

Contour line construction for a new rectangular facility in an existing layout with rectangular departments

by

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Abstract

In a recent paper, Savas, Batta and Nagi [12] consider the optimal placement of a finite-sized facility in the presence of arbitrarily-shaped barriers under rectilinear travel. Their model applies to a layout context, since barriers can be thought to be existing departments and the finite-sized facility can be viewed as the new department to be placed. In a layout situation, the existing and new departments are typically rectangular in shape. This is a special case of the Savas *et al.* paper. However the resultant optimal placement may be infeasible due to practical constraints like aisle locations, electrical connections, etc. Hence there is a need for the development of contour lines, i.e. lines of equal objective function value. With these contour lines constructed, one can place the new facility in the best manner. This paper deals with the problem of constructing contour lines in this context. This contribution can also be viewed as the finite-size extension of the contour line result of Francis [3].

Keywords: Contour Line, Facility Layout, Facility Location.

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1 Introduction

According to Francis, McGinnis and White [4] and Bindschedler and Moore [2], a facilities layout problem may arise because of a change in the design of the product, the addition or deletion of a product from a company's product line, a significant increase or decrease in the demand for a product, changes in the design of the process, etc. Sometimes the layout has to be redesigned to include a new facility such as a single machine, cell or a department. Placement of a new facility in the presence of existing facilities can be considered as a "restricted layout problem" since in a plant layout the existing facilities will act as *barriers* where travel and new facility placement are not permitted.

The facilities location literature discusses problems in which restrictions are imposed upon where facilities can be placed (Hamacher [5], Nickel [10], Hamacher and Schöbel [6] and Klamroth [7]). Recognizing the practical relevance of facility size consideration, Savas *et al.* [12] consider the optimal placement of a finite-sized facility in the presence of arbitrarily-shaped barriers with the median objective and rectilinear distance metric. In a layout context, barriers may be thought of as existing facilities which are usually rectangular. Therefore a special case of their model in which the barriers and the facility are assumed to be rectangular, may be applied to a layout problem where a new rectangular facility has to be optimally placed in the presence of other rectangular facilities. In a layout context, the optimal site may not be always suitable for facility placement. For example the optimal site may pose concerns due to sharp corners, jogs or narrowing of material handling aisles. Hence there is a need to find a nearby location that is usable. Contour lines, that are lines of equal objective function value, help to evaluate the costs of locations other than optimal sites. They help to find the next best solution for an existing layout problem, when the new facility cannot be placed at its intended optimal location. Francis [3] has considered this problem in the context where facilities are points. The finite-area case is more appropriate for facilities layout.

The remainder of this paper is organized as follows. In Section 2, we describe and define the problem. In Section 3, we briefly visit the grid construction procedure of Larson and Sadiq [8] and the Equal Travel Time Line concept of Batta, Ghose and Palekar [1]. We then illustrate some new properties of Equal Travel Time Lines in Section 4. Section 5 illustrates the contour line construction procedure which is followed by a numerical example in Section 6. Section 7 describes the complexity of our solution procedure. Conclusions and directions for future research are presented in Section 8.

2 Problem Definition

We are given a finite number of rectangular existing facilities (EFs) in a 2D plane in which a rectangular new facility (NF) has to be placed. We assume that the NF is oriented with its sides parallel to the X and Y axes, and one of the four possible orientations is chosen. The procedure can be repeated for the remaining three orientations. Each EF is characterized by its four corner vertices and has one or more I/O points on its boundary. These EFs are labeled EF_1, \dots, EF_m . Material flow between the NF and the EFs takes place through a single I/O point X located on the NF boundary. Let (x, y) represent the coordinates of the I/O point of the NF, D be the set of all EF I/O points, and S be the rectangular space representing the shop floor area. Let $d(i, j : X)$ be the shortest feasible rectilinear distance between I/O points i and j ($i, j \in D \cup X$) given that X is located at (x, y) . Here the term feasible implies a path that does not penetrate the interior of the NF or EFs, which serve as impenetrable barriers to travel.

There are two types of material flows in the problem. One is the flow between the EFs and the NF and the other is the flow between all pairs of EFs. All flows takes place through a shortest feasible rectilinear distance path. In computing the distance, we permit travel on a EF or NF boundary but not inside. It certainly makes good sense to recognize that the placement of the NF may block shortest path movements between pairs of EF I/O points, as Fig. 1 illustrates. The same is true for movement between X and an EF I/O point. At the NF position in Fig. 1, the NF interferes with the flow between I/O points 1-3 and 1- X .

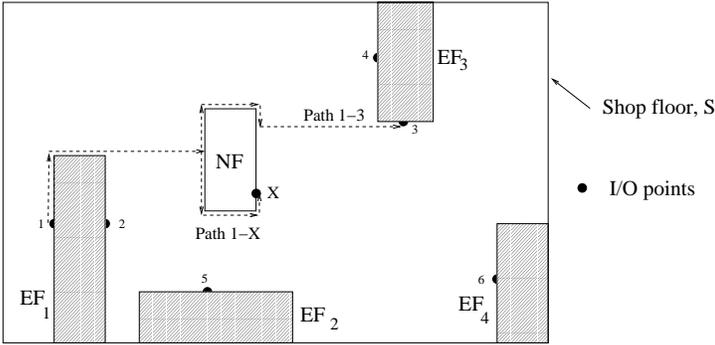


Figure 1: Example to illustrate that NF placement can affect the flow distance between a pair of EFs I/O points and between the NF I/O point and an EF I/O point

The NF needs to be placed such that: (a) the NF does not intersect the union of the interiors of

the EFs, and (b) the NF is contained in S . Placing the NF reduces to locating the NF I/O point X . We refer to the set of all feasible locations for X as F . Note that feasible placements allow the boundaries of the NF and EFs to coincide. Having a single I/O point for the NF is a modeling limitation made to avoid complications that can arise due to the flow assignment choices of EF I/O points to the NF I/O points.

For a given X , the total weighted travel distance between EFs and the NF is: $\sum_{i \in D} w_i d(i, X : X)$, where w_i is the rate of material flow between the i^{th} EF I/O point and I/O point of the NF at X . Similarly, the total weighted travel distance between all EFs is: $\sum_{i \in D} \sum_{j \in D} u_{ij} d(i, j : X)$, where u_{ij} is the rate of material flow between I/O points i and j of a pair of EFs. The objective function is the total material distance traveled per unit time between pairs of EF I/O points and between the NF I/O point and EF I/O points, and is given by:

$$f(X) = \sum_{i \in D} w_i d(i, X : X) + \sum_{i \in D} \sum_{j \in D} u_{ij} d(i, j : X).$$

For optimal placement (Savas *et al.* [12]), the problem is to find $X^* \in F$ such that $f(X^*) \leq f(X), \forall X \in F$. In the present paper we are interested in finding the set of all points that satisfy $f(X) = z$, where z is a constant. Such a set of points is referred to as a contour line in Francis [3].

