An integrated model for space determination and site selection of distribution centers

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Abstract

In this paper we present an integrated distribution center site selection and space requirement problem on a two-stage network in which products are shipped from plants to distribution centers, where they are stored for an arbitrary period of time and then delivered to retailers. The objective of the problem is to minimize total inbound and outbound transportation costs and total distribution center construction cost – which includes fixed costs related to their locations and variable costs related to their space requirements for given service levels. Each distribution center is modeled as an $M/G/c$ queueing system, in which each server represents a storage slot. We formulate this problem as a nonlinear mixed integer program with a probabilistic constraint. Two cases are considered. For the continuous unbounded size case, we find an approximate formula for the overflow probability and restructure this model into a connection location problem. For the discrete size option case, we reformulate the problem into a capacitated connection location problem with discrete size options. Computational results and a comparison of the two cases are provided.

Keywords: Distribution center sizing, distribution center site selection.

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Introduction

Consider a typical centralized two-stage distribution center (DC) system in which the distribution of products is carried out as follows: From production plants the products are transported to a certain number of DCs and then delivered to geographically dispersed retailers. Such a distribution network is depicted in Figure 1. The planning of the distribution network involves decisions on: (i) the location and product routing for DCs; and (ii) the size of each DC to satisfy a pre-determined service level, where the service level is measured by the probability that an arriving product finds no storage slot available at a DC.

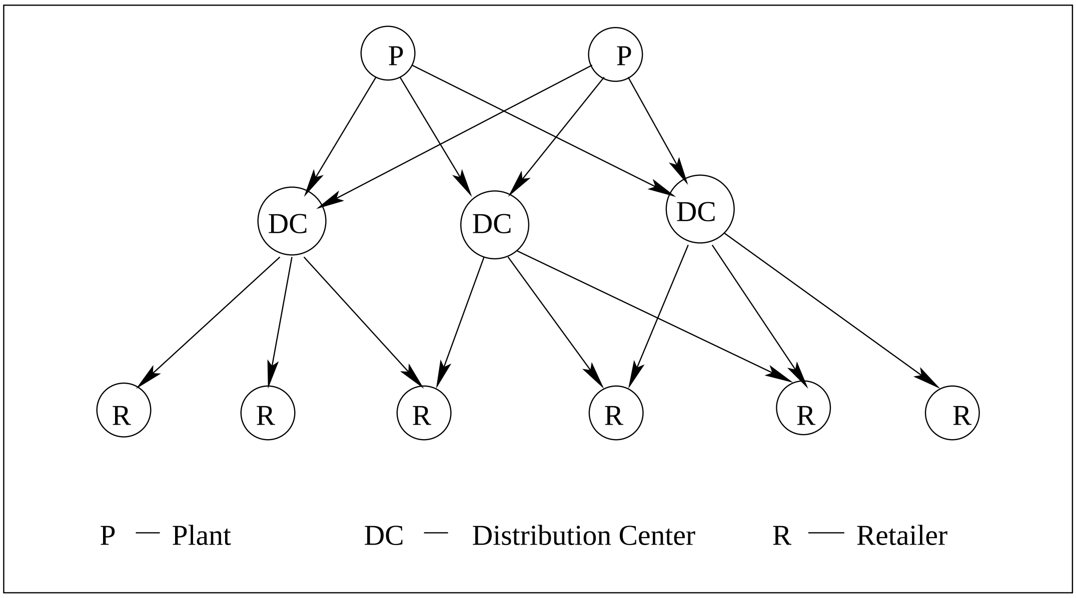

Figure 1: Typical two-stage distribution system.

The DC *location and product routing* problem has received considerable attention. It focuses on the determination of the number and locations of the DCs, as well as the product flow assignments in order to minimize the transportation cost and the fixed location cost. Reviews on the plant location problem can be found in Krarup and Pruzan (1983), Mirchandani and Francis (1990) and Sridharan (1995). Recent related research can be found in Nozick and Turnquist (2001) and Shen, Coullard, and Daskin (2003). In this class of problems, space requirements are not explicitly considered. Capacitated versions of the problem have been analyzed, which implicitly try to account for space restrictions. Nevertheless, DC capacities are an input to these models, whereas space requirements are an output for our models.

The DC *sizing* problem has also received considerable attention. It seeks to find a DC size which either allows a required service level to be attained or minimizes total costs. Under the assumption of constant product demand, Cormier and Gunn (1996a) and Cormier and Gunn (1996b) coordinated DC size and inventory policy. Sung and Han (1992) presented a problem of determining the optimal size of an AS/RS by analyzing some queueing models. Multi-period DC leasing problems were investigated by White and Francis (1971), Lowe, Francis, and Reinhardt (1979) and Rao and Rao (1998). Jucker, Carlson, and Kropp (1982) considered a multi-location DC leasing problem under uncertain demand. Roll and Rosenblatt (1983) and Rosenblatt and Roll (1988) developed simulation models to measure the relationship among DC size, inventory policy, and other various parameters. We note, however, that all of these models assume that locations have been selected.

Talavera (2002) presents a simulation study that demonstrates that the simultaneous consideration of DC location/routing and size yields significantly better results than a sequential method. Motivated by this empirical finding, we develop an analytical approach to this integrated decision making of DC location/routing and size. The problem is formulated as a nonlinear mixed integer program with a probabilistic constraint. For the continuous unbounded size case, we find an approximate formula for the overflow probability and restructure this model into the connection location problem with a concave cost function – which is solved by a column generation method. For discrete size option case, we reformulate the problem into the capacitated connection location problem with discrete size options – which can be solved by a Largrangian relaxation approach.

