalent to the related K-median problem. This result is in the following theorem.

Theorem 1: With known optimal assignment variables w^*_{ijk}, (P_1) is equivalent to (RP_1). Furthermore, (RP_1) can be reduced to K independent 1-median problems.

Proof: The proof can be easily done by rewriting the objective function of (P_1) and switching the order of summations. Supposing that $X^*_k, \forall\, k$ and $w^*_{ijk}, \forall\, i, j, k$, is an optimal solution of the problem (P_1), we can rewrite the objective function as follows:

$$Z^{P_1}(\boldsymbol{x^*}, \boldsymbol{w^*}) = \sum_{i=1}^{M} \sum_{k=1}^{K} (\sum_{j=1}^{M} w^*_{ijk}) d(Ex_i, X^*_k) + \sum_{j=1}^{M} \sum_{k=1}^{K} (\sum_{i=1}^{M} w^*_{ijk}) d(X^*_k, Ex_j)$$

$$= \sum_{i=1}^{M} \sum_{k=1}^{K} (\sum_{j=1}^{M} w^*_{ijk}) d(Ex_i, X^*_k) + \sum_{i=1}^{M} \sum_{k=1}^{K} (\sum_{j=1}^{M} w^*_{jik}) d(X^*_k, Ex_i)$$

$$= \sum_{i=1}^{M} \sum_{k=1}^{K} \bar{w}^*_{ik} d(Ex_i, X^*_k).$$

Since $(\boldsymbol{x^*}, \boldsymbol{w^*})$ is optimal for the problem (P_1), $\boldsymbol{x^*}$ is optimal for the problem (RP_1) with $\bar{w}^*_{ik} = \sum_{j=1}^{M} w^*_{ijk} + \sum_{j=1}^{M} w^*_{jik}$, $i \in \{1, \ldots, M\}$, $k \in \{1, \ldots, K\}$. Otherwise, we can find better connection locations, X_k, for the problem (P_1).

On the other hand, if we know the optimal assignment variables w^*_{ijk} and let $\bar{w}_{ik} = \sum_{j=1}^{M} w^*_{ijk} + \sum_{j=1}^{M} w^*_{jik}$, by the same argument, all optimal solutions of the problem (RP_1) are optimal for the corresponding restricted problem (P_1). Since we assume in this model that there is no interaction between the connections, they can reduce to K independent 1-median problems. Thus the result follows. ■

The following theorem allows us to search for the optimal solution of the uncapacitated K-connection location problem within a specified region.

Theorem 2: Let $d(\cdot)$ be a distance function such that the set of optimal solutions of every related 1-median problem with distance function $d(\cdot)$ lies within the convex hull of all existing facilities. Then the set of optimal solutions of the uncapacitated K-connection location problem has the same property, i.e., all optimal connection locations lie within the convex hull of the existing facilities.

Proof: Let $(\boldsymbol{x}^*, \boldsymbol{w}^*)$ be an optimal solution of the uncapacitated K-connection location problem. Then $\boldsymbol{x}^* = \{X^*_1, ..., X^*_K\}$ is also an optimal solution of the related K-median problem (Theorem 1) and each $X^*_k, \forall\, k$, is the optimal solution of the related 1-median problem. We can conclude that each $X^*_k, \forall\, k$, has to be located in the convex hull of the existing facilities. ■

Lemma 1: Let $d(\cdot)$ be a distance function such that at least one optimal solution of every related 1-median problem with distance function $d(\cdot)$ lies within the convex hull of all existing facilities. Then at least one optimal solution of the uncapacitated K-connection location problem has the same property, i.e., for at least one optimal connection location all connection locations lie within the convex hull of the existing facilities.

Proof: Since the related K-median problem reduces to K independent related 1-median problems with distance function $d(\cdot)$, there exists an optimal connection location $X^*_k, \forall\, k$, in the convex hull of the existing facilities for all $k = 1, ..., K$. Recalling that there is no interaction between the connections, the result follows. ■

Since the connections are uncapacitated, if connection locations are fixed, the optimal flow assignment can be obtained by assigning flows to the nearest connections. Thus, we have the following corollary.

Corollary 1: If connection locations are given, the uncapacitated K-connection location problem reduces to a series of shortest path problems, one for each pair of existing facilities.

According to Corollary 1, we do not obtain any benefit by assigning a flow to more than one connection. In order to have a theoretical lower bound to compare the results obtained for a given problem instance by several different solution methods, we state the following lemma.

