

# Contour line construction for a new rectangular facility in an existing layout with rectangular departments

by

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## Abstract

In a recent paper, Savas, Batta and Nagi [14] consider the optimal placement of a finite-sized facility in the presence of arbitrarily-shaped barriers under rectilinear travel. Their model applies to a layout context, since barriers can be thought to be existing departments and the finite-sized facility can be viewed as the new department to be placed. In a layout situation, the existing and new departments are typically rectangular in shape. This is a special case of the Savas *et al.* paper. However the resultant optimal placement may be infeasible due to practical constraints like aisle locations, electrical connections, etc. Hence there is a need for the development of contour lines, i.e. lines of equal objective function value. With these contour lines constructed, one can place the new facility in the best manner. This paper deals with the problem of constructing contour lines in this context. This contribution can also be viewed as the finite-size extension of the contour line result of Francis [7].

**Keywords:** Contour Line, Facility Layout, Facility Location.

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# 1 Introduction

According to Francis *et al.* [8] and Bindschedler and Moore [2], a facilities layout problem may arise because of a change in the design of the product, the addition or deletion of a product from a company's product line, a significant increase or decrease in the demand for a product, changes in the design of the process, etc. Sometimes the layout has to be redesigned to include a new facility such as a single machine, cell or a department. Placement of a new facility in the presence of existing facilities can be considered as a "restricted layout problem" since in a plant layout the existing facilities will act as *barriers* where travel and new facility placement are not permitted.

There has been significant recent work in the area of planar facility location with barriers. The reader is referred to a recent book by Klamroth [10], a recent chapter by Drezner *et al.* [6], articles by Dearing, Hamacher and Klamroth [3], Dearing and Segars [4], Frieß, Klamroth and Sprau [9], Dearing, Klamroth and Segars [5], and Nandikonda, Batta and Nagi [12]. Recognizing the practical relevance of facility size consideration, Savas *et al.* [14] consider the optimal placement of a finite-sized facility in the presence of arbitrarily-shaped barriers with the median objective and rectilinear distance metric. In a layout context, barriers may be thought of as existing facilities which are usually rectangular. Therefore a special case of their model in which the barriers and the facility are assumed to be rectangular, may be applied to a layout problem where a new rectangular facility has to be optimally placed in the presence of other rectangular facilities. In a layout context, the optimal site may not be always suitable for facility placement. For example the optimal site may pose concerns due to sharp corners, jogs or narrowing of material handling aisles. Hence there is a need to find a nearby location that is usable. Contour lines, that are lines of equal objective function value, help to evaluate the costs of locations other than optimal sites. They help to find the next best solution for an existing layout problem, when the new facility cannot be placed at its intended optimal location. Francis [7] has considered this problem in the context where facilities are points. The finite-area case is more appropriate for facilities layout.

The remainder of this paper is organized as follows. In Section 2, we describe and define the problem, which is defined using the rectangular metric. In Section 3, we briefly visit the grid construction procedure of Larson and Sadiq [11] and the Equal Travel Time Line concept of Batta *et al.* [1]. We then illustrate some new properties of Equal Travel Time Lines in Section 4. Section 5 illustrates the

contour line construction procedure which is followed by a numerical example in Section 6. Section 7 describes the complexity of our solution procedure. Conclusions and directions for future research are presented in Section 8.

## 2 Problem Definition

We are given a finite number of rectangular existing facilities (EFs) in a 2D plane in which a rectangular new facility (NF) has to be placed. The shape of the NF is given and fixed. We assume that the NF is oriented with its sides parallel to the  $X$  and  $Y$  axes, and one of the four possible orientations is chosen. The procedure can be repeated for the remaining three orientations. Each EF is characterized by its four corner vertices and has one or more I/O points on its boundary. These EFs are labeled  $EF_1, \dots, EF_m$ . Material flow between the NF and the EFs takes place through a single I/O point  $X$  located on the NF boundary. Let  $(x, y)$  represent the coordinates of the I/O point of the NF,  $D$  be the set of all EF I/O points, and  $S$  be the rectangular space representing the shop floor area. Let  $d(i, j : X)$  be the shortest feasible rectilinear distance between I/O points  $i$  and  $j$  ( $i, j \in D \cup X$ ) given that  $X$  is located at  $(x, y)$ . Here the term feasible implies a path that does not penetrate the interior of the NF or EFs, which serve as impenetrable barriers to travel.