The rest of the paper is organized as follows. Section 1 presents a mathematical formulation of the integrated DC site selection and space requirement problem. Section 2 details the continuous unbounded size case. Some properties and a column generation approach are established. Section 3 analyzes the discrete size option case. A Largrangian relaxation heuristic approach is suggested for the problem. Section 4 reports computational results and provides a comparison of the two cases. Section 5 contains a summary and suggests directions for future work.

1 Formulation

Tables 1 and 2 summarize the parameters and decision variables for our model, respectively. We assume that each product has a standard unit volume size and is produced by at least one plant. Demand forecast from retailers gets transformed to a production schedule at the plants. We assume that the flow of product out of a plant occurs in a Poisson manner, and the processes at each plant are independent of each other. Thus the flow of products into each DC is given by a Poisson random variable. Each DC is modeled by an $M/G/c$ queue, where c is expressed in terms of the number of storage slots required to store products at a DC. In other words, a storage slot is a "server" of the queueing system. If an arriving product finds that all storage slots are occupied at a DC, it has to be sent to a leased slot, which is much more expensive, until a slot is available at this DC. The DC size is measured as the total number of storage spaces (each assumed to be of equal size). The amount of time that a product stays in a DC (its service time) depends on when this product is shipped to a retailer. We allow this service/storage time to follow a general distribution with a known mean. Figure 2 depicts such a system. We assume that the total construction cost of DC k is $F_k + t_k c_k$. A regression analysis by Ashayeri, Gelders, and Wassenhove (1985) showed that this assumption is very reasonable.

A strength of our modeling approach is its simplicity in data requirements. All we need is the mean storage time at each DC (which can be derived from inventory turn data) and the mean rate of production at each plant (which can be derived from production planning data).

Table 1: Model Parameters

Symbol	Meaning
$I = \{i : i = 1, 2, \cdots, n\}$	set of plants
$J = \{j : j = 1, 2, \cdots, m\}$	set of retailers
$K = \{k : k = 1, 2, \cdots, K\}$	set of candidate sites for DCs
$1/\mu_k$	mean storage time of a product at DC k
t_k	unit space construction and operational cost at DC k
F_k	fixed cost if a DC is located at candidate site k
f_{ij}	mean demand for products from plant i at retailer j
u_{ik}	unit shipping cost from plant i to DC k
v_{kj}	unit shipping cost from DC k to retailer j
β_k	threshold overflow probability at DC k

Due to the demand and supply processes the inventory level fluctuates. We estimate the storage space requirement such that the storage space suffices for at least a fraction $0 < 1 - \beta < 1$ of the time. In other words, the probability that an arriving product finds

Table 2: Decision Variables

Symbol	Meaning
x_{ijk}	fraction of demand for products from plant i at retailer j shipped via DC k
y_k	1 if a DC is located at candidate site k, and 0 otherwise
c_k	space requirement for DC k for given threshold overflow probability β_k

no storage slot available at a DC is less than a pre-determined level β, which is the overflow probability of an $M/G/c$ queue.

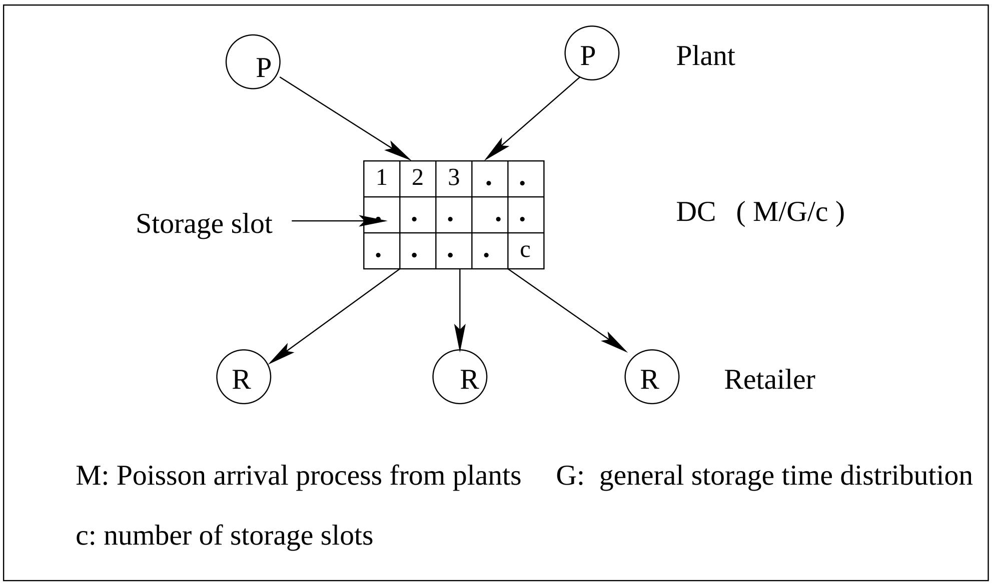

Figure 2: DC as an $M/G/c$ queueing system

Using notation $Z^P(\boldsymbol{x})$ to denote the objective function of a problem (P) with a vector of decision variables $\boldsymbol{x}$ and NA_k to the fact that an arriving product finds no storage slot available at DC k, we arrive at the following formulation:

$$(P) \qquad \min_{\boldsymbol{x},\boldsymbol{y},\boldsymbol{c}} Z^P(\boldsymbol{x},\boldsymbol{y},\boldsymbol{c}) = \sum_{i,j}\sum_k f_{ij}(u_{ik}+v_{kj})x_{ijk} + \sum_k (F_k + t_k c_k) y_k \tag{1}$$

$$\text{subject to} \qquad \sum_k x_{ijk} = 1, \qquad \forall\, i,j, \tag{2}$$

$$x_{ijk} \leq y_k, \quad \forall\, i, j, k, \tag{3}$$

$$Pr(NA_k) \leq \beta_k, \quad \forall\, k, \tag{4}$$

$$x_{ijk} \geq 0, \quad y_k \in \{0, 1\}, \quad c_k \geq 0, \qquad \forall\, i, j, k. \tag{5}$$

The objective function (1) minimizes the total inbound and outbound shipping costs and total DC construction cost. Constraint (2) stipulates that the product demands can only be shipped via DCs. Constraint (3) states that the product demand can only ship via the DC selected. Constraint (4) is a probabilistic capacity constraint, which forces the the overflow probability at a DC k to be less than or equal to β_k. Constraint (5) represents the integrality and non-negativity constraints.