3 Background

We divide the plane into regions and establish that a contour line is represented by a straight line in each region. The slope of the line potentially changes as we move from one region to the next. This is similar to the situation when EFs and the NF are points (refer Francis [3]). Though the method to find the slope in a region remains similar, the regions are more complex to determine in this case. Like the point case, a grid construction procedure is employed to determine the regions in a plane. Section 3.1 illustrates this. But in our case, additional regions need to be determined because of the effect of finite-sized NF and EFs on the traversal path of material flow. A finite-sized NF may interfere with the traversal path of flow in some regions of the plane. These interference regions are discussed in Section 3.2. In addition, EFs may create alternate traversal paths for material flow in regions of the layout. This is illustrated in Section 3.3.

3.1 Grid construction and cell formation

If the EFs and NF are points, the grid is obtained by drawing horizontal and vertical lines through each EF. When the EFs have finite dimensions, Larson and Sadiq’s approach [8] is needed. Specializing their approach for rectangular EFs, the grid can be constructed by drawing horizontal and vertical traversal lines from all the corners and the I/O points of the EFs, with each line terminating at the first EF encountered or at an edge of S . Figure 2 illustrates the grid lines constructed. Let L_h denote the set of horizontal traversal lines and L_v denote the set of vertical traversal lines. Also, let $L = L_h \cup L_v$ be the set of all *EF traversal lines*. The EFs and L divide the region $S - \bigcup_{i=1}^m \text{EF}_i$ into a number of *cells*. A cell is defined as a closed region in the grid that is not an EF. Cells have the property that a shortest feasible rectilinear path from an EF I/O point to a point located in the cell is passes through one of the cell corners (see Larson and Sadiq [8]), and that the length of this path is concave over the cell. Since we consider rectangular EFs, all the cells formed will also be rectangular. When a finite-sized NF is placed, a new set of traversal lines passing through its corners and I/O point are introduced. This new set of lines are referred to as *NF traversal lines*, $L'(X)$, when the NF I/O point is located at X .

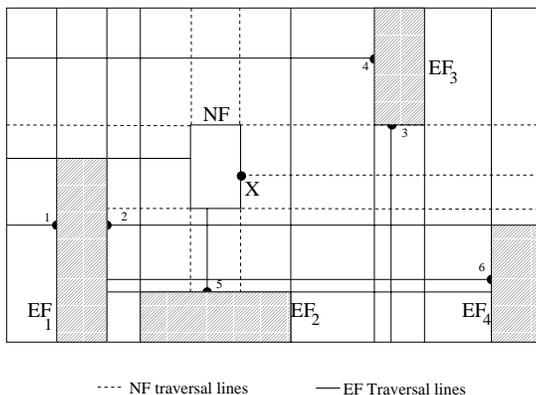


Figure 2: EF and NF traversal lines

3.2 Interference sets \mathcal{Q}

If a rectangular NF is fully-contained in a cell, the shortest feasible rectilinear path from its I/O point to an EF I/O point is still concave. In fact, the distance function is identical to that of a point NF located at X . However, when the rectangular NF intersects one or more EF traversal lines, distance measurement from its I/O point to an EF I/O point can be affected. The disruption can be

characterized provided the NF intersects a distinct set of grid lines as X is varied. This forms the motivation of interference sets. An interference set \mathcal{Q} is a set of placements X obtained by moving the NF such that the NF intersects a distinct set of grid lines (Savas *et al.* [12]). More precisely, for a set of grid lines $\mathcal{S} \subset L$, $\mathcal{Q}_{\mathcal{S}}$ is a closed region whose boundary is composed from two types of segments: (a) locations X such that the NF boundary intersects with some EF boundary, and/or (b) locations X such that some grid line(s) in set \mathcal{S} coincide with some NF traversal line(s). An example for the formation of sets \mathcal{Q} is illustrated in Fig. 3. Consider a NF being moved around a cell such that the NF intersects the subset of the traversal lines $\{h_i, h_{i+1}, v_j\}$. As illustrated, \mathcal{Q}_{h_i} , $\mathcal{Q}_{h_{i+1}}$, \mathcal{Q}_{h_i, v_j} , \mathcal{Q}_{v_j} and $\mathcal{Q}_{h_{i+1}, v_j}$ are the sets of X locations such that the NF intersects the sets of traversal lines $\{h_i\}$, $\{h_{i+1}\}$, $\{h_i, v_j\}$, $\{v_j\}$ and $\{h_{i+1}, v_j\}$ respectively. It is noted that each intersection set \mathcal{Q} is rectangular and mutually exclusive. \mathcal{Q}_{h_i, v_j} is shaded for ease of visualization (while the others are defined by their corners). For example, point 6 of \mathcal{Q}_{h_i, v_j} is generated when grid line h_i is coincident with NF traversal line h'_1 and grid line v_j is coincident with NF traversal line v'_2 .

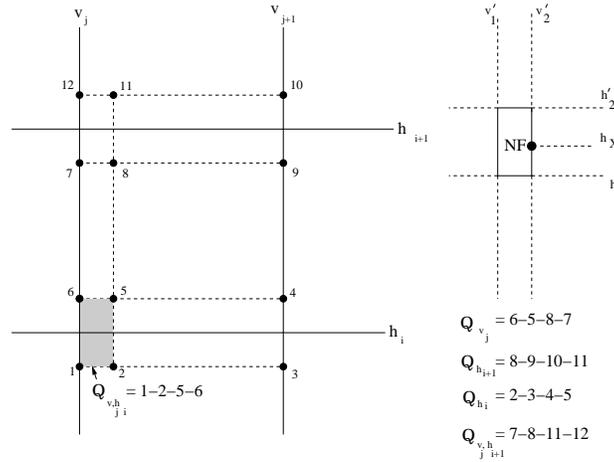


Figure 3: Interference Sets \mathcal{Q}

3.3 Equal travel time lines

As stated in Section 3.1, a shortest feasible rectilinear path from an EF I/O point to a point NF inside a cell passes through the corner of the cell. In other words, for a fixed NF location within a cell C , $X \in C$, the EF I/O points may be “assigned” to the nearest corners of the cell for the purpose of shortest distance measurement. However for any EF I/O point i , the assignment of the cell corner may change upon moving the NF within the cell. This results in the creation of an *Equal Travel Time*

Line for EF I/O point i within cell C (Batta *et al.* [1]). Figure 4 illustrates a case when an ETTL is generated. Consider a cell C having dimensions a and b and with corners C_1, C_2, C_3 and C_4 . Here, the absolute value of the difference between the distances from I/O point 1 to the point NF through the corners C_1 , and through C_4 , is less than a , the length of the edge connecting corners C_1 and C_4 , i.e., $|d(1, C_1) - d(1, C_4)| < a$. Therefore for a chosen point NF location within cell C , $X \in C$, the travel from I/O point 1 to X is through corners C_1 or C_4 . Figure 4 shows the corresponding ETTL generated for I/O point 1. Along the ETTL, the shortest feasible rectilinear path from X to I/O point 1 passes either through corner C_1 or C_4 . The ETTL generated divides the cell into two subcells SC_1 and SC_2 . Within SC_1 the shortest feasible rectilinear path to I/O point 1 is through corner C_1 , whereas in SC_2 the shortest path is through corner C_4 .