There are two types of material flows in the problem. One is the flow between the EFs and the NF and the other is the flow between all pairs of EFs. All flows takes place through a shortest feasible rectilinear distance path. In computing the distance, we permit travel on a EF or NF boundary but not inside. It certainly makes sense to recognize that the placement of the NF may block shortest path movements between pairs of EF I/O points, as Fig. 1 illustrates. The same is true for movements between  $X$  and an EF I/O point. At the NF position in Fig. 1, the NF interferes with the flow between I/O points 1-3 and 1- $X$ .

The NF needs to be placed such that: (a) the NF does not intersect the union of the interiors of the EFs, and (b) the NF is contained in  $S$ . Placing the NF reduces to locating the NF I/O point  $X$ . We refer to the set of all feasible locations for  $X$  as  $F$ . Note that feasible placements allow the boundaries of the NF and EFs to coincide. Having a single I/O point for the NF is a modeling limitation made to avoid complications that can arise due to the flow assignment choices of EF I/O points to the NF I/O points. We note here that it would be much harder to consider two or more I/O points for the new facility, since that makes up a new location problem of locating two or more

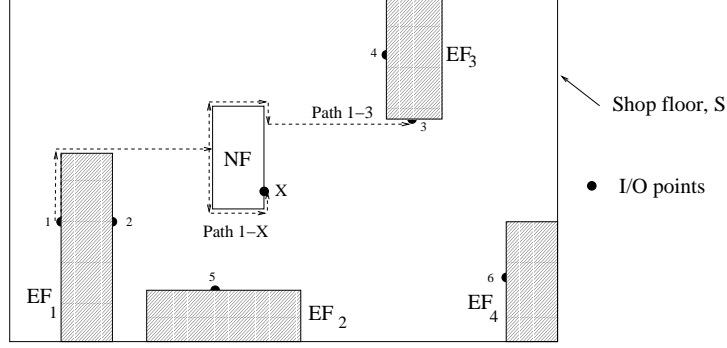


Figure 1: Example to illustrate that NF placement can affect the flow distance between a pair of EFs I/O points and between the NF I/O point and an EF I/O point

related points.

For a given  $X$ , the total weighted travel distance between EFs and the NF is:  $\sum_{i \in D} w_i d(i, X : X)$ , where  $w_i$  is the rate of material flow between the  $i^{\text{th}}$  EF I/O point and I/O point of the NF at  $X$ . Similarly, the total weighted travel distance between all EFs is:  $\sum_{i \in D} \sum_{j \in D} u_{ij} d(i, j : X)$ , where  $u_{ij}$  is the rate of material flow between I/O points  $i$  and  $j$  of a pair of EFs. The objective function is the total material distance traveled between pairs of EF I/O points and between the NF I/O point and EF I/O points, and is given by:

$$f(X) = \sum_{i \in D} w_i d(i, X : X) + \sum_{i \in D} \sum_{j \in D} u_{ij} d(i, j : X).$$

For optimal placement (Savas *et al.* [14]), the problem is to find  $X^* \in F$  such that  $f(X^*) \leq f(X), \forall X \in F$ . In the present paper we are interested in finding the set of all points that satisfy  $f(X) = z$ , where  $z$  is a constant. Such a set of points is referred to as a contour line in Francis [7].

### 3 Background

We divide the plane into regions where the objective function is linear and establish that a contour line is represented by a straight line in each region. The slope of the line potentially changes as we move from one region to the next. This is similar to the situation when EFs and the NF are points (refer Francis [7]). Though the method to find the slope in a region remains similar, the regions are more complex to determine in this case. Like the point case, a grid construction procedure is employed to determine the regions in a plane. Section 3.1 illustrates this. But in our case, additional regions need to be determined because of the effect of finite-sized NF and EFs on the traversal path

of material flow. A finite-sized NF may interfere with the traversal path of flow in some regions of the plane. These interference regions are discussed in Section 3.2. In addition, EFs may create alternate traversal paths for material flow in regions of the layout. This is illustrated in Section 3.3.