A major difficulty is that a direct algebraic formula of overflow probability in an $M/G/c$ queue system is not available. We therefore proceed by using a suitable approximation to this probability.

2 Continuous size case

Here we assume that the size variable c_k is continuous and unbounded for all k. First, we develop an approximate formula for c_k for the pre-determined β_k. Using the approximation, we reformulate the problem as a set covering problem and solve it by a column generation algorithm.

2.1 An approximate formula for the required DC size

Here, we develop an approximate formula using a result from Parikh (1977). Consider an $M/G/c$ queueing system. Let λ and μ be, respectively, the system arrival and service rate, with load $\rho = \lambda/\mu$. Let $Pr(\rho, c)$ be the overflow probability for the $M/G/c$ queueing system, which refers to the probability that all servers in the system are busy. According to the definition of the space requirement at a DC, for the pre-determined β, the required number of servers, c, is a value that satisfies:

$$Pr(\rho, c) \leq \beta, \text{ and } Pr(\rho, c-1) > \beta.$$

From Parikh (1977), empirically observed lower and upper bounds for c, given β, are as follows:

$$c_N \leq c \leq c_M, \tag{6}$$

where c_M is the required number of servers for an $M/M/c$ queueing system for the given β, and

$$c_N \approx \rho + q_\beta \rho^{1/2} + 0.5. \tag{7}$$

Here, q_β is the $(\beta)th$ percentile of the standard normal distribution, i.e., $Pr(Z \geq q_\beta) = \beta$ and $Z \sim N(0, 1)$.

Empirical results for different sets of parameters from Parikh (1977) demonstrate that the bounds are very tight. In fact, with $\beta \leq 0.10$, $c_M - c_N$ almost always equals to 1 if we round up c_N to an integer. Expression (7) has also been used to approximate c_M by Kolesar and Green (1998). They provide details on the insights of this approximation. We select c_N (as opposed to c_M) as the approximate formula of c for a pre-determined β value due to its relative ease in mathematical analysis.

In our model, let λ_k and ρ_k be, respectively, the mean arrival rate and the load of DC k. It follows that:

$$\lambda_k = \sum_{i,j} f_{ij} x_{ijk}, \qquad \forall\, k \in K,$$

and

$$\rho_k = \sum_{i,j} \frac{f_{ij}}{\mu_k} x_{ijk}, \qquad \forall\, k \in K.$$

Therefore, from (7),

$$c_k \approx \sum_{i,j} \frac{f_{ij}}{\mu_k} x_{ijk} + q_{\beta_k} \sqrt{\sum_{i,j} \frac{f_{ij}}{\mu_k} x_{ijk}} + 0.5. \tag{8}$$

It is easy to see that c_k is a concave function of assignment variable $\boldsymbol{x}$. Therefore, (8) gives a concave shape to the total DC cost to accommodate economies of scale in the construction of DCs, which allows some insightful analysis to be carried out.

2.2 An approximate formulation

We can use the expression for c_k in equation (8) in the objective function and apply the inequality in constraint (4) to obtain:

$$Z^P(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{c}) = \sum_{i,j} \sum_k f_{ij}(u_{ik} + v_{kj} + \frac{t_k}{\mu_k}) x_{ijk} + \sum_k (F_k + 0.5 t_k) y_k + \sum_k t_k q_{\beta_k} \sqrt{\sum_{i,j} \frac{f_{ij}}{\mu_k} x_{ijk}}.$$

Note that when we substitute c_k into the objective function, we will no longer need the probabilistic capacity constraint (4) according to the definition of c from Section 2.1. We let $\alpha_{ijk} = f_{ij}(u_{ik} + v_{kj} + \frac{t_k}{\mu_k})$, $\gamma_k = \frac{t_k q_{\beta_k}}{\sqrt{\mu_k}}$, $F_k' = F_k + 0.5 t_k$ and $Q_k(\boldsymbol{x}) = \sqrt{\sum_{i,j} f_{ij} x_{ijk}}$. Then, the approximate formulation of the original problem is as follows:

$$(P_1) \min_{\boldsymbol{x}, \boldsymbol{y}} Z^{P_1}(\boldsymbol{x}, \boldsymbol{y}) = \sum_{i,j} \sum_k \alpha_{ijk} x_{ijk} + \sum_k F_k' y_k + \sum_k \gamma_k Q_k(\boldsymbol{x}) \tag{9}$$

$$\text{subject to} \qquad \sum_k x_{ijk} = 1, \qquad \forall\, i, j, \tag{10}$$

$$x_{ijk} \leq y_k, \qquad \forall\, i, j, k, \tag{11}$$

$$x_{ijk} \geq 0, \qquad y_k \in \{0, 1\}, \qquad \forall\, i, j, k. \tag{12}$$

2.3 Properties

In this subsection we develop a key property for our model. This property enables the development of an efficient solution method.