Lemma 2: For a given uncapacitated K-connection location problem, a lower bound for the optimal objective function value is

$$\sum_{i=1}^{M}\sum_{j=1}^{M} w_{ij} d(Ex_i, Ex_j).$$

Proof: The result follows from a relaxation in the number of connections such that each flow is permitted to travel on its shortest path. ■

This means that if all flows are allowed to go along shortest paths, the resulting objective function value is the smallest.

2.3 Discretization result for polyhedral gauges

For polyhedral gauges, the uncapacitated K-connection location problem can be reduced to a discrete K-median problem with a finite dominating set. This result has the following advantages:

- Efficient heuristic solutions for the discrete K-median problem can be found in, for example, Daskin (1995) and Mirchandani and Francis (1990).
- The discrete K-median problem can be formulated as a linear integer problem, which can be solved optimally for problems of a relatively small size by a standard solver like CPLEX. This allows us to benchmark the heuristic algorithm described later.
- It helps us to search a good set of initial solutions in our solution approach.

A polyhedral gauge is defined by its unit ball, which is a convex polyhedron P in the plane $\Re^2$ containing the origin $O = (0, 0)$. Then

$$\gamma(X) := inf\{\lambda > 0 : X \in \lambda P\}, \forall X \in \Re^2.$$

is the corresponding polyhedral gauge defining a distance function by

$$d(X,Y) := |\gamma(X) - \gamma(Y)|.$$

Let $exp(P) = \{v_1, \ldots, v_N\}$ be the set of extreme points of P. Then Ward and Wendell (1985) and Nickel (1995)) showed that

$$\gamma(X) := min\{\sum_{l=1}^{N} \lambda_l : X = \sum_{l=1}^{N} \lambda_l v_l \text{ and } \lambda_l \geq 0, \forall l = 1, \cdots, N\}.$$

The extreme points of P define fundamental directions as the half-lines starting at the origin with direction v_l. The difference between a polyhedral gauge and a norm is that the symmetry assumption is dropped in the definition of a distance measure when the polyhedral gauge is defined. An example of a polyhedral gauge with five fundamental vectors and an example of the Manhattan norm are given in Figure 1.

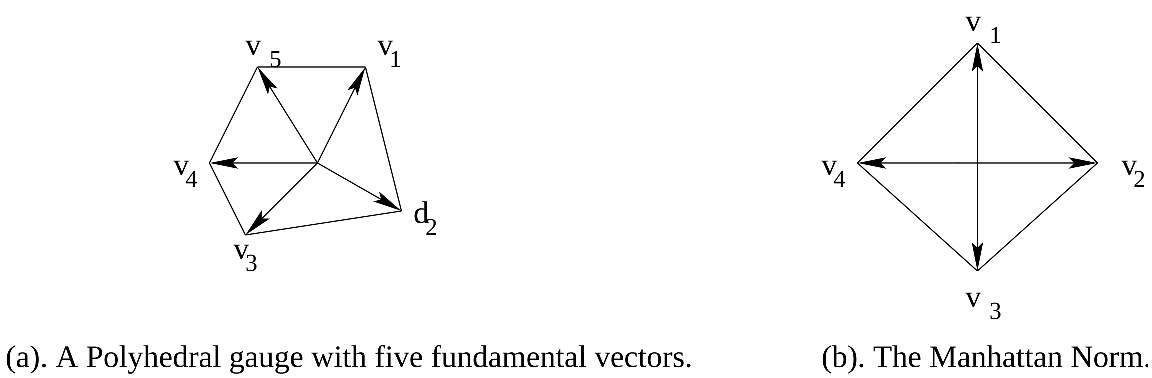

Figure 1: An example of a polyhedral gauge with five fundamental vectors and an example of a Manhattan norm

Durier and Michelot (1985) derived a discretization result for the unrestricted 1-median problem with polyhedral gauges. Rooting the fundamental directions as construction lines at each existing facility $Ex_i, \forall\, i$, a grid tesselation G of the plane is defined as follows:

Let $X + v_l := \{X + \lambda v_l : \lambda \in \Re\}$. Then the grid G is given by

$$G := \cup_{i=1}^{M} \cup_{k=1}^{N} (Ex_i + v_k).$$

Using this grid, the result of Durier and Michelot (1985) can be used to derive a similar property for the uncapacitated K-connection location problem with polyhedral gauges:

Lemma 3: The set of all optimal locations of the 1-median problem with polyhedral gauges consists of complete cells of the grid, lines connecting two adjacent grid points of a cell, and single grid points.