### 3.1 Grid construction and cell formation

The EFs and the I/O points of the EFs are needed for the grid construction process. A grid is formed with  $S$  as follows: Pass lines parallel to the  $x$  and  $y$  axes through all EF corners and through all EF I/O points, with each line terminated at the first interior point of an EF that it encounters or at an edge of  $S$ . Figure 2 illustrates a grid. Let  $L_h$  denote the set of horizontal traversal lines and  $L_v$  denote the set of vertical traversal lines. Also, let  $L = L_h \cup L_v$  be the set of all *EF traversal lines*. The EFs and  $L$  divide the region  $S - \bigcup_{i=1}^m \text{EF}_i$  into a number of *cells*. A cell is defined as a closed region in the grid that is not an EF. Cells have the property that a shortest feasible rectilinear path from an EF I/O point to a point located in the cell passes through one of the cell corners (see Larson and Sadiq [11]). Furthermore, the function which defines the length of a shortest feasible rectilinear path is concave over the domain of the cell (the variable being the location of the destination point in the cell). Since we consider rectangular EFs and  $S$  is assumed to be rectangular, all the cells formed will also be rectangular. When a finite-sized NF is placed, a new set of traversal lines passing through its corners and I/O point are introduced. This new set of lines are referred to as *NF traversal lines*,  $L'(X)$ , when the NF I/O point is located at  $X$ . We note that it is possible that some lines are both EF and NF traversal lines.

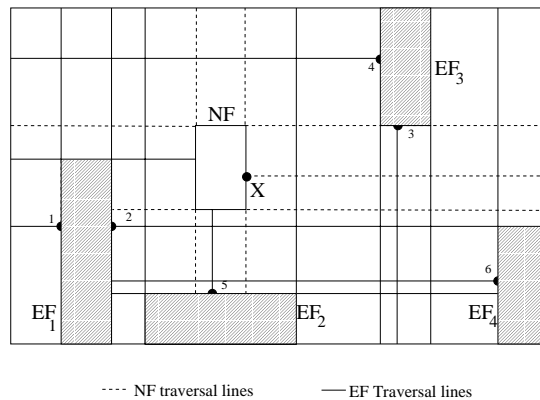


Figure 2: EF and NF traversal lines

### 3.2 Interference sets $\mathcal{Q}$

If a rectangular NF is fully-contained in a cell, the shortest feasible rectilinear path from its I/O point to an EF I/O point is still concave. In fact, the distance function is identical to that of a point NF located at  $X$ . This is shown in Lemma 2 and Theorem 1 of Savas *et al.* [14]. However, when the rectangular NF intersects one or more EF traversal lines, distance measurement from its I/O point to an EF I/O point can be affected. The disruption can be characterized provided the NF intersects a distinct set of grid lines as  $X$  is varied. This forms the motivation of interference sets. Consider a set of grid lines  $\mathcal{S} \subset L$ . We seek a maximal set of locations  $\mathcal{Q}_{\mathcal{S}}$  such that when the NF I/O location  $X \in \mathcal{Q}_{\mathcal{S}}$  the NF intersects the grid lines in the set  $\mathcal{S}$  and does not intersect the grid lines in the set  $L - \mathcal{S}$ . The set  $\mathcal{Q}_{\mathcal{S}}$  is a closed region whose boundary is composed from two types of segments: (a) locations  $X$  such that the NF boundary intersects with some EF boundary, and/or (b) locations  $X$  such that some grid line(s) in set  $\mathcal{S}$  coincide with some NF traversal line(s). An example for the formation of sets  $\mathcal{Q}$  is illustrated in Fig. 3. Consider a NF being moved around a cell such that the NF intersects the subset of the traversal lines  $\{h_i, h_{i+1}, v_j\}$ . As illustrated,  $\mathcal{Q}_{h_i}$ ,  $\mathcal{Q}_{h_{i+1}}$ ,  $\mathcal{Q}_{h_i, v_j}$ ,  $\mathcal{Q}_{v_j}$  and  $\mathcal{Q}_{h_{i+1}, v_j}$  are the sets of  $X$  locations such that the NF intersects the sets of traversal lines  $\{h_i\}$ ,  $\{h_{i+1}\}$ ,  $\{h_i, v_j\}$ ,  $\{v_j\}$  and  $\{h_{i+1}, v_j\}$  respectively. It is noted that each interference set  $\mathcal{Q}$  is rectangular and non-intersecting with another interference set.  $\mathcal{Q}_{h_i, v_j}$  is shaded for ease of visualization (while the others are defined by their corners). For example, point 6 of  $\mathcal{Q}_{h_i, v_j}$  is generated when grid line  $h_i$  is coincident with NF traversal line  $h'_1$  and grid line  $v_j$  is coincident with NF traversal line  $v'_2$ .