Property 1. There exists an optimal solution of the problem (P_1) for which each demand for product from plant i at retailer j is allocated to a single DC, i.e., $x_{ijk} = 0$ or $1, \forall\, i, j, k$.

Proof: See Appendix.

By Property 1, to solve the problem, we can divide the whole demand flow set $\{(i,j) : \forall\, i \in I, j \in J\}$ into different subsets and find the best partition of the demand flow set. This can be formulated as the set covering problem.

2.4 Set covering model and column generation approach

We now reformulate our problem as a set covering problem. Since an optimal solution to the problem (P_1) consists of a partition of demand flow (i,j) into nonempty subsets, we can find the partition from all nonempty subsets of the demand flow set by solving the set covering problem. Let $\mathcal{S}$ be the collection of all nonempty subsets of the demand flow set, i.e., $\mathcal{S} = \{W_1, W_2, \cdots, W_s, \cdots\}$. Let a_{ijs} be a constant that is equal to 1 if demand flow (i,j) is included in subset W_s and 0 otherwise, and $c_{s,k}$ be the related cost if W_s is assigned to candidate DC k. Then,

$$c_{s,k} = F_k^{'} + \sum_{i,j} a_{ijs}\alpha_{ijk} + \gamma_k \sqrt{\sum_{i,j} f_{ij} a_{ijs}}.$$

We define c_s to be the lowest cost of having one DC serve the demand flow set W_s, i.e., $c_s = min_k c_{s,k}$.

Let decision variable $z_s = 1$ if the demand flow set W_s is selected to be served by a

DC and 0 otherwise. The problem (P_1) can be reformulated into a set covering problem as follows:

$$(SC) \qquad \min_{\boldsymbol{z}} Z^{SC}(\boldsymbol{z}) = \sum_{W_s \in \mathcal{S}} c_s z_s \tag{13}$$

$$\text{subject to} \qquad \sum_{W_s \in \mathcal{S}} z_s \leq K, \tag{14}$$

$$\sum_{W_s \in \mathcal{S}} a_{ijs} z_s \geq 1, \qquad \forall\, i, j, \tag{15}$$

$$z_s \in \{0, 1\}, \qquad \forall\, W_s \in \mathcal{S}. \tag{16}$$

Constraint (14) ensures that the total number of the selected demand flow sets is less than the total number of the candidate DC sites. Constraint (15) guarantees that each demand flow belongs to at least one selected demand flow set. Constraint (16) is the integrality constraint. For this set covering problem, we obtain the following property:

Property 2. There is no optimal solution for the problem (SC) such that W_s and $W_{s'}$ are assigned to the same DC, where W_s and $W_{s'}$ are any two demand flow sets.

Proof: See Appendix for details.

This property guarantees that only one demand flow set will be assigned to a selected DC in the optimal solution.

The number of columns involved in this formulation is exponential. Neither the set covering problem nor its linear programming relaxation can be solved by a method that first generates all feasible columns explicitly. We therefore resort to a column generation approach.

Let $(\bar{SC})$ be the linear programming relaxation of (SC) and $(\bar{SC}_{\mathcal{S}'})$ be the master problem of $(\bar{SC})$ in which a subset $\mathcal{S}'$ of $\mathcal{S}$ is available. Thus, the master problem $(\bar{SC}_{\mathcal{S}'})$ is as follows:

$$(\bar{SC}_{\mathcal{S}'}) \qquad \min_{\boldsymbol{z}} Z^{\bar{SC}_{\mathcal{S}'}}(\boldsymbol{z}) = \sum_{W_s \in \mathcal{S}'} c_s z_s \tag{17}$$

$$\text{subject to} \qquad \sum_{W_s \in \mathcal{S}'} z_s \leq K, \tag{18}$$

$$\sum_{W_s \in \mathcal{S}'} a_{ijs} z_s \geq 1, \qquad \forall\, i, j, \tag{19}$$

$$0 \leq z_s \leq 1 \qquad \forall\, W_s \in \mathcal{S}^{'}. \tag{20}$$

The primary issue is the design of the pricing algorithm. We know that a solution to a minimization problem is optimal if the reduced cost of each variable is nonnegative. To test whether the current solution is optimal, we determine if there exists a $W_s \in \mathcal{S}$ with negative reduced cost, which leads to the pricing problem (SC_{P_k}) for each candidate DC site k.

$$(SC_{P_k}) \qquad \min \qquad (F_k^{'} - \eta) + \sum_{i,j} (\alpha_{ijk} - \pi_{ij}) x_{ijk} + \gamma_k \sqrt{\sum_{i,j} f_{ij} x_{ijk}} \tag{21}$$

$$\text{subject to} \qquad x_{ijk} \in \{0, 1\} \qquad \forall\, i, j, \tag{22}$$

where η and π_{ij} are corresponding optimal dual costs associated with (18) and (19). The objective function of (SC_{P_k}) is a concave function of $\boldsymbol{x}$. Coincidently, the pricing problem is very similar to the one in Shen, Coullard, and Daskin (2003). They developed an effective approach to solve such a pricing problem. We follow their idea to solve the problem (SC_{P_k}). For given DC site k, $(F_k^{'} - \eta)$ is a constant. We essentially need to solve the following pricing problem:

$$(P_k) \qquad \min_{\boldsymbol{x}} Z^{P_k}(\boldsymbol{x}) = \sum_{i,j} (\alpha_{ijk} - \pi_{ij}) x_{ijk} + \gamma_k \sqrt{\sum_{i,j} f_{ij} x_{ijk}} \tag{23}$$

$$\text{subject to} \qquad x_{ijk} \in \{0, 1\} \qquad \forall\, i, j. \tag{24}$$

Let $\boldsymbol{x}^*$ be an optimal solution to the problem (P_k) and the minimum reduced-cost set $W_s = \{(i,j) : x^*_{ijk} = 1\}$, and $G^*_k = (F'_k - \eta) + Z^{P_k}(\boldsymbol{x}^*)$. If $G^*_k \geq 0$, then we can conclude that there is no set W_s assigning to DC k with negative reduced cost. If $\forall k$, $G^*_k \geq 0$, we can conclude that there is no set $W_s \in \mathcal{S}$ with negative reduced cost.