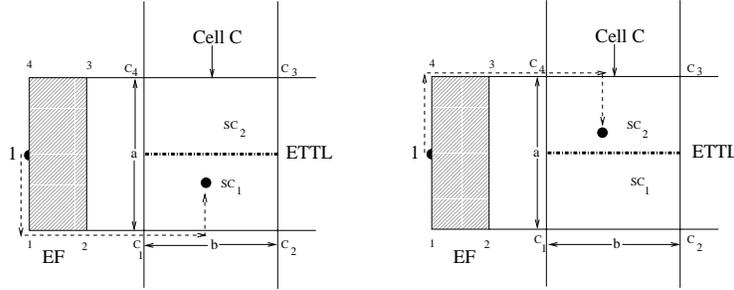


Figure 4: Equal Travel Time Line

For any I/O point, an ETTL may potentially be generated between any pair of corners of a cell. Between adjacent corners, the ETTL generated will be perpendicular to the edge connecting the corners. However between diagonally-opposite corners, it will be inclined at 45 degree with the edges which contains one of the two diagonally-opposite corners. Refer to Batta *et al.* [1] for details regarding construction of ETTLs.

Result 1: An I/O point i generates at most one ETTL in a rectangular cell (Nandikonda *et al.* [9]).

Result 2: ETTLs between both pairs of diagonally opposing cell corners in a rectangular cell cannot exist (Lemma 4.3.1 in Sarkar *et al.* [11]).

In the traditional definition an ETTL is generated because of an EF. However, if a finite-sized NF is placed inside the cell then the shortest rectilinear paths to the I/O point may change. This may cause the ETTL to move to a new location. The concept of *Moving ETTLs* has been described in Sarkar *et al.* [11].

Result 3: An ETTL may move (from the ETTL position of the point NF) if the rectangular NF blocks the cell corners between which the ETTL is generated (Sarkar *et al.* [11]). By “block” we imply interference to the shortest rectilinear path.

Consider placing a rectangular NF inside cell C mentioned in Fig. 4. Figure 5 shows the ETTL after the placement of the NF. As can be observed the ETTL has now moved to a new location. We note that the ETTL will move only when the shortest rectilinear path is interfered or blocked. This necessarily requires, that the NF intersects some grid line segment, which means that X is placed inside some interference set \mathcal{Q} . Argued in the opposite way, if $X \notin \mathcal{Q}$, the ETTL will not move. In the next section we illustrate some new properties of ETTLs which are necessary for our analysis.

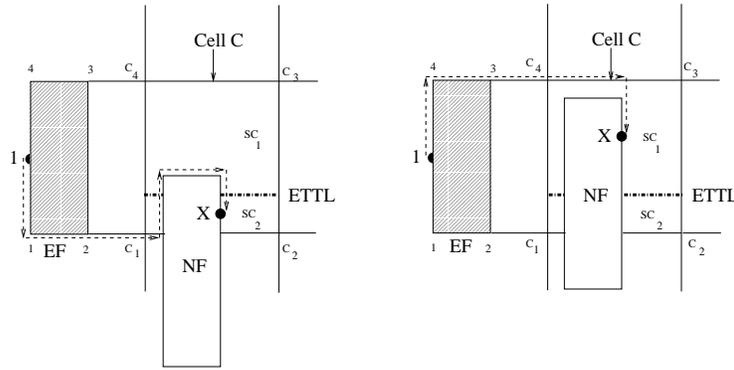


Figure 5: Illustration of a Moving ETTL

4 Additional Properties of ETTLs

Like EFs, a rectangular NF can generate ETTLs when the NF interferes with the shortest rectilinear path between a pair of EF I/O points, or between an EF I/O point and the NF I/O point. This is possible only when $X \in \mathcal{Q}$ associated with the affected rectilinear path.

Lemma 4.1 *For any EF-NF rectilinear flow which is interfered by the NF, the ETTL generated is parallel to the affected traversal path. Also, the edges of \mathcal{Q} and the NF which are parallel to the affected traversal path coincide with each other.*

Proof: Consider an NF which interferes with the shortest rectilinear path from its I/O point to an I/O point i of an EF. Let the NF have length H , width W and its I/O point at a distance Y , ($Y > H/2$) on the longer side as shown in Fig. 6. The NF interferes with the traversal path h_i of the flow in the region \mathcal{Q} , where \mathcal{Q} is the union of sets \mathcal{Q}_{h_i} and \mathcal{Q}_{h_i, v_j} , associated with the affected

traversal path h_i . Let X be such that the edges of \mathcal{Q} and the NF which are parallel to the traversal path h_i coincide with each other. At this position the distance traveled along the path $i - 1 - 2 - X$ is $Y + H - Y + const = H + const$ and along $i - 4 - 3 - X$ is $H - Y + Y + const = H + const$. This is illustrated in Fig. 6a. Therefore at this position the flow from $i - X$ can occur through corners 1 - 2 or 4 - 3 of the NF. This remains true for any position along a line through this point and parallel to the affected traversal path. Therefore the ETTL generated will be parallel to the affected traversal path and the edges of \mathcal{Q} and the NF which are parallel to the affected traversal path coincide with each other. The ETTL generated divides the region \mathcal{Q} into \mathcal{Q}_1 and \mathcal{Q}_2 . Inside \mathcal{Q}_1 the flow is through corners 4 - 3 of the NF whereas inside \mathcal{Q}_2 the flow is through corners 1 - 2. \square