Here, the set of cells of G is defined as the set of all polyhedra in $\Re^2$ induced by G, the nonempty interior of which is not intersected by a line segment in G. A grid point is then an extreme point of a cell (which is equivalent to saying that it is an intersection point of grid lines), and a facet of the grid is a facet of a cell.

Theorem 3: The set of optimal locations of each connection $X_k, \forall\, k = \{1, ..., K\}$, of the uncapacitated K-connection location problem with polyhedral gauges can be partitioned into subsets that are either complete cells of G, facets of G, or grid points of G.

Proof: Let X_k^* and $w_{ijk}^*, \forall\, i, j, k$, with objective function value $Z^{P_1}(\boldsymbol{x^*}, \boldsymbol{w^*})$ be an optimal solution of the problem (P_1) with polyhedral gauges. Following Theorem 1 there exist K independent 1-median problems with objective function $Z_k^{RP_1}(\cdot)$ such that $X_k^*, \forall\, k$, is an optimal solution with respect to $Z_k^{RP_1}(\cdot)$ and

$$Z^{P_1}(\boldsymbol{x^*}, \boldsymbol{w^*}) = \sum_{k=1}^{K} Z_k^{RP_1}(X_k^*).$$

For each $k \in \{1, ..., K\}$ the set of optimal locations of the related 1-median problem consists of complete cells of the grid, lines connecting two adjacent grid points of a cell, and single grid points. Since all these optimal locations have the same objective function value $Z_k^{RP_1}(X_k^*)$, they are also optimal locations for the uncapacitated K-connection location problem with polyhedral gauges. ■

The following discretization result is now an immediate consequence of the above theorem and Lemma 1.

Corollary 2: There exists at least one optimal solution of the uncapacitated K-connection location problem with polyhedral gauges for which each connection point is located on a grid point of $G \cap conv\{Ex_i : i \in \{1, ..., M\}\}$.

This corollary turns the continuous uncapacitated K-connection location problem into a discrete version of the uncapacitated K-connection location problem whose finite dominating set contains those grid points of G that are located within the convex hull of the existing facilities. For a given polyhedral gauge γ with N fundamental directions, this candidate set is at most of size $O(M^2N^2)$, where M is the total number of existing facilities. Selecting K locations out of $O(M^2N^2)$ candidates is a hard combinatorial problem. Thus the need for a heuristic solution procedure for the planar case becomes apparent.

2.4 Heuristic algorithm under the Manhattan distance

The method of sequential location and allocation (SLA) is used here to develop our heuristic algorithm. Usually, this approach terminates with a local minimum and there could be a large number of local minima even for moderate sized cases: see Eilon, Watson-Gandy, and Christofides (1971) and Brandeau and Chiu (1993). The efficiency of the solution depends mainly on the selection of initial locations. If the proper initial locations are selected, the solution could be very good, even optimal. In the following, we try to find a good set of initial locations for the connections based on the structure of the uncapacitated K-connection location problem. For simplicity, we develop the heuristic algorithm under the Manhattan distance, which is a special case of polyhedral gauges. The idea of the algorithm can be extended to the general polyhedral gauge case.

To facilitate the development of the heuristic algorithm under the Manhattan distance for (P_1), the grid construction for the Manhattan distance is as follows: Consider the smallest rectangle (*bounding rectangle*) that encloses all existing facilities and connections and whose sides are parallel to the x and y axes. Within this bounding rectangle, the grid is formed by lines parallel to the x and y axes through all existing facility nodes. A grid point is an intersection point of any two lines.

In order to select K relatively efficient initial connection locations, we define the *shortest path flow set* for each grid point as follows. Let (i,j) refer to a pair of existing facilities Ex_i and Ex_j with coordinates (x_i, y_i) and (x_j, y_j), respectively, and s refer to any grid point with location coordinates (x_s, y_s), then the *shortest path flow set* P_s is given by:

$$P_s := \{(i,j) : x_i \leq x_s, y_i \geq y_s, x_j \geq x_s, y_j \leq y_s\} \cup \{(i,j) : x_i \leq x_s, y_i \leq y_s, x_j \geq x_s, y_j \geq y_s\}$$

$$\cup\{(i,j) : x_i \geq x_s, y_i \geq y_s, x_j \leq x_s, y_j \leq y_s\} \cup \{(i,j) : x_i \geq x_s, y_i \leq y_s, x_j \leq x_s, y_j \geq y_s\}.$$

Under the Manhattan distance, if we select s as a connection location, then for all $(i,j) \in P_s$ the shortest path from Ex_i to Ex_j passes through s. Figure 2 provides examples of sets P_s.