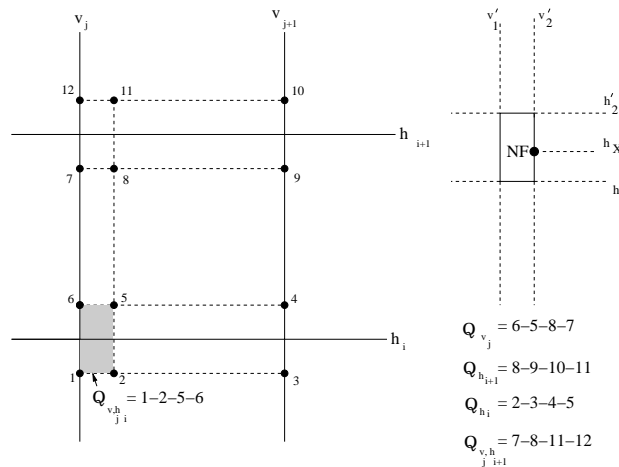


Figure 3: Interference Sets  $\mathcal{Q}$

### 3.3 Equal travel time lines

We now focus on the computation of distance from a given EF I/O point  $i$  outside cell  $C$  to a point  $X$  which belongs to cell  $C$ . We know that for a given point  $X \in C$  a shortest feasible rectilinear path from point  $i$  to  $X$  passes through a corner of the cell  $C$  (Section 3.1). However, there is no guarantee that the specific cell corner of  $C$  through which this path traverses remains the same for different points  $X$  in the cell  $C$ . An Equal Travel Time Line (ETTL) induced by  $i$  onto cell  $C$  allows us to partition  $C$  into two sub-cells such that the cell corner assignment remains the same within each sub-cell. The term ETTL arises from the fact that for all points  $X$  that lie on the ETTL itself two alternate cell corners can be used to reach point  $i$ ; i.e., they have equal travel time. The following material closely follows the presentation in Nandikonda *et al.* [12]. Consider the cell  $C$  shown in Figure 4. Let the coordinates of the four cell corners of  $C$ , labeled 1, 2, 3, and 4, be  $(0, 0)$ ,  $(0, b)$ ,  $(a, b)$ , and  $(a, 0)$ , respectively. Let  $d_{ij}$  denote the length of a shortest feasible rectilinear path from point  $i$  to cell corner  $j$ . If  $|d_{i1} - d_{i2}| < b$  then  $i$  will generate an ETTL via corners 1 and 2, which will be the line segment (parallel to the edges 1 – 2 and 3 – 4 of  $C$ ) joining  $(0, (|d_{i1} - d_{i2}| + b)/2)$  and  $(a, (|d_{i1} - d_{i2}| + b)/2)$ . If  $|d_{i1} - d_{i3}| < (a + b)$ , then  $i$  will generate an ETTL of a different type through corners 1 and 3, which will be the line segment joining  $(a - (|d_{i1} - d_{i3}| + a + b)/2, b)$  and  $(a, b - (|d_{i1} - d_{i3}| + a + b)/2)$ . This ETTL is at a  $45^\circ$  angle with sides 2 – 3 and 3 – 4 of the cell. Similarly, ETTLs can be constructed via corners 3 and 4, 2 and 4, 1 and 4, and 2 and 3. The following results hold.

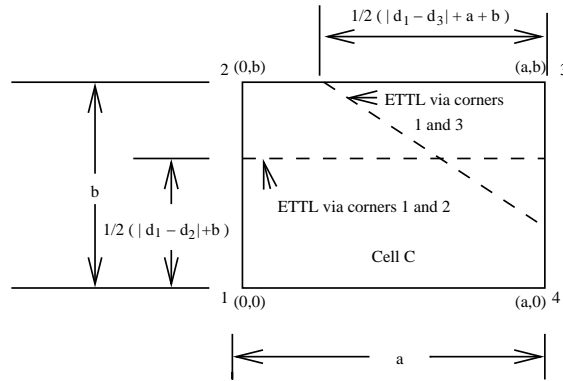


Figure 4: ETTLs in a cell  $C$

**Result 1:** An I/O point  $i$  generates at most one ETTL in a rectangular cell (Theorem 2 in Nandikonda *et al.* [12]).

**Result 2:** ETTLs between both pairs of diagonally opposing cell corners in a rectangular cell cannot exist (Lemma 4.3.1 in Sarkar *et al.* [13]).

In the traditional definition an ETTL is generated because of an EF. However, if a finite-sized NF is placed inside the cell then the shortest rectilinear paths to the I/O point may change. This may cause the ETTL to move to a new location. The concept of *Moving ETTLs* has been described in Sarkar *et al.* [13].

**Result 3:** An ETTL may move (from the ETTL position of the point NF) if the rectangular NF blocks the cell corners between which the ETTL is generated (Sarkar *et al.* [13]). By “block” we imply interference to the shortest rectilinear path.