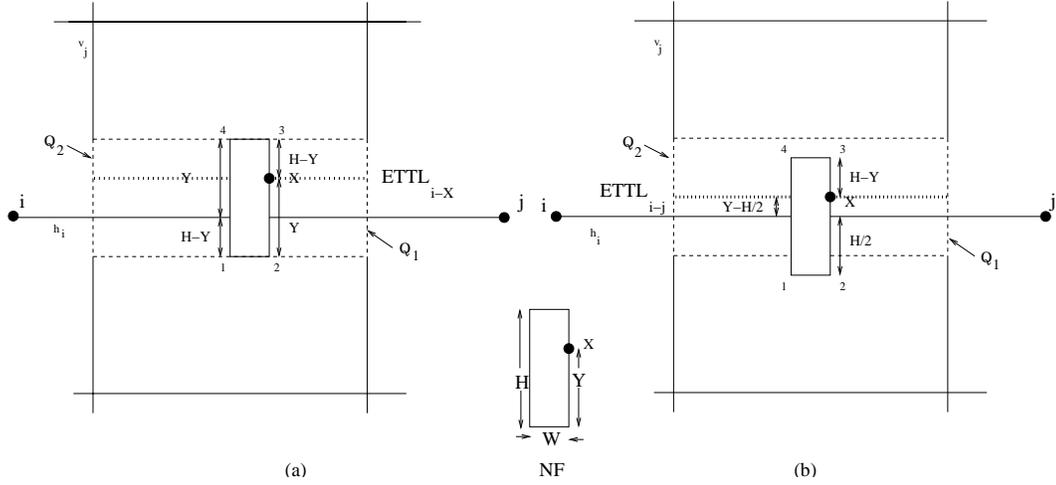


Figure 6: Illustrations for Lemmas 4.1 and 4.2

Lemma 4.2 *For rectilinear flow between a pair of EF I/O points which is interfered with by an NF, the ETTL generated will be at the center of the region \mathcal{Q} associated with the affected traversal path and parallel to the path.*

Proof: Suppose that the NF interferes with the shortest rectilinear path between I/O points i and j . The NF interferes with the traversal path h_i of the flow in the region \mathcal{Q} , where \mathcal{Q} is the union of sets \mathcal{Q}_{h_i} and \mathcal{Q}_{h_i, v_j} , associated with the affected traversal path h_i . Consider the NF being placed at the center of \mathcal{Q} . Since X is not at the center ($Y > H/2$), the center of \mathcal{Q} will be offset from the traversal path by $(Y - H/2)$. At this position the distance travelled along the path $i - 1 - 2 - j$ is $2 * H/2 + const = H + const$ and along $i - 4 - 3 - j$ is $2 * (H - Y + Y - H/2) + const = H + const$. This is illustrated in Fig. 6b. Therefore at this position the flow from $i - j$ can occur through corners 1 - 2

or 4 – 3 of the NF. This remains true for any position along a line through this point and parallel to the affected traversal path. The line divides the region \mathcal{Q} into $\mathcal{Q}1$ and $\mathcal{Q}2$. Inside $\mathcal{Q}1$ the flow is through corners 4 – 3 of the NF whereas inside $\mathcal{Q}2$ the flow is through corners 1 – 2. Therefore the ETTL generated will be at the center of the region \mathcal{Q} associated with the affected traversal path and parallel to it. \square

In summary the different types of ETTLs that can be generated are as follows:

- Region inside \mathcal{Q}
 - ETTLs due to an EF I/O point for flow between the EF I/O point and X (Section 3.3).
 - ETTLs due to the NF for flow between an EF I/O point and X (Lemma 4.1).
 - ETTLs due to the NF for flow between a pair of EF I/O points (Lemma 4.2).
- Region outside \mathcal{Q} (i.e., NF fully contained in a cell)
 - ETTLs due to an EF I/O point for flow between the EF I/O point and X . In this case the finite size of the facility will not cause the ETTL to move from the original point NF case.

We illustrate these types of ETTLs using the example shown in Fig. 7. There are two EF I/O points, i and j . As illustrated, i generates an ETTL for flow between $i - X$ inside the cell. The NF also generates ETTLs for flow between $i - X$ and $i - j$. Inside the region $abcd$ the flow from $i - X$ is through the upper corners $m - n$ of the EF_1 . Inside $dclk$ it is through the lower corners $p - o$ of the EF_2 . Inside $dclk$ the flow is through either corners $q - r$ or corners $t - s$ of the NF. Inside $efgh$ the flow is through the lower corners $t - s$ of the NF. However inside $hglk$ it is through the upper corners $q - r$ of the NF. Armed with the above results we can proceed with constructing contour lines.

5 Contour Line Construction

For the function $f(X)$ the contour line of value z is represented as $L(z)$ where $L(z) = \{X \in F : f(X) = z\}$. A contour set whose boundary is a contour line is the set of all points having values of $f(X) \leq z$. Francis [3] has shown that for point-sized NF and EFs, the contour line is continuous, with the corresponding contour set being convex. However the finite sizes of the NF and the EFs may present complications as given below:

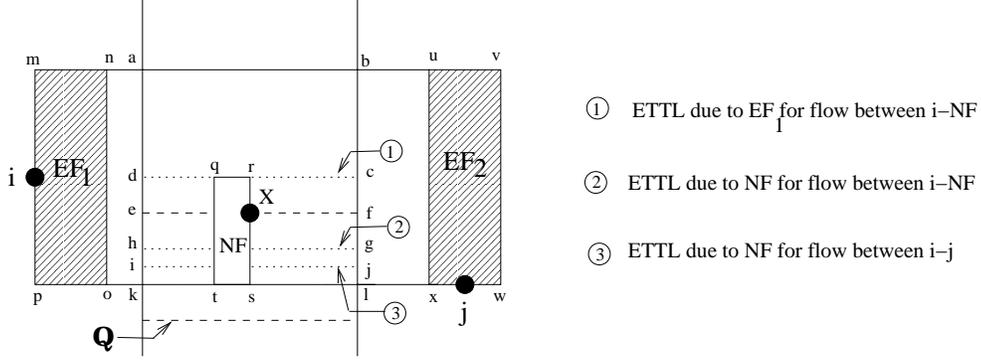


Figure 7: Different types of ETTLs

1. Contour lines may be intercepted by an EF and hence could be disconnected.
2. The finite-sized NF may interfere with the traversal path of flow between facilities in the region \mathcal{Q} associated with the affected traversal path. Therefore points inside \mathcal{Q} may have an objective function value higher than that of the nearby points. This can make the contour line disconnected.

In the case of point NF and point EFs, if one starts with a location X of objective function value z , then the slope in the cell to which X belongs will cause the contour line to be incident on some adjacent cells and the process can be repeated until the contour is closed. In our case, the contour line and set for a particular value z can only be constructed after evaluating all the cells, at least implicitly, to identify those cells which contain an X such that $f(X) = z$. Let S_z denote the set of cells which contain an X such that $f(X) = z$. The NF may interfere with the material flow in some cells while in others it may not. In those cells where the NF interferes with the flow it does so only in the region \mathcal{Q} associated with the affected traversal path. Therefore the set S_z may be classified as shown in Fig. 8. In our analysis, we consider the objective function and the method to calculate the slope of the contour line in each of them separately. We start by illustrating our methodology through a numerical example where an NF has to be placed in a shop floor having two EF I/O points, i and j . The weights of the flow between facilities are as shown in Fig. 9. We then present a general procedure to construct the contour line for a given problem instance.