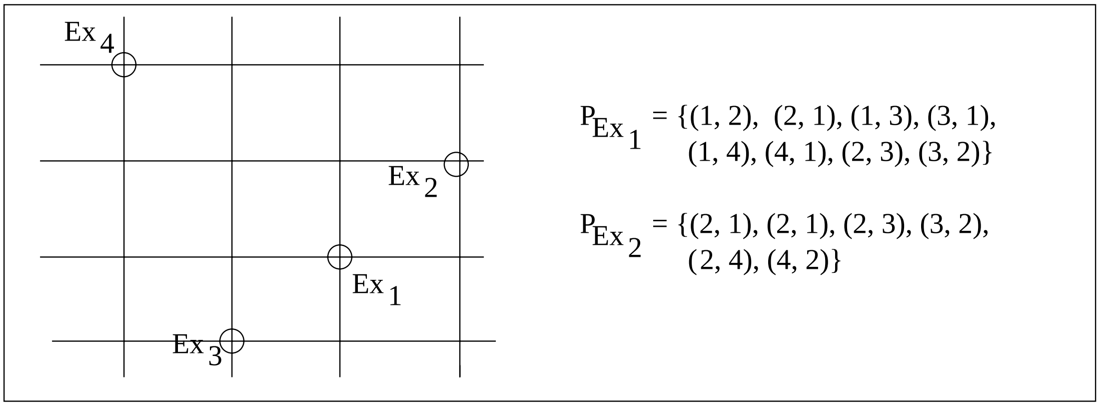

Figure 2: An illustration of the shortest path flow set

Let $w_s = \sum_{(i,j)\in P_s} w_{ij}$. Then w_s is the total flow amount by way of s with the shortest distance. Intuitively, the larger w_s is, the more important the grid point s is. Thus, the basic idea for this heuristic algorithm is to use the order of w_s to find an initial location solution. Then according to Corollary 1 we can allocate flows to the connection locations. Finally, by Theorem 1, solving K independent 1-median problems gives the relocation of the connections. This cycle of allocating and relocating is repeated until no further improvement can be made.

We define F to be the set of all pairs (i,j) and F^u to be the set of (i,j) that are currently not assigned to any connection. The heuristic algorithm is as follows:

- Step 1 (Location). Initialize $S = \{s : s \text{ is a grid point}\}$ and $F^u := F$. Calculate $w_s = \sum_{(i,j)\in P_s \cap F^u} w_{ij}, \forall s \in S$. Pick the grid point s with the largest w_s value as a candidate connection location and assign all $(i,j) \in P_s \cap F^u$ to this connection. Update $F^u := F^u \setminus P_s$ and $S := S \setminus \{s\}$, and repeat the procedure until we have picked K connection locations or F^u is empty. If F^u is empty, then stop; the solution is optimal due to Lemma 2. Otherwise, go to step 2.

- Step 2 (Allocation). For the selected K candidate connections, assign the flows by solving the corresponding shortest path problems (see Corollary 1) and obtain the w_{ijk}

values.

- Step 3 (Relocation). Using the w_{ijk} found in step 2, solve the related 1-median problems to relocate the connection locations (Theorem 1) and obtain K new candidate connection locations. If no further improvement can be made in the value of objective function, stop; otherwise, go to Step 2.

The computational performance of the algorithm is presented in Section 4.

3 Capacitated K-connection location problem

In this section we take capacities of the connections into account. The capacity of a connection is defined as the maximum unit time rate at which flows can be expected to traverse the connection. We note that the capacitated planar location-allocation problem is seldom considered in the open literature. One exception is the paper by Cooper (1972) which discusses such a problem on the plane in a transportation-location context. However, this paper only considers problems of a relatively small size. In the following, we build some properties and develop an algorithm which is able to solve very large problems efficiently and accurately.