Consider the situation in Figure 5 where the I/O point of an NF,  $X$ , is located in cell  $C$ . In the absence of the NF, I/O point 1 of the EF generates an ETTL in cell  $C$  via corners  $c_1$  and  $c_4$ . While accounting for the presence of the NF, the shortest feasible path from 1 to  $X$  can be altered or interfered due to NF. We note that this can only happen if the NF intersects at least one grid line. If  $\mathcal{S}$  is the set of grid lines intersected by the NF, then there exists an interference set  $\mathcal{Q}_{\mathcal{S}}$ . In such a situation, the ETTL in cell  $C$  can be shifted or moved due to the presence of the NF (refer to Figure 5). Argued in the opposite way, if  $X \notin \mathcal{Q}$ , the ETTL will not move.

In addition to ETTLs that are generated due to the flow between EF I/O points and the NF I/O point, there can be ETTLs between pairs of EF I/O points if the finite-sized NF causes flow interference. Specifically, this can happen when the NF intersects grid lines, or  $X \in \mathcal{Q}$ . These ETTLs are analogous to the ones already described in the sense that they partition the region  $\mathcal{Q}$  (opposed to cell  $C$ ) such that the EF I/O point flows are directed through a specific corner of  $\mathcal{Q}$ . To avoid repetitious exposition we do not elaborate on these further (see also Section 4). In the next section we illustrate some new properties of ETTLs which are necessary for our analysis.

## 4 Additional Properties of ETTLs

Like EFs, a rectangular NF can generate ETTLs when the NF interferes with the shortest rectilinear path between a pair of EF I/O points, or between an EF I/O point and the NF I/O point. This is



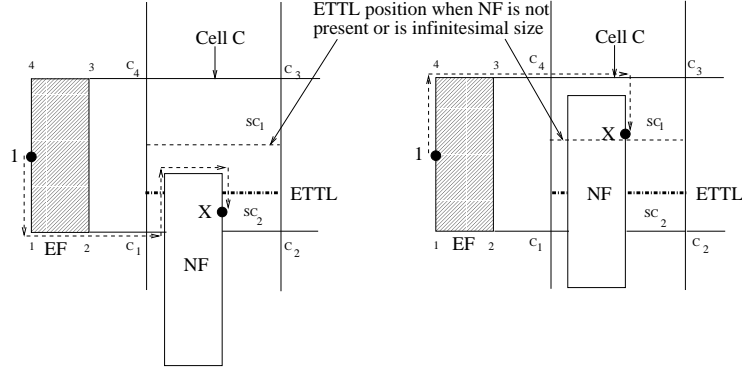


Figure 5: Illustration of a Moving ETTL

possible only when  $X \in \mathcal{Q}$  associated with the affected rectilinear path.

**Lemma 4.1** *Consider an EF-NF rectilinear flow which is interfered by the NF. If an ETTL is generated it is parallel to the affected traversal path and the edges of  $\mathcal{Q}$  and the NF which are parallel to the affected traversal path coincide with each other.*

**Proof:** Consider an NF which interferes with the shortest rectilinear path from its I/O point to an I/O point  $i$  of an EF. Let the NF have length  $H$ , width  $W$  and its I/O point at a distance  $Y$ , ( $Y > H/2$ ) on the longer side as shown in Fig. 6. The NF interferes with the traversal path  $h_i$  of the flow in the region  $\mathcal{Q}$ , where  $\mathcal{Q}$  is the union of sets  $\mathcal{Q}_{h_i}$  and  $\mathcal{Q}_{h_i, v_j}$ , associated with the affected traversal path  $h_i$ . Let  $X$  be such that the edges of  $\mathcal{Q}$  and the NF which are parallel to the traversal path  $h_i$  coincide with each other. At this position the distance traveled along the path  $i - 1 - 2 - X$  is  $Y + H - Y + const = H + const$  and along  $i - 4 - 3 - X$  is  $H - Y + Y + const = H + const$ . This is illustrated in Fig. 6a. Therefore at this position the flow from  $i - X$  can occur through corners  $1 - 2$  or  $4 - 3$  of the NF. This remains true for any position along a line through this point and parallel to the affected traversal path. Therefore, the ETTL generated will be parallel to the affected traversal path and the edges of  $\mathcal{Q}$  and the NF which are parallel to the affected traversal path coincide with each other. The ETTL generated divides the region  $\mathcal{Q}$  into  $\mathcal{Q}_1$  and  $\mathcal{Q}_2$ . Inside  $\mathcal{Q}_1$  the flow is through corners  $4 - 3$  of the NF whereas inside  $\mathcal{Q}_2$  the flow is through corners  $1 - 2$ .  $\square$