5.1 Identification of cells

Inside a cell, the candidate points for the minima of the objective function of a point NF are the corners of the cell. Let z_p^i represent the objective function value of a point-sized NF located at the

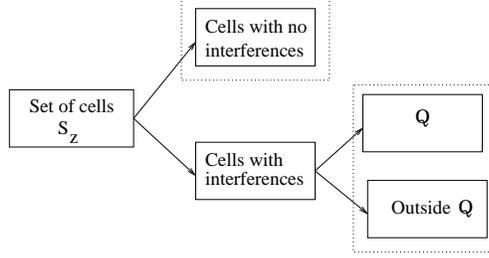


Figure 8: Classification of cells

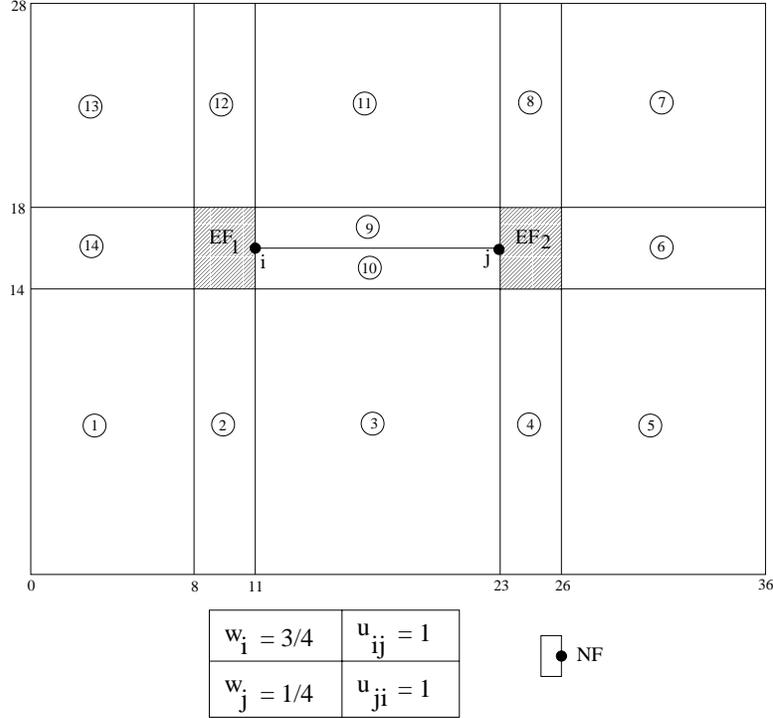


Figure 9: Numerical Example

i^{th} corner of a cell, where $i = 1, 2, 3, 4$. In a cell if the minimum value of z_p , i.e. $\min_i \{z_p^i\}$, is greater than z then no point can be found inside the cell with value z since the placement of a rectangular NF will only increase the objective function value. If the NF is not interfering with any flow then it may be considered as a point inside the cell. Then values of z_p^i at the corners may be used to identify cells which contain the objective function value z . However if the NF interferes with the flow at any of the corners and $\min_i \{z_p^i\}$ is less than z , then a point with value less than z may or may not be present in these cells. The presence of such a point depends on the effect of the NF on the material flow. Based on the above observations we present an algorithm to determine the set of candidate cells, S_z , which contain contour line segments of value z .

Algorithm for determining the set of cells S_z :

Input: Set of cells from the grid construction procedure of Section 3.1

Initialize: $S_z = \emptyset$

For each cell C :

If $\min_i \{z_p^i\} \leq z$

 If no interferences

 If $\max_i \{z_p^i\} \geq z$

$$S_z \leftarrow S_z \cup C$$

 Else

$$S_z \leftarrow S_z \cup C$$

Output: Set of cells S_z

Applying this algorithm to the numerical example, for $z = 35$, the set of cells is $S_{35} = \{1, 2, 3, 9, 10, 11, 12, 13, 14\}$.

5.2 Objective function

In this section we analyze the objective function and illustrate the method to calculate slope of the line in each of the classifications of set S_z .

5.2.1 Cells with no interferences

Consider a cell C where an NF does not interfere with any of the material flow. Consider a shortest feasible rectilinear path from an I/O point i to the NF as illustrated in Fig. 10. The NF does not affect the length of the path from I/O point i to the NF, and hence, the NF may be considered as a point inside the cell. For a point NF, irrespective of its position inside cell C the total weighted travel distance between a pair of EF I/O points will be a constant. However the total weighted travel distance between EF I/O points and X will vary. The EF I/O points may be assigned to appropriate corners of the cell. It has to be noted that even though there is no effect of the NF inside the cell, an EF could generate an ETTL as defined in Section 3.3. Therefore the assignment of the EF I/O points may vary inside the cell.

Let w_1, w_2, w_3, w_4 be the weights of the I/O points assigned to the corresponding corner of a cell/subcell for the purpose of shortest distance measurement. Note that corners are numbered from

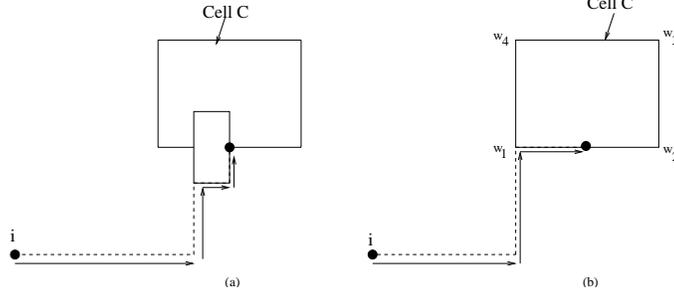


Figure 10: Cell with no interferences

the lower left corner and moving counterclockwise. The objective function may be written as:

$$f(x, y) = w_1(x - x_{min}) + w_2(x_{max} - x) + w_3(x_{max} - x) + w_4(x - x_{min}) + w_1(y - y_{min}) + w_2(y - y_{min}) + w_3(y_{max} - y) + w_4(y_{max} - y) + const,$$

where $(x_{min}, y_{min}), (x_{max}, y_{min}), (x_{max}, y_{max}), (x_{min}, y_{max})$ are the corners of the cell/subcell. As can be observed the objective function is linear in x and y . Therefore the part of the contour line inside this cell/subcell will be a line segment of the following functional form:

$$y = x \left\{ -\frac{w_1 + w_4 - w_3 - w_2}{w_1 + w_2 - w_3 - w_4} \right\} + const.$$

The coefficient of x yields the slope of the contour line where the numerator is the difference between the sum of the weights of the I/O points served through corners 1 and 4, and the sum of the weights of the I/O points served through corners 2 and 3; and the denominator is the difference between the sum of the weights of the I/O points served through corners 1 and 2, and the sum of the weights of the I/O points served through corners 3 and 4. This is similar to the results for point-sized NF and EFs as defined in Francis [3].