Now, let

$$\frac{\alpha_{i_1 j_1 k} - \pi_{i_1 j_1}}{f_{i_1 j_1}} \leq \frac{\alpha_{i_2 j_2 k} - \pi_{i_2 j_2}}{f_{i_2 j_2}} \leq \cdots \leq \frac{\alpha_{i_h j_h k} - \pi_{i_h j_h}}{f_{i_h j_h}}.$$

Then, the following Theorem is an immediate consequence from Shen, Coullard, and Daskin (2003).

Theorem There is an optimal solution x^*_{ijk} to the problem (P_k) in which the following properties hold:

1. If $\alpha_{ijk} \geq \pi_{ij}$, then $x^*_{ijk} = 0, \forall\, i, j$.

2. If $x^*_{i_t j_t k} = 1$, for some $t \in \{1, 2, \cdots, h\}$, then $x^*_{i_l j_l k} = 1$, for all $l \in \{1, 2, \cdots, t-1\}$.

By the Theorem, we can develop an algorithm to solve the pricing problems in polynomial time. In fact, we can solve the problem (P_k) by enumeration, i.e., by generating all solutions with the properties and selecting the one with the lowest objective function value. The computational complexity for sorting is $O(mnlog(mn))$, m and n are the number of retailers and plants, respectively. There are total mn such solutions for the pricing problem. Therefore, the computational complexity for the pricing problem is $O((mn)^2 log(mn))$.

3 Discrete size case

For simplicity in presentation, we assume that there are an equal number of size options for each DC (this can be achieved by using an infinite cost when fewer options are provided). Suppose that we have total L pre-selected size options for each DC k, denoted by c_{kl}, $l = 1, 2, \cdots, L$. The corresponding DC total construction cost is $F_{kl} = F_k + t_k c_{kl}$. For a size option c_{kl}, the probability of needing more than c_{kl} units of space at DC k is an increasing function of arrival rate. Therefore, for a given threshold overflow probability β_k, if the arrival rate is less than a threshold value, the probabilistic constraint (4) will be satisfied. Let λ_{kl} be this threshold arrival rate. From Section 2.1, we know that the arrival rate in DC k can be written as $\lambda_k = \sum_{i,j} f_{ij} x_{ijk}$. Thus, the constraint (4) is equivalent to the following if the DC k size option is c_{kl}:

$$\sum_{i,j} f_{ij} x_{ijk} \leq \lambda_{kl}.$$

In order to reformulate the problem (P) into a mixed integer linear program, we introduce a DC size selection variable, y_{kl}. Let $y_{kl} = 1$ if a DC is located at candidate site k with size option l, and 0 otherwise. The original problem (P) can then be formulated as a capacitated connection location problem with discrete size options as follows:

$$(P_2) \qquad \min_{\boldsymbol{x}, \boldsymbol{y}} Z^{P_2}(\boldsymbol{x}, \boldsymbol{y}) = \sum_{i,j} \sum_k f_{ij}(u_{ik} + v_{kj}) x_{ijk} + \sum_k \sum_l F_{kl} y_{kl} \tag{25}$$

$$\text{subject to} \qquad \sum_k x_{ijk} = 1, \qquad \forall\, i, j, \tag{26}$$

$$\sum_{i,j} f_{ij} x_{ijk} \leq \sum_l \lambda_{kl} y_{kl}, \qquad \forall\, k, \tag{27}$$

$$\sum_l y_{kl} \leq 1, \qquad \forall k, \tag{28}$$

$$x_{ijk} \geq 0, \qquad y_{kl} \in \{0, 1\}, \qquad \forall\, i, j, k, l. \tag{29}$$

The objective function (25) minimizes the total cost, which is the sum of the DC construction costs and the shipping cost. Constraint (26) stipulates that the product demands only shipping via DCs. Constraint (27) is a size capacity constraint. Constraint (28) assures that only one size option is selected for each DC. Constraints (29) are the non-negativity and integrality constraints.

Therefore, the original problem (P) can be solved by the following two-step procedure:

- Step 1. Find the threshold arrival rates, λ_{kl}, for each options.

- Step 2. Solve the problem (P_2).

In order to find the threshold arrival rates, λ_{kl}, either approximation or simulation approaches can be employed. In the approximation approach, λ_{kl} can be obtained from (8) for given size option and overflow probability. For given c_{kl} and β_k, we obtain:

$$\lambda_{kl} = \mu_k c_{kl} + 0.5\mu_k(q_{\beta_k}^2 - \mu_k q_{\beta_k}\sqrt{q_{\beta_k}^2 + 2c_{kl} - 1} - 1).$$

For solving the problem (P_2), we use the Lagrangian decomposition method presented in Huang, Batta, and Nagi (2003). They relax the capacity constraint (27) and decompose (P_2) into two relatively easy subproblems. Their computational experiments showed that the approach can deal with problems having up to 3000 flows, 200 candidate connection sites with 6 size options in about one hour of CPU time. The average heuristic gap for all combined data was found to be less than 2%.