**Lemma 4.2** *For rectilinear flow between a pair of EF I/O points which is interfered with by an NF, the ETTL generated is in the interior of the region  $\mathcal{Q}$  associated with the affected traversal path and parallel to the path.*

**Proof:** Suppose that the NF interferes with the shortest rectilinear path between I/O points  $i$  and

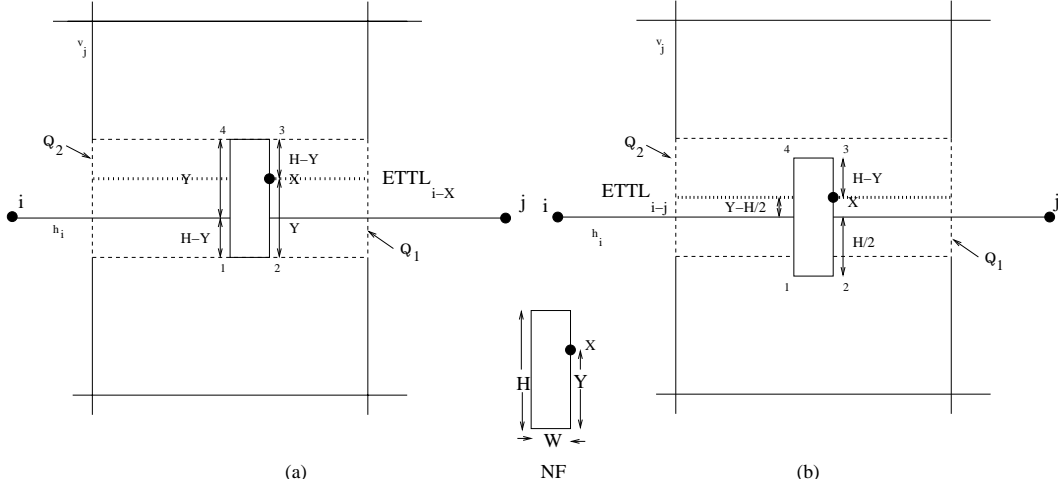


Figure 6: Illustrations for Lemmas 4.1 and 4.2

$j$ . The NF interferes with the traversal path  $h_i$  of the flow in the region  $\mathcal{Q}$ , where  $\mathcal{Q}$  is the union of sets  $\mathcal{Q}_{h_i}$  and  $\mathcal{Q}_{h_i, v_j}$ , associated with the affected traversal path  $h_i$ . Consider the NF being placed at the center of  $\mathcal{Q}$ . Since  $X$  is not at the center ( $Y > H/2$ ), the center of  $\mathcal{Q}$  will be offset from the traversal path by  $(Y - H/2)$ . At this position the distance travelled along the path  $i - 1 - 2 - j$  is  $2 * H/2 + const = H + const$  and along  $i - 4 - 3 - j$  is  $2 * (H - Y + Y - H/2) + const = H + const$ . This is illustrated in Fig. 6b. Therefore at this position the flow from  $i - j$  can occur through corners  $1 - 2$  or  $4 - 3$  of the NF. This remains true for any position along a line through this point and parallel to the affected traversal path. The line divides the region  $\mathcal{Q}$  into  $\mathcal{Q}_1$  and  $\mathcal{Q}_2$ . Inside  $\mathcal{Q}_1$  the flow is through corners  $4 - 3$  of the NF whereas inside  $\mathcal{Q}_2$  the flow is through corners  $1 - 2$ . Therefore the ETTL generated will be at the center of the region  $\mathcal{Q}$  associated with the affected traversal path and parallel to it.  $\square$

In summary the different types of ETTLS that can be generated are as follows:

- Region inside  $\mathcal{Q}$ 
  - ETTLS due to an EF I/O point for flow between the EF I/O point and  $X$  (Section 3.3).
  - ETTLS due to the NF for flow between an EF I/O point and  $X$  (Lemma 4.1).
  - ETTLS due to the NF for flow between a pair of EF I/O points (Lemma 4.2).
- Region outside  $\mathcal{Q}$  (i.e., NF fully contained in a cell)
  - ETTLS due to an EF I/O point for flow between the EF I/O point and  $X$ . In this case

the finite size of the facility will not cause the ETTL to move from the original point NF case.