In the numerical example, cells 1, 2, 3, 11, 12, 13 and 14 of the set S_{35} have no interferences due to the NF. Inside cell 14, I/O points i and j generate ETTLs for flow to X . Therefore the assignment of the I/O points i and j to cell corners varies inside cell 14. However the NF may be considered as a point inside these and the slope may be found using the results for a point NF inside a subcell.

5.2.2 Inside \mathcal{Q}

Consider cells 9 and 10 in the numerical example where the NF interferes with the traversal path h_i of the flow between $i - X$ and $i - j$.

1. Flow between an EF I/O point and X : Figure 11 shows the \mathcal{Q} associated with the traversal path h_i . Let the four corners of \mathcal{Q} be a, b, c and d . The ETTL for flow between $i-X$ divides the region \mathcal{Q} into $\mathcal{Q}1$ and $\mathcal{Q}2$, referred to as subcells. As shown in Fig. 11a, inside $\mathcal{Q}1$ a shortest rectilinear path from the I/O point i to X is through $i-4-3-X$. Therefore the extra distance traveled due to the NF is $Y_{h'_2} - Y_{h_i} + Y_{h'_2} - Y_X$. The distance $Y_{h'_2} - Y_X$ related to path $3-X$ is a constant. The distance that varies as we move the NF inside $\mathcal{Q}1$ is $Y_{h'_2} - Y_{h_i}$. Further, $Y_{h'_2} - Y_{h_i}$ is equal to $Y_X - Y_{ab}$ and $Y_{h'_2} - Y_X$ is equal to $Y_{h_i} - Y_{ab}$. Therefore the path $i-4-3-X$ may be considered equivalent to the path from i through corner a to a point facility at X . Hence inside $\mathcal{Q}1$, the rectilinear path from the I/O point i to X can be interpreted as a path to a point-sized facility through the corner of the region $\mathcal{Q}1$. Similarly in $\mathcal{Q}2$, a shortest rectilinear path from the I/O point i to X is through $i-1-2-X$ as shown in Fig. 11b, which is equivalent to the path from i through corner d to a point-sized facility at X .

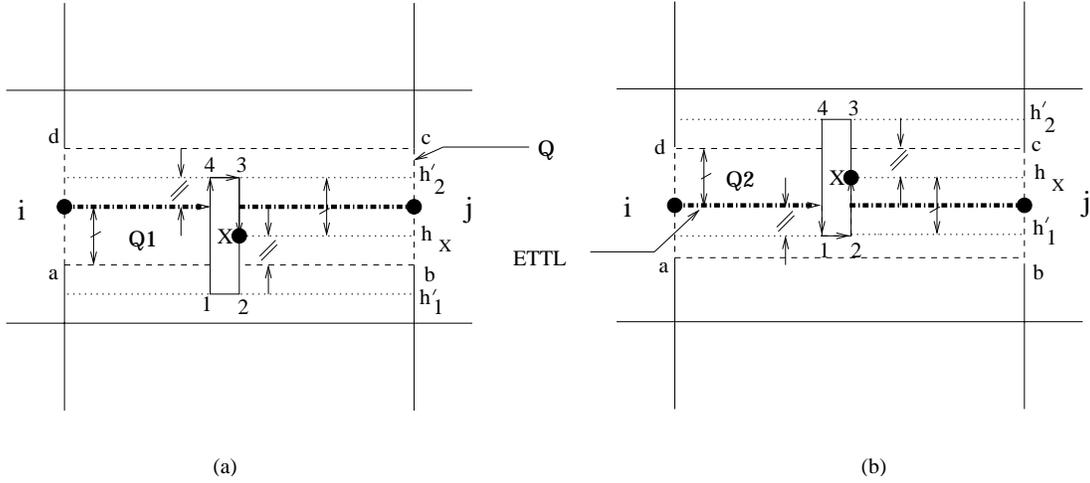


Figure 11: Shortest rectilinear paths in $\mathcal{Q}1$ and $\mathcal{Q}2$

In summary, when \mathcal{Q} is split into subcells by an ETTL generated due to I/O point i , the shortest rectilinear distance from i to X in a subcell can be measured from the subcell corner opposite the ETTL and closer to i . Let w_1, w_2, w_3, w_4 be the weights of the I/O points not affected by the NF and assigned to the corresponding corner of a cell/subcell for the purpose of shortest distance measurement. Then the total weighted travel distance between EF-NF inside $\mathcal{Q}1$ (and $\mathcal{Q}2$) is:

$$f_{\mathcal{Q}1}^{EF-NF}(x, y) = w_1(x - x_{min}) + w_1(y - y_{min}) + w_2(x_{max} - x) + w_2(y - y_{min}) + w_3(x_{max} - x) + w_3(y_{max} - y) + w_4(x - x_{min}) + w_4(y_{max} - y) + w_i(x - x_{min}) + w_i(y - y_{min}) + const.$$

$$= x(w_1 + w_i + w_4 - w_3 - w_2) + y(w_1 + w_i + w_2 - w_3 - w_4) + const.$$

$$f_{Q2}^{EF-NF}(x, y) = w_1(x - x_{min}) + w_1(y - y_{min}) + w_2(x_{max} - x) + w_2(y - y_{min}) + w_3(x_{max} - x) + w_3(y_{max} - y) + w_4(x - x_{min}) + w_4(y_{max} - y) + w_i(x - x_{min}) + w_i(y_{max} - y) + const.$$

$$= x(w_1 + w_i + w_4 - w_3 - w_2) + y(w_1 + w_2 - w_3 - w_4 - w_i) + const.$$

2. Flow between a pair of EF I/O points: Figure 12 shows the ETTL for flow between $i - j$ in the region \mathcal{Q} associated with the traversal path h_i . It divides the region \mathcal{Q} into $\mathcal{Q}1$ and $\mathcal{Q}2$. Inside $\mathcal{Q}1$ a shortest rectilinear path from the I/O point i to the point j is through $i - 4 - 3 - j$. Therefore the extra distance traveled due to the NF is $2(Y_{h'_2} - Y_{h_i})$. As shown in Fig. 12a, $Y_{h'_2} - Y_{h_i}$ is equal to $Y_X - Y_{ab}$. Similarly in $\mathcal{Q}2$, the shortest rectilinear path from the I/O point i to j is through $i - 1 - 2 - j$. Therefore the extra distance travelled due to the NF is $2(Y_{h_i} - Y_{h'_1})$. As shown in Fig. 12b, $Y_{h_i} - Y_{h'_1}$ is equal to $Y_{cd} - Y_X$.