3.1 Formulation

Let the capacities of the connections be c_1, c_2, ..., c_K, respectively. The capacitated K-connection location problem can be formulated as follows:

$$(P_2) \qquad \min_{\boldsymbol{x},\, \boldsymbol{w}} Z^{P_2}(\boldsymbol{x}, \boldsymbol{w}) = \sum_{i=1}^{M} \sum_{j=1}^{M} \sum_{k=1}^{K} w_{ijk}(d(Ex_i, X_k) + d(X_k, Ex_j)) \tag{5}$$

$$\begin{aligned} \text{subject to} \quad \sum_{k=1}^{K} w_{ijk} &= w_{ij}, \qquad \forall\, i, j, && (6) \\ \sum_{i=1}^{M} \sum_{j=1}^{M} w_{ijk} &\leq c_k, \qquad \forall\, k, && (7) \\ w_{ijk} &\geq 0, \qquad \forall\, i, j, k. && (8) \end{aligned}$$

Constraint (7) is a capacity constraint. This mathematical formulation is a continuous programming problem with neither convex nor concave objective function and linear constraints.

We now develop some properties and an algorithm for the problem.

3.2 General properties

Since the objective function of the capacitated K-connection location problem is the same as that of the uncapacitated K-connection location problem, they share some similar characteristics. On the other hand, some properties are not valid in the capacitated model.

3.2.1 Properties related to the uncapacitated case

Both the capacitated and uncapacitated versions share the same objective function and hence Theorem 1 is still true in the capacitated case. We therefore obtain the following discretization result:

Theorem 4: There exists at least one optimal solution of the capacitated K-connection location problem with polyhedral gauges for which each connection point is located on a grid point of $G \cap conv\{Ex_i : i \in \{1, ..., M\}\}$.

Since the feasible set of the uncapacitated model contains that of the capacitated case and they have the same objective function we obtain Theorem 5.

Theorem 5: The optimal objective function value of the problem (P_1) is a lower bound of that of the problem (P_2).

The following corollary is an immediate consequence of Lemma 2 and Theorem 5.

Corollary 3: For a given capacitated K-connection location problem, a lower bound for the optimal objective function value is

$$\sum_{i=1}^{M} \sum_{j=1}^{M} w_{ij} d(Ex_i, Ex_j).$$

3.2.2 Other properties

In Corollary 1, we prove that if connection locations are given, the uncapacitated K-connection location problem reduces to a shortest path problem for each pair of existing facilities. This is not true in the capacitated model because of the capacity restriction on the connections. The following example (see Figure 3) shows that.

Suppose that there are two flows, $A-B$ (2 units) and $C-D$ (4 units), in the system. We have to locate two connections. The capacities for the connections are 1 unit and 5 units, respectively. Thus, the optimal locations are as follows: One connection (1 unit) is located

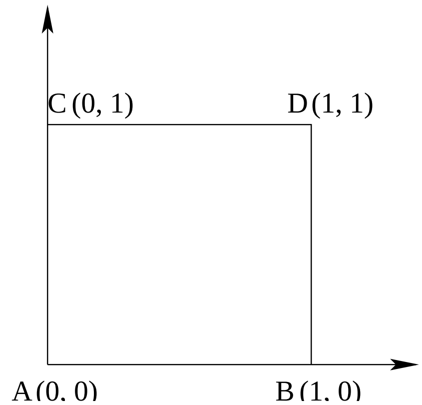

Figure 3: Example

on the line AB and the other (5 units) is located on the line CD. But one unit of flow $A-B$ does not go through the nearest connection. However, we have the following theorem:

Theorem 6: Given the locations of the connections, the capacitated K-connection location problem reduces to a transportation problem.

Proof: If the locations, X_k, are given, then all possible travel costs in the objective function are a set of constant costs. Let $d_{ijk} = d(Ex_i, X_k) + d(X_k, Ex_j))$, then the problem (P_2) becomes:

$$\min_{\boldsymbol{w}} Z(\boldsymbol{w}) = \sum_{i=1}^{M}\sum_{j=1}^{M}\sum_{k=1}^{K} d_{ijk}w_{ijk} \tag{9}$$

$$\begin{aligned}
\text{subject to} \qquad \sum_{k=1}^{K} w_{ijk} &= w_{ij}, && \forall\, i,j, && (10)\\
\sum_{i=1}^{M}\sum_{j=1}^{M} w_{ijk} &\le c_k, && \forall\, k, && (11)\\
w_{ijk} &\ge 0, && \forall\, i,j,k. && (12)
\end{aligned}$$