We illustrate these types of ETTLs using the example shown in Fig. 7. There are two EF I/O points,  $i$  and  $j$ . As illustrated,  $i$  generates an ETTL for flow between  $i - X$  inside the cell. The NF also generates ETTLs for flow between  $i - X$  and  $i - j$ . Inside the region  $abcd$  the flow from  $i - X$  is through the upper corners  $m - n$  of the  $EF_1$ . Inside  $dclk$  it is through the lower corners  $p - o$  of the  $EF_2$ . Inside  $dclk$  the flow is through either corners  $q - r$  or corners  $t - s$  of the NF. Inside  $efgh$  the flow is through the lower corners  $t - s$  of the NF. However inside  $hglk$  it is through the upper corners  $q - r$  of the NF. Armed with the above results we can proceed with constructing contour lines.

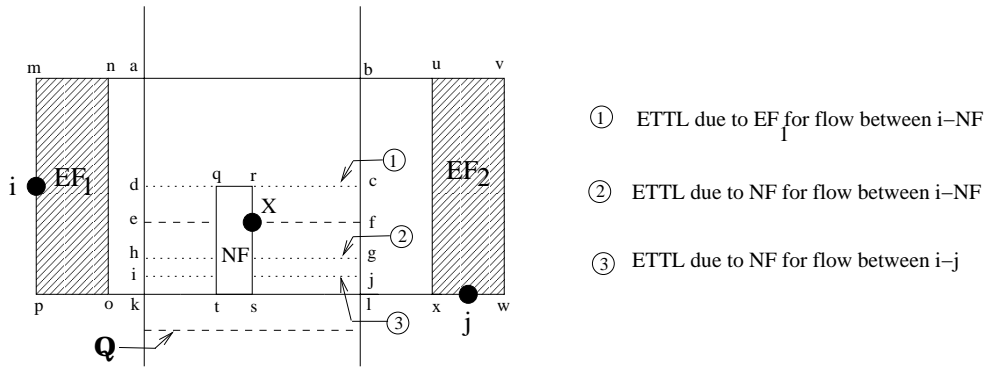


Figure 7: Different types of ETTLs

## 5 Contour Line Construction

For the function  $f(X)$  the contour line of value  $z$  is represented as  $L(z)$  where  $L(z) = \{X \in F : f(X) = z\}$ . A contour set whose boundary is a contour line is the set of all points having values of  $f(X) \leq z$ . Francis [7] has shown that for point-sized NF and EFs, the contour line is continuous, with the corresponding contour set being convex. However the finite sizes of the NF and the EFs may present complications as given below:

1. Contour lines may be intercepted by an EF and hence could be disconnected.
2. The finite-sized NF may interfere with the traversal path of flow between facilities in the region  $\mathcal{Q}$  associated with the affected traversal path. Therefore points inside  $\mathcal{Q}$  may have an objective function value higher than that of the nearby points. This can make the contour line disconnected.

In the case of point NF and point EFs, if one starts with a location of objective function value  $z$ , then the slope in the cell to which  $X$  belongs will cause the contour line to be incident on some adjacent cells and the process can be repeated until the contour is closed. In our case, the contour line and set for a particular value  $z$  can only be constructed after evaluating all the cells, at least implicitly, to identify those cells which contain an  $X$  such that  $f(X) = z$ . We start the implicit enumeration by eliminating certain cells and thus identify a set of candidate cells  $S_z$  which *potentially* contain an  $X$  such that  $f(X) = z$ . The NF may interfere with the material flow in some cells while in others it may not. In those cells where the NF interferes with the flow it does so only in the region  $Q$  associated with the affected traversal path. Therefore the set  $S_z$  may be classified as shown in Fig. 8. In our analysis, we consider the objective function and the method to calculate the slope of the contour line in each of them separately. We start by illustrating our methodology through a numerical example where an NF has to be placed in a shop floor having two EF I/O points,  $i$  and  $j$ . The weights of the flow between facilities are as shown in Fig. 9. We then present a general procedure to construct the contour line for a given problem instance.