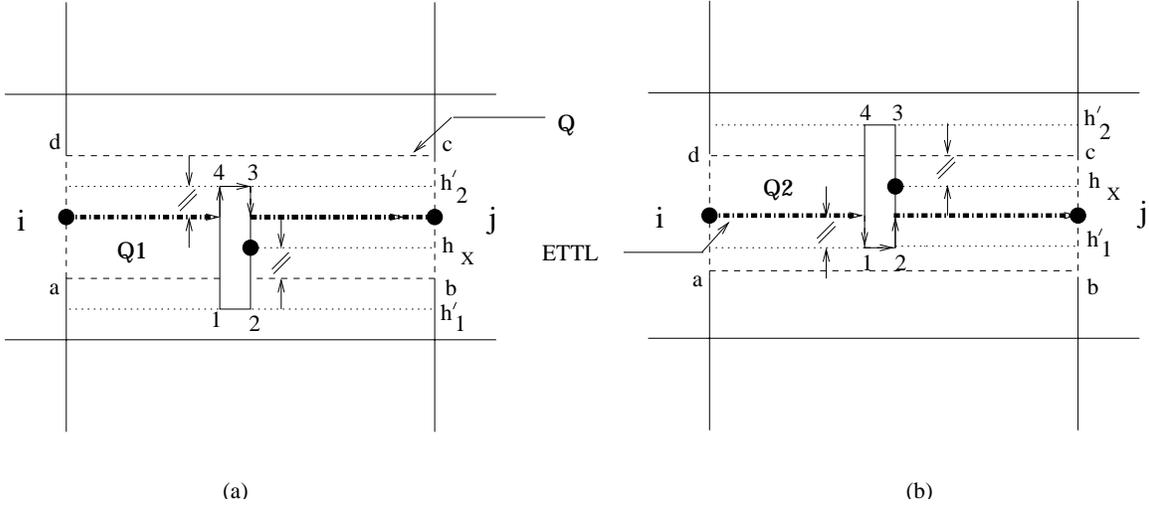


Figure 12: Shortest rectilinear paths in $\mathcal{Q}1$ and $\mathcal{Q}2$

Then the total weighted travel distance between EF-EF inside $\mathcal{Q}1$ (and $\mathcal{Q}2$) is:

$$f_{Q1}^{EF-EF}(x, y) = 2u_{ij}(y - y_{min}) + 2u_{ji}(y - y_{min}) + const$$

$$= y(2u_{ij} + 2u_{ji}) + const.$$

$$f_{Q2}^{EF-EF}(x, y) = 2u_{ij}(y_{max} - y) + 2u_{ji}(y_{max} - y) + const$$

$$= y(-2u_{ij} - 2u_{ji}) + const.$$

3. Objective function: Inside \mathcal{Q} the ETTLs for flow between $i - X$ and $i - j$ are at the same location as shown in Fig. 13. ETTLs divide \mathcal{Q} into $\mathcal{Q}1$ and $\mathcal{Q}2$. Considering $\mathcal{Q}1$, the objective function may be written as:

$$f_{\mathcal{Q}1}(x, y) = x(w_1 + w_i + w_4 - w_3 - w_2) + y(w_1 + w_i + w_2 - w_3 - w_4) + y(2u_{ij} + 2u_{ji}) + const.$$

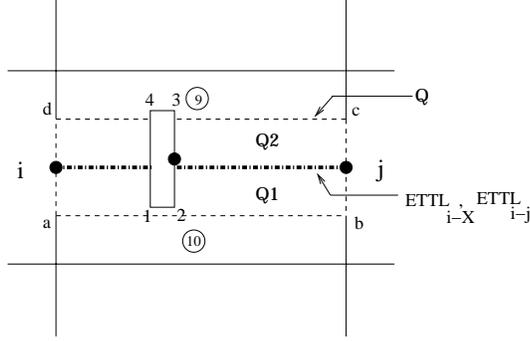


Figure 13: Subcells

Once again, the objective function is linear in x and y . Therefore the part of the contour line inside subcell $\mathcal{Q}1$ will be a line segment of the following functional form:

$$y = x \left\{ -\frac{w_1 + w_i + w_4 - w_3 - w_2}{w_1 + w_i + 2u_{ij} + 2u_{ji} - w_3 - w_4} \right\} + const.$$

The coefficient of x gives the slope of the contour line inside $\mathcal{Q}1$. The slope of the line inside $\mathcal{Q}2$ may be found in a similar manner. Similar to the discussion in Section 5.2.1, an EF I/O point could generate an ETTL for flow between an EF I/O point and X inside \mathcal{Q} . This may alter the traversal path from EF to the NF. Therefore the assignment of I/O points have to be done accordingly.

5.2.3 Outside \mathcal{Q}

Consider the region outside \mathcal{Q} in cell 9. As illustrated in Fig. 14, the NF does not interfere with any of the material flow in the region outside \mathcal{Q} and hence it may be considered as a point inside the region. For a point-sized facility, irrespective of its position inside the region, the total weighted travel distance between a pair of EF I/O points is a constant. However the total weighted travel distance between an EF I/O point and X will vary and the weights of the I/O points may be assigned to the corners of the region. Region outside \mathcal{Q} may also be considered as a subcell. In this example, i may be assigned to the lower left corner and j to the lower right corner of the subcell. The slope may then be found by applying the results discussed in Section 5.2.1 for a point-sized NF.

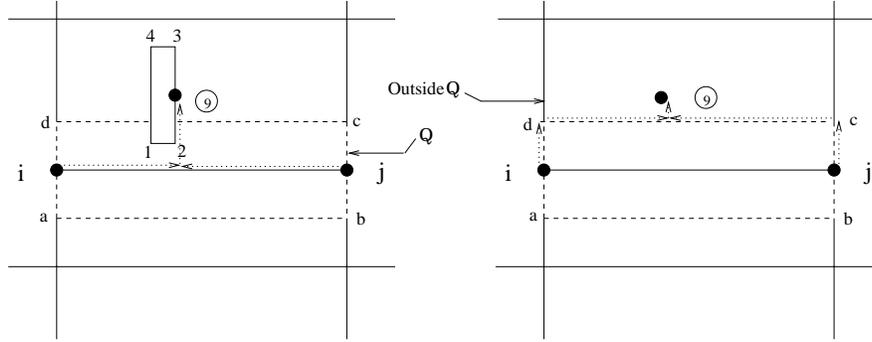


Figure 14: Outside Q

5.3 Contour line construction method

We summarize the contour line construction procedure as follows:

Consider starting at a feasible point $X \in F$ which has an objective function value of z .