If we treat each origin-destination pair (i, j) as a demand source and each connection as a supply, then the above formulation is a usual form of the transportation problem. ■

3.3 Heuristic algorithm

For simplicity, we develop heuristic algorithms under the Manhattan distance. The heuristic algorithm for the capacitated case is similar to that for the uncapacitated problem. We use the method of sequential location and allocation (SLA) and the same grid structure described in Section 2.4. Since we have the discretization result (Theorem 4), the *shortest path flow set* algorithm for the uncapacitated case can be used here. In the capacitated case, however, we may not be able to assign all $(i, j) \in P_s$ to a single connection as we did in Section 2.4 because of the capacity limit. Therefore, minor modifications are needed for the capacitated case.

For any flow (i, j) with the location coordinates (x_i, y_i) and (x_j, y_j), respectively, we define $A_{ij} = |x_i - x_j| \cdot |y_i - y_j|$,where A_{ij} is the area of a rectangle that has two diagonally opposite vertices (x_i, y_i) and (x_j, y_j). We denote the rectangle as R_{ij}. If a connection is located within this rectangle, the travel distance between the existing facilities Ex_i and Ex_j by way of the connection is the shortest one. On the other hand, $\forall(i, j) \in P_s$, R_{ij} contains s. In general, if A_{ij} is bigger, there could be more grid points within R_{ij}, and (i, j) could belong to more *shortest path flow sets*. Now, for any two flows $(i, j) \in P_s$ and $(k, l) \in P_s$, if $A_{ij} \leq A_{kl}$, then there are more opportunities for flow (k, l) to go through the shortest route. Therefore, when we are not able to assign all $(i, j) \in P_s$ to a single connection as we did in Section 2.4, we assign the flows (i, j) with smaller A_{ij} first.

Let $C = \{c_1, c_2, \cdots, c_K\}$. Also, we define F to be the set of all pairs (i, j) and F^u to be the set of (i, j) that are not assigned to any connection. Thus, the heuristic algorithm is as follows:

- Step 1 (Location). Initialize $S = \{s : s \text{ is a grid point}\}$ and $F^u := F$. Calculate $w_s = \sum_{(i,j)\in P_s \cap F^u} w_{ij}, \forall s \in S$. Pick the grid point s with the largest w_s as a candidate connection location and the largest capacity in C, say c_b. If $w_s \leq c_b$, assign all $(i, j) \in P_s \cap F^u$ to the selected candidate connection s, update $F^u := F^u \setminus P_s$, $C := C \setminus \{c_b\}$ and $S := S \setminus \{s\}$. If $w_s > c_b$, assign as many as possible flows till the capacity constraint is reached in the order of smallest A_{ij} first (ties broken arbitrarily), $\forall(i, j) \in P_s \cap F^u$. If the last flow assigned is partially assigned, then change it to the remaining unassigned amount. Let P_s' be the set of the fully assigned flows, update $F^u := F^u \setminus P_s'$, $C := C \setminus \{c_b\}$ and $S := S \setminus \{s\}$. Repeat the procedure until we have picked K connection locations or F^u is empty. If F^u is empty, then stop; the solution is optimal for the problem (since the value of the objective function equals to the lower bound, see

Corollary 3). Otherwise, go to step 2.

- Step 2 (Allocation). For the selected K connection locations, assign all the flows by solving the related transportation problem (Theorem 6) and obtain the w_{ijk} values.

- Step 3 (Relocation). Using the w_{ijk} found in Step 2, solve the related 1-median problems to relocate the connection locations (Theorem 1). If no further improvement can be made in the value of the objective function, stop; otherwise, go to Step 2.

The computational performance of this algorithm is presented in the next section.

4 Computational experience

In this section, we test the performance of the algorithms for both the uncapacitated and the capacitated models. The efficiency of the algorithms is tested by solving randomly generated problems of different sizes. Then, for small problems, we assess the accuracy of the heuristic solutions with the optimal solutions or linear relaxation lower bound, which can be found by solving discrete uncapacitated median problems (for the uncapacitated case) or capacitated median problems (for the capacitated case) using CPLEX 7.5 on a Dell X86 Pentium 3 with 256MB RAM. The algorithms were coded in C++.

4.1 Data generation

First, we generated each existing facility's location, which was given by its x and y coordinates. These coordinate values were randomly selected from $U(0, 1000)$, where U denotes a uniform distribution. For each pair of origin-destination facilities, the amount of flow was randomly drawn from $U(5, 30)$. The parameter values for test problems are summarized in Table 1.