4 Computational results

In this section, we design experiments to test the performance of the column generation approach developed in Section 2.4, to compare the results of the continuous and discrete models, and to show the benefit of the simultaneous consideration of DC location/routing and size. All algorithms were coded in C++ and tested on a Dell Precision 330 Pentium 4 with 1700 MHZ CPU and 512 MB RAM. The solver for the linear- and integer-program problems is CPLEX 7.1.

The test problems were randomly generated as follows. First, we generated plants, retailers and candidate DC sites' locations, which were decided by their x- and y-coordinates. These coordinate values were randomly selected from $U(0, 200)$, where U denotes a uniform distribution. For each pair of demand flow (i, j), the amount of its mean demand was randomly drawn from $U(1, 10)$. We assume $t_k = 1$ and $\mu_k = 1$. In all cases, we let the threshold overflow probability $\beta_k = 0.05$. The fixed DC construction cost F_k was randomly drawn from different ranges: $U(200, 500)$, $U(1000, 2500)$, and $U(2000, 5000)$, in order to test how the fixed DC construction cost affects the solution difficulty. The unit shipping costs, u_{ik} and v_{kj}, were set to be d_{ik} and d_{kj}, respectively, where d_{ik} is the rectlinear distance between plant i and candidate DC site k and d_{kj} is the rectlinear distance between candidate DC site k and retailer j.

The headers of the columns in Tables 3 through 6 are:

- flow #: number of plant-retailer pairs;
- plant #: number of plants;

- DC #: number of DCs;
- retailer #: number of retailers;
- CPU (s): CPU times (seconds) consumed for solving the instance;
- CG #: total number of columns added;
- GAP: (best solution value - lower bound)/lower bound * 100;
- Objective Value: objective function value in the solution;
- # of DC opened: total number of DCs opened in the solution.

4.1 Computational results for the column generation algorithm

Here we report the performance of the column generation algorithm for the continuous case. The initial columns were obtained by solving the shortest path problem for the problem instance. We iteratively added columns for all k with negative reduced cost to the linear program after having solved the pricing algorithm. We stopped generating new columns when the gap was less than 1%. For the different range of fixed DC construction cost, the computational results for this case are showed in Table 3, 4 and 5. Figure 3 summarizes the results. We can see that the problem becomes difficult to solve if the fixed DC construction cost increases as other parameters remain unchanged. We can expect this since the pricing problem usually needs more time to add columns if the fixed DC construction costs are large.

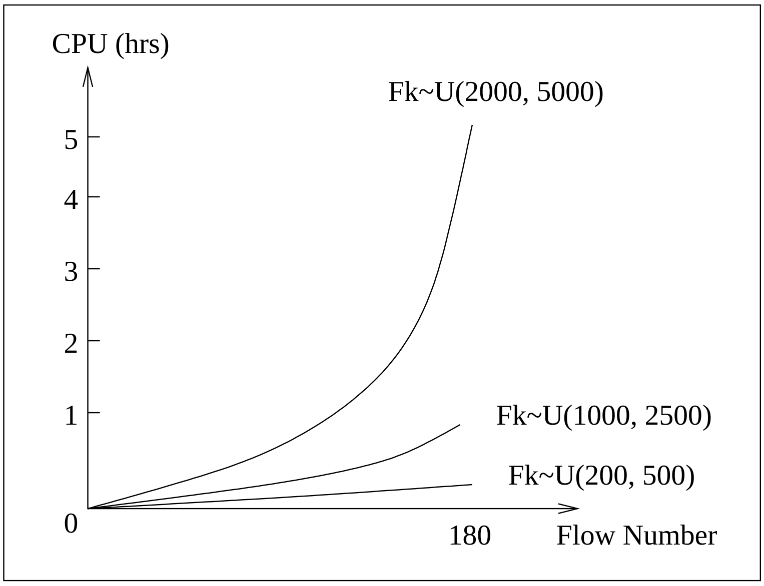

Figure 3: Computational results for different fixed DC construction costs

4.2 Comparison of continuous and discrete cases

In this subsection, we compare the solution results of the continuous and discrete models. The motivation for this is as follows: (i) the column generation method might not solve large problems (see Table 5) while the discrete model may be able to; and (ii) if we discretize the continuous case to a discrete model, what is the solution gap between them and is the gap consistent? The problem we selected for the comparison has a size of 180 flows, 4 plants, 15 connections, and 45 retailers, with fixed cost from $U(2000, 5000)$. A problem with this size is difficult for the continuous case. We randomly generated 20 problems of this size.

Some additional parameters have to be set for the discrete case. We let the total number of size options $L = 3$ and $F_{kl} = F_k + t_k c_{kl}$. Here, the fixed DC construction cost F_k was randomly drawn from $U(2000, 5000)$. In order to generate feasible test problems for the discrete case, we first generated the threshold arrival rate, λ_{kl}, then using the approximate equation (8) we obtained the size options c_{kl}. The λ_{kl} was randomly decided in the following

Table 3: Computational results when fixed cost is drawn from $U(200, 500)$

	flow #	plant #	DC #	retailer #	CPU (s)	CG #	GAP (%)
1	20	2	5	10	0.30	325	0.01
2	30	2	5	15	0.16	813	0.30
3	40	2	5	20	0.27	1252	0.99
4	50	3	10	25	10.58	6983	0.64
5	60	3	10	30	11.84	7391	0.62
6	105	3	10	35	45.56	11436	0.96
7	120	4	15	30	48.66	16919	0.69
8	140	4	15	35	81.47	19106	0.86
9	160	4	15	40	274.41	28019	0.81
10	180	4	15	45	254.11	27518	0.82