1. Draw the EF traversal lines and form the grid.
2. Identify the the cell C which contains X .
3. Identify the the subcells in cell C .
 - (a) Identify ETTLs formed due to EF I/O points, for flow between EF I/O points and X in cell C and determine its subcells. Note that the NF is finite-sized and could cause the ETTL to move from the point NF case (Result 3).
 - (b) Identify the interference sets Q_s , where the NF interferes with any of the material flows. Identify the ETTLs formed, due to the NF for material flow between a pair of EF I/O points or an EF I/O point and X , inside cell C and hence determine the subcells.
4. For cell C and its subcells, assign weights of the unaffected EF I/O point to X material flow to the appropriate cell corner. For affected material flow between a pair of EF I/O points and between an EF I/O point and X , assign weights to the corner opposite to the corresponding ETTL and nearest to the EF (Section 5.2.2).
5. For cell C and its subcells, the slope of the contour line is determined as follows:
 - (a) From step 4, let,

w_i be the sum of the weights of the material flow between EF I/O points and X assigned to corner i ,

u_{v_j} and u_{h_k} be the sum of the weights of the material flow between pairs of EF I/O points affected along the vertical and horizontal path(s),

let 1, 2, 3, 4 be the corners of the cell/subcell starting from the lower left corner and moving anticlockwise, then,

$$a = (w_1 + w_4 + 2u_{v_1} + 2u_{v_4}) - (w_2 + w_3 + 2u_{v_2} + 2u_{v_3}), \text{ and}$$

$$b = (w_1 + w_2 + 2u_{h_1} + 2u_{h_2}) - (w_3 + w_4 + 2u_{h_3} + 2u_{h_4})$$

(b) Slope of the contour line = $-\frac{a}{b}$

6. Proceed to construct the contour line that passes through X , using the slope computed in Step 5. If this line terminates (in either direction) at the boundary of an adjacent cell/subcell, it identifies a point in this adjacent cell/subcell that has an objective function value of z . In this case, recompute the slope using Step 5 and then proceed to draw the continuation of this contour line. It is also possible that the line terminates at the boundary of the feasible region F . In this case, examine cells in set S_z (determined using the algorithm of Section 5.1) through which this contour line has not yet passed, and attempt to find a new starting point that has objective function value z . To do this one can take advantage of the fact that inside a cell/subcell the assignment of EF I/O points to the corners of the cell/subcell remains unchanged. Hence the determination of a point of objective function value z can be set up as a linear program with no objective, and with constraints that limits the location of X to be within the cell/subcell as well as a constraint that specifies that the objective function value for the chosen X is z . If this linear program is infeasible, the cell/subcell contains no point with objective function value z . Otherwise, it returns one such point. The procedure is then restarted with this new location and its associated cell C' .

6 Numerical Example

Figure 15 shows the contour lines constructed for $z = 31, 35, 41$ and 43 using the above procedure. As can be observed the contour lines for $z = 31$ and 35 are intercepted by the EFs making them disconnected.

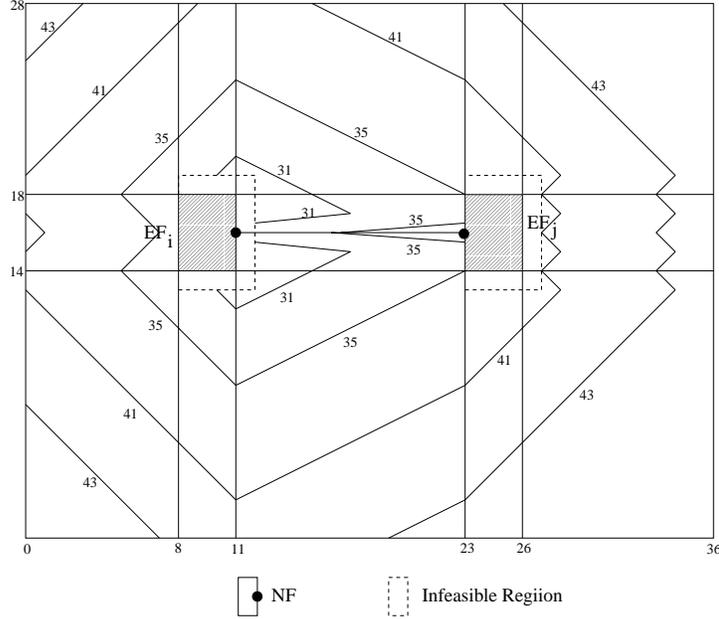


Figure 15: Contour lines

7 Solution Complexity

Our solution methodology proceeds in two steps. First the number of cells in which objective function value z is potentially present has to be identified using the algorithm of Section 5.1. Each EF generates 2 horizontal and 2 vertical traversal lines from its boundaries. Hence if there are m EFs, $2m$ horizontal and $2m$ vertical lines will be produced. Similarly n I/O points can generate at most n horizontal and n vertical lines. Hence the maximum number of cells generated is $O(m^2 + n^2)$.

The contour line is then constructed by calculating the slope of the line in each of the identified cells. Here the complexity depends on the number of subcells generated inside each cell. Result 2 states that ETTLs between diagonally-opposite corners for rectangular EFs are not possible. Nandikonda *et al.* [9] have shown that an I/O point of an EF will generate at most one ETTL. Hence n I/O points of EFs can generate n ETTLs, in the worst case generating $O(n^2)$ subcells per cell. In addition to this, subcells may be generated due to ETTLs for material flow between a pair of EF I/O points and between an EF I/O point and X . If, in a cell, the NF interferes the material flow with k ($k \leq n$) EF I/O points, then the maximum number of ETTLs formed inside the cell for flow between EF I/O points and X will be k . Additionally the flow between k I/O points and the rest of the I/O points may be affected by the NF. This can generate a maximum of $k(n - k)$ ETTLs for the flows between

pairs of EF I/O points. In addition to the ETTLs the boundaries of \mathcal{Q} s will also divide the cell into subcells. Hence the number of subcells that can be generated inside a cell is $O(n^2)$. Therefore the total number of subcells that can be generated is $O(m^2n^2 + n^4)$, which is polynomially bound.

8 Conclusions

In summary, this paper addresses the contour line construction procedure for a finite-sized rectangular new facility to be placed in a layout having other existing rectangular facilities. Optimal placement of a finite-sized new facility in the presence of other facilities has been studied by Savas *et al.* [12]. However due to other considerations the optimal site may not be always suitable for placement of the new facility. This necessitates the new facility to be placed at alternate locations and provides the motivation for this paper. Contour lines helps to place the I/O point of the new facility at locations other than the optimal site with characterized increase in the objective function value. To draw the contour lines, we divide the plane into regions and formulate a method to calculate the slope of the contour line in each of these regions. Our work can be viewed as an extension to the work by Francis [3] who developed the contour line procedure for locating a point NF in the presence of point EFs. Francis [3] also use the concept of dividing the plane into regions and calculating the slope of the contour line in each of those regions. This work is applicable for a facilities location problem. To extend this to a layout context we have to assume finite size for the new facility and the presence of other finite-sized existing facilities. However, the finite size of the new facility and the presence of other facilities makes the problem complex to solve since travel is not permitted in these regions and the rectilinear metric gets destroyed. By a systematic division of the plane into subregions where the slope of the line remains constant, we develop a procedure for constructing layout contour lines. Unlike the location case, layout contour lines are non-convex and disconnected. Complexity analysis establishes that the methodology developed is polynomially bound.

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