For small-sized problems, the accuracy of the algorithms is assessed by heuristic gap, which is defined by *(heuristic solution value - lower bound)/lower bound * 100.* For larger problems, the CPU time is used to evaluate the efficiency of our algorithms. The stopping rule is that the improvement of the objective value is less than 0.0001.

Table 1: Parameter values for test problems

	parameter	small	medium	large
1	number of facilities	10-30	55-65	70-80
2	number of connections	5-15	30-40	45-55
4	number of non-zero flows	45-435	1485-2080	2415-3160

4.2 Uncapacitated K-connection location problem

According to Corollary 2, the planar problem can be discretized and the resulting problem can be transformed into a discrete uncapacitated median problem. The optimal solutions or linear relaxation lower bounds for small problems are found by using the CPLEX solver.

Table 2 shows the computational result for small problems. All these randomly generated problems can be solved within one second. The average heuristic gap is 0.99%. Table 3 shows the computational result for medium and large problems. We failed to obtain linear relaxation lower bounds because of insufficient computer memory to load the problem. But the CPU times demonstrate the efficiency of the algorithm.

Table 2: Computational results for the uncapacitated case (small size)

Facility #	Connection #	Flow #	Heuristic Obj. value	Heuristic CPU (s)	Lower Bound	Gap (%)	Optimal Value
10	5	45	114449	0.01	113761	0.60	113893
10	5	45	106173	0.01	104579	1.52	104859
10	5	45	102071	0.01	100823	1.24	101757
10	5	45	118733	0.01	115869	2.47	116481
20	10	190	455900	0.08	452100	0.84	
20	10	190	471442	0.08	469618	0.39	
20	10	190	468621	0.08	466511	0.45	
20	10	190	490026	0.08	478878	2.33	
30	15	435	1058624	0.27	1051594	0.67	
30	15	435	997987	0.27	993295	0.47	
30	15	435	1084293	0.27	1079285	0.46	
30	15	435	947359	0.27	942765	0.49	
					Average	0.99	

Table 3: Computational results for the uncapacitated case (medium and large size)

Facility #	Connection #	Flow #	Heuristic	
			Obj. value	CPU (s)
55	30	1485	3615871	2.473
60	35	1770	3783055	2.984
65	40	2080	4957825	5.679
70	45	2415	5538463	6.69
75	50	2775	6279571	8.903
80	55	3160	7743327	13.299

4.3 Capacitated K-connection location problem

For the capacitated case, the dicretization result is still available by Theorem 4. We assume that all capacities are equal in order to decrease the number of integer variables in the discretized problem The capacity is set to $1.5 \sum_{i=1}^{M} \sum_{j=1}^{M} w_{ij}/K$, which is 1.5 times the average flow per connection.

Table 4 shows the computational result for small problems. As in the uncapacitated model, all randomly generated problems can be solved within one second. The average heuristic gap is 1.70. The optimal solutions or linear relaxation lower bounds for small problems are found by CPLEX (the value with * is the best objective value obtained after one hour CPU time). Table 5 shows the computational result for medium and large problems. We failed to obtain a linear relaxation lower bound because of insufficient computer memory. However, the relatively low CPU times demonstrate the efficiency of the algorithm.

5 Conclusions and future work

In this work we have studied a K-connection location problem that may be encountered during the development of plant layout, urban planning, or the design of telecommunication and distribution networks. Both uncapacitated and capacitated models were investigated and the mathematical formulations were developed. For both models, we established discretization results for general polyhedral gauges as well as other relevant properties. This discretization leads to an NP-hard problem (NB: we are unaware of the NP-hardness of the

Table 4: Computational results for the capacitated case (small size)

Facility #	Connection #	Flow #	Heuristic Obj. value	Heuristic CPU (s)	Lower Bound	Gap (%)	Optimal Value
10	5	45	109818	0.01	105978	3.62	106188
10	5	45	89791	0.02	87197	2.97	87531*
10	5	45	94553	0.01	93685	0.93	93811
10	5	45	116505	0.01	113365	2.77	113461
20	10	190	467582	0.07	462080	1.12	
20	10	190	422553	0.06	418079	1.07	
20	10	190	452155	0.07	440847	2.57	
20	10	190	451348	0.08	443668	1.73	
30	15	435	947773	0.31	940287	0.80	
30	15	435	1124328	0.35	1111252	1.18	
30	15	435	1046923	0.30	1040269	0.64	
30	15	435	996091	0.40	986365	0.99	
					Average	1.70	