Table 4: Computational results when fixed cost is drawn from $U(1000, 2500)$

	flow #	plant #	DC #	retailer #	CPU (s)	CG #	GAP (%)
1	20	2	5	10	0.08	285	0.02
2	30	2	5	15	0.17	813	0.54
3	40	2	5	20	0.95	2399	0.98
4	50	3	10	25	14.84	8098	0.82
5	60	3	10	30	90.91	15574	0.76
6	105	3	10	35	114.59	18292	0.69
7	120	4	15	30	821.02	36899	0.87
8	140	4	15	35	1132.72	49451	0.95
9	160	4	15	40	2006.61	55835	0.93
10	180	4	15	45	1815.47	49007	0.99

manner:

$\lambda_{k1} \sim U(0.7\rho, 1.0\rho)$, $\lambda_{k2} \sim U(1.1\rho, 1.4\rho)$, $\lambda_{k3} \sim U(1.5\rho, 1.80\rho)$;

where $\rho = 2\sum_{i,j} f_{ij}/|K|$, and $|K|$ is the total number of candidate DC sites. The term $\sum_{i,j} f_{ij}/|K|$ is the average flow for the total number of candidate DC sites.

The discrete model was solved by the CPLEX solver optimally instead of by the Lagrangian relaxation method. By doing this, we can eliminate the heuristic gap of the method. For all problems, CPU time for the discrete model is about 2 seconds, the average solution

Table 5: Computational results when fixed cost is drawn from $U(2000, 5000)$

	flow #	plant #	DC #	retailer #	CPU (s)	CG #	GAP (%)
1	20	2	5	10	0.38	676	0.87
2	30	2	5	15	0.91	1502	0.59
3	40	2	5	20	0.50	1185	0.80
4	50	3	10	25	88.48	14235	0.54
5	60	3	10	30	380.96	19588	0.85
6	105	3	10	35	313.20	20833	0.94
7	120	4	15	30	1222.10	42828	0.98
8	140	4	15	35	4909.27	45806	0.97
9	160	4	15	40	6943.07	51747	0.93
10	180	4	15	45	19109	123682	1.09

gap between continuous and discrete cases is 0.9%, and the standard deviation is 0.0008. The result provides evidence that discretizing the continuous model is relatively accurate and consistent, and that larger dimension problems can be solved by the discrete model.

4.3 Benefit of simultaneous consideration of DC location/routing and size

The goal of this subsection is to compare the integrated model with a sequential method. The objective value of the integrated model was obtained by solving the continuous unbounded size case. The procedure of the sequential method here was as follows: (1) solve the related uncapacitated facility location problem for given fixed cost F_k, obtain the objective value Z^u and $\lambda_k = \sum_{i,j} f_{ij} x_{ijk}$, and (2) find the DC size c_k by (8) and the total cost Z^s by $Z^s = Z^u + t_k c_k$. The related uncapacitated facility location problem was solved optimally by the CPLEX solver. For each problem, we randomly generated the unit space cost t_k from three different ranges: $U(1, 50)$, $U(50, 100)$, and $U(100, 150)$. The fixed cost F_k was drawn randomly from $U(200, 500)$. The problem size and the results are showed in Table 6. The benefit ranges from 0.8% to 14.9%. The average benefit for all the problems is about 4%. The reason for the benefit is intuitive since the sizing cost is ignored in the location decision procedure of the sequential method.

Table 6: Benefit of the integrated model

	flow #	plant #	DC #	retailer #	$t_k \sim U(1, 50)$ %	$t_k \sim U(50, 100)$ %	$t_k \sim U(100, 150)$ %
1	20	2	5	10	4.4	4.3	3.2
2	30	2	5	15	5.0	5.6	1.0
3	40	2	5	20	2.1	14.9	1.6
4	50	3	10	25	10.0	2.7	1.3
5	60	3	10	30	7.3	2.7	4.2
6	105	3	10	35	2.0	0.8	2.3

5 Conclusions and future work

In this paper an integrated DC site selection and space determination problem has been presented. The problem involves decisions on the number and locations of DCs, the amount of each DC space to be constructed in order to satisfy a pre-determined service level, as well as product routing through DCs. The objective of the problem is to minimize the sum of inbound/outbound transportation costs and DC construction costs, for given DC service levels. A key feature is the explicit treatment of a probabilistic constraint that guarantees a specified DC service level. Two cases are analyzed. In the continuous size case, an approximate expression is used and the problem transforms into a concave minimization problem, which is solved using a column generation method. In the discrete size option case, we first find a set of maximum arrival rates corresponding to the given size options. This converts the probabilistic constraint into a linear capacity constraint, thereby transforming the problem into a capacitated connection location problem with discrete size options. Computational results show that the continuous approach has smaller error gap and that the discrete approach can deal with larger size problems. Compared to a sequential procedure, the savings of the simultaneous consideration of DC location/routing and size are significant.

Further research on the integrated DC location and space requirement problem could consider a multiple period planning problem with different lease ending time. In this case, the decisions on DC include when and how long to lease DCs. Modeling the DCs as an $M/G/1$ queueing system with finite waiting space could be another approach. Yet another method would be to use an $M/G/c$ loss system with explicit leasing costs. Both these methods are worth exploring.

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Appendix

The following Lemma is needed to prove Property 1.