Table 5: Computational results for the capacitated case (medium and large size)

Facility #	Connection #	Flow #	Heuristic Obj. value	Heuristic CPU (s)
55	30	1485	3670076	6.029
60	35	1770	4301842	6.92
65	40	2080	4768813	7.521
70	45	2415	5676977	15.022
75	50	2775	6459764	21.111
80	55	3160	7308221	35.441

original continuous problem). Heuristic algorithms based on the *shortest path flow set* were proposed. The computational performances of the algorithms were studied by comparision of heuristic results and exact solutions or linear relaxation lower bounds for small randomly generated problems. For large problems, the algorithms were assessed in terms of their CPU time. Computational results showed that they were efficient and accurate (whenever it has been possible to verify).

There are several directions for further research. First, barriers, where not only placement but also trespassing of regions is forbidden, could be considered. The corresponding connection location models are mathematically challenging. A different algorithm should be developed since the shortest path set is not suitable in this case. Second, connections located in an arbitrary shaped region can be investigated. Connections restricted to be on a curve would appear to be an important special case. Third, the factors of spatial size and specific site of connection in the capacitated case could be considered. Finally, the number of connections can be also a decision variable and the installation cost of connections could be considered. The goal here would be to minimize the sum of transportation cost and installation cost.

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References

Aykin, T. (1988). On the location of hub facilities. *Transportation Science 22*, 155–157.

Aykin, T. and Brown, G. F. (1992). Interacting new facilities and location-allocation problem. *Transportation Science 26*, 212–222.

Brandeau, M. L. and Chiu, S. S. (1993). Sequential location and allocation: Worst case performance and statistical estimation. *Location Science 1*, 289–298.

Cooper, L. (1963). Location-allocation problems. *Operations Research 11*, 331–343.

Cooper, L. (1964). Heuristic methods for location-allocation problems. *SIAM Review 6*, 37–53.

Cooper, L. (1972). The transportation-location problem. *Operations Research 20*, 94–108.

Daskin, M. S. (1995). *Network and Discrete Location: Models, Algorithms, and Applications*. Wiley, New York.

Durier, R. and Michelot, C. (1985). Geometrical properties of the Fermat-Weber problem. *European Journal of Operational Research 20*, 332–343.

Eilon, S., Watson-Gandy, C. D. T. and Christofides, N. (1971). *Distribution Management: Mathematical Modelling and Practical Analysis*. Hafner, New York.

Francis, R. L., Lowe, T. J. and Rayco, M. B. (1996). Row-column aggregation for rectilinear distance p-median problems. *Transportation Science 30*, 160–174.

Hodgon, M. J., Rosing, K. E. and Shmulevitz, F. (1993). A review of location-allocation applications literature. *Studies in Locational Analysis 5*, 2–29.

Huang, S., Batta, R. and Nagi, R. (2002). Selection and sizing of congested connections for a transportation network. *Operations Research, submitted*.

Huang, S., Batta, R. and Nagi, R. (2003). Variable capacity sizing and selection of connections in a facility layout. *IIE Transactions 35*, 49–59.

Mirchandani, P. B. and Francis, R. L. (1990). *Discrete Location Theory*. John Wiley, New York.

Montreuil, B. and Ratliff, H. D. (1988). Optimizing the location of input/output stations within facilities layout. *Engineering Costs and Production Economics 14*, 177–187.

Nickel, S. (1995). *Discretization of Planar Location Problems*. Shaker Verlag, Aachen.

O'Kelly, M. (1986). The location of interacting hub facilities. *Transportation Science 20*, 92–106.

O'Kelly, M. (1992). A clustering approach to the planar hub location problem. *Annals of Operations Research 40*, 339–353.

Plastria, F. (1995). Continuous location problems. In Z. Drezner, (Ed.), *Facility Location: A Survey of Applications and Methods*, pp. 225–260. Springer, NY.

Robenhymer, B. and Estrada, S. (1998). Designing transit at the world's busiest international border crossing. *Presented at the 68th Annual Meeting of the Institute of Transportation Engineers, Toronto*.

Ward, J. E. and Wendell, R. E. (1985). Using block norms for location modelling. *Operations Research 33*, 1074–1090.