Lemma 1. If an optimal solution for the problem (P_1) exists where a demand for product from plant i at retailer j is assigned to two DCs k and l, then we have

$$\gamma_k \frac{\partial Q_k(\boldsymbol{x})}{\partial x_{ijk}} + \alpha_{ijk} = \gamma_l \frac{\partial Q_l(\boldsymbol{x})}{\partial x_{ijl}} + \alpha_{ijl}, \qquad \forall\, i, j. \tag{30}$$

Proof: If the statement is not true, then we arbitrarily assume that

$$\gamma_k \frac{\partial Q_k(\boldsymbol{x})}{\partial x_{ijk}} + \alpha_{ijk} < \gamma_l \frac{\partial Q_l(\boldsymbol{x})}{\partial x_{ijl}} + \alpha_{ijl}, \qquad \forall\, i, j. \tag{31}$$

If we perturb x_{ijk} by $\delta \geq 0$ and x_{ijl} by $-\delta$, then the net improvement of the objective function value equals to (assuming that both DCs k and l are still open after the perturbation):

$$\alpha_{ijk}(x_{ijk} + \delta) + \gamma_k Q_k(x_{ijk} + \delta) + \alpha_{ijl}(x_{ijl} - \delta) + \gamma_l Q_l(x_{ijl} - \delta)$$

$$-(\alpha_{ijk} x_{ijk} + \gamma_k Q_k(\boldsymbol{x}) + \alpha_{ijl} x_{ijl} + \gamma_l Q_l(\boldsymbol{x}))$$

$$= \delta[\gamma_k \frac{Q_k(x_{ijk} + \delta) - Q_k(x_{ijk})}{\delta} + \alpha_{ijk} - (\gamma_l \frac{Q_l(x_{ijl} + \delta) - Q_l(x_{ijl})}{\delta} + \alpha_{ijl})],$$

where $Q_k(x_{ijk} + \delta)$ denotes that only variable x_{ijk} is perturbed by δ in the function $Q_k(\boldsymbol{x})$ and all other variables are kept unchanged. Since $\frac{\partial Q_k(\boldsymbol{x})}{\partial x_{ijk}} = lim_{\delta \to 0} \frac{Q_k(x_{ijk}+\delta) - Q_k(x_{ijk})}{\delta}$, by (31), the net improvement of the objective function value will be negative when $\delta \to 0$. The statement follows from the ensuing contradiction. ■

Property 1. There exists an optimal solution of the problem (P_1) for which each demand for product from plant i at retailer j is allocated to a single DC, i.e., $x_{ijk} = 0$ or $1, \forall\, i, j, k$.

Proof: Suppose that there is an optimal solution for the problem (P_1) with a demand for product from plant i at retailer j is allocated to at least two different DCs, say k and l. Then both x_{ijk} and x_{ijl} are positive. Let us arbitrarily increase x_{ijk} by an amount $\delta = x_{ijl}$ and decrease x_{ijl} to zero. Since $Q_k(\boldsymbol{x})$ and $Q_l(\boldsymbol{x})$ are concave, we obtain,

$$Q_k(x_{ijk}+\delta) - Q_k(x_{ijk}) \le \delta \frac{\partial Q_l(\boldsymbol{x})}{\partial x_{ijk}},$$

and

$$Q_k(x_{ijl}) - Q_k(x_{ijl}-\delta) \ge \delta \frac{\partial Q_l(\boldsymbol{x})}{\partial x_{ijl}}.$$

From the proof of Lemma, we know the net improvement of the objective function value is

$$\delta[\gamma_k \frac{Q_k(x_{ijk}+\delta) - Q_k(x_{ijk})}{\delta} + \alpha_{ijk} - (\gamma_l \frac{Q_l(x_{ijl}+\delta) - Q_l(x_{ijl})}{\delta} + \alpha_{ijl})]$$

$$\le \delta[\gamma_k \frac{\partial Q_k(\boldsymbol{x})}{\partial x_{ijk}} + \alpha_{ijk}] - \delta[\gamma_l \frac{\partial Q_l(\boldsymbol{x})}{\partial x_{ijl}} + \alpha_{ijl}] = 0.$$

The last equation follows because of Lemma 1. Thus, a new feasible solution is obtained which is at least as good as the original optimal solution, and which has one less non-zero variable. We repeat the process until all flows are completely allocated to a single DC. The result follows. ∎

Property 2. There is no optimal solution for the problem (SC) such that W_s and $W_{s'}$ are assigned to the same DC, where W_s and $W_{s'}$ are any demand flow sets.

Proof: Suppose that there is an optimal solution for the problem (SC) such that W_s and $W_{s'}$ are assigned to the same DC k. According to the definition of c_s, we obtain,

$$c_s + c_{s'} = c_{s,k} + c_{s',k}$$

$$= F_k^{'} + \sum_{i,j} a_{ijs}\alpha_{ijk} + \gamma_k \sqrt{\sum_{i,j} f_{ij}a_{ijs}} + F_k^{'} + \sum_{i,j} a_{ijs'}\alpha_{ijk} + \gamma_k \sqrt{\sum_{i,j} f_{ij}a_{ijs'}}.$$

Let $W_u = W_s \cup W_{s'}$, assign W_u to DC k and drop W_s and $W_{s'}$. Then the related cost becomes,

$$c_u = F_k' + \sum_{i,j} a_{ijs}\alpha_{ijk} + \sum_{i,j} a_{ijs'}\alpha_{ijk} + \gamma_k \sqrt{\sum_{i,j} f_{ij}a_{ijs} + \sum_{i,j} f_{ij}a_{ijs'}}.$$

Since $\sqrt{x}+\sqrt{y} \geq \sqrt{x+y}$, for any $x \geq 0$ and $y \geq 0$, it is easy to see that $c_u \leq c_s + c_{s'}$. By doing this and keeping everything else unchanged, we obtain a new feasible solution, which is better than the old one. The result follows. ■

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