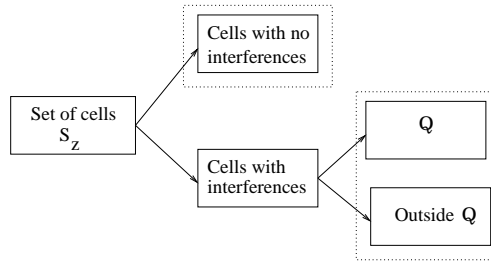


Figure 8: Classification of cells

## 5.1 Identification of cells

Inside a cell, the candidate points for the minima of the objective function of a point NF are the corners of the cell. Let  $z_p^i$  represent the objective function value of a point-sized NF located at the  $i^{th}$  corner of a cell, where  $i = 1, 2, 3, 4$ . In a cell if the minimum value of  $z_p$ , i.e.  $\min_i \{z_p^i\}$ , is greater than  $z$  then no point can be found inside the cell with value  $z$  since the placement of a rectangular NF will only increase the objective function value. If the NF is not interfering with any flow then it may be considered as a point inside the cell. In this case the maximum value of  $z_p$  at the cell corners, i.e.  $\max_i \{z_p^i\}$  should be greater than  $z$  for the cell to be a candidate. However if the NF interferes

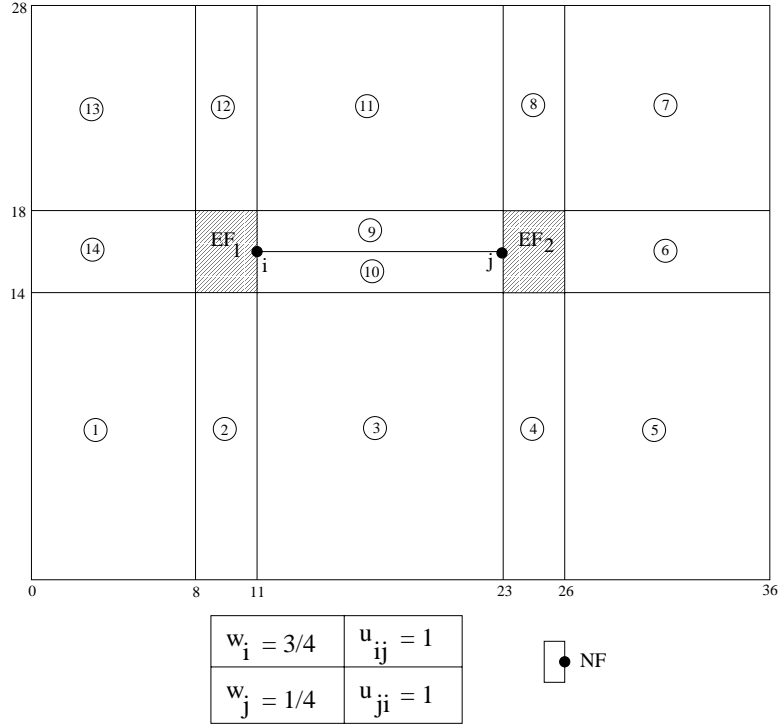


Figure 9: Numerical Example

with the flow at any of the corners and  $\min_i \{z_p^i\}$  is less than  $z$ , then a point with value  $z$  may or may not be present in these cells. The presence of such a point depends on the effect of the NF on the material flow. Based on the above observations we present an algorithm to determine the set of candidate cells,  $S_z$ , which potentially contain contour line segments of value  $z$ .

**Algorithm for determining the set of cells  $S_z$ :**

Input: Set of cells from the grid construction procedure of Section 3.1

Initialize:  $S_z = \emptyset$

For each cell  $C$ :

If  $\min_i \{z_p^i\} \leq z$

If no interferences

If  $\max_i \{z_p^i\} \geq z$

$S_z \leftarrow S_z \cup C$

Else

$S_z \leftarrow S_z \cup C$

Output: Set of cells  $S_z$

Applying this algorithm to the numerical example, for  $z = 35$ , the set of cells is  $S_{35} = \{1, 2, 3, 9, 10, 11, 12, 13, 14\}$ .

## 5.2 Objective function

In this section we analyze the objective function and illustrate the method to calculate slope of the line in each of the classifications of set  $S_z$ .

### 5.2.1 Cells with no interferences

Consider a cell  $C$  where an NF does not interfere with any of the material flow. Consider a shortest feasible rectilinear path from an I/O point  $i$  to the NF as illustrated in Fig. 10. The NF does not affect the length of the path from I/O point  $i$  to the NF, and hence, the NF may be considered as a point inside the cell. For a point NF, irrespective of its position inside cell  $C$  the total weighted travel distance between a pair of EF I/O points will be a constant. However the total weighted travel distance between EF I/O points and  $X$  will vary. The EF I/O points may be assigned to appropriate corners of the cell. It has to be noted that even though there is no effect of the NF inside the cell, an EF could generate an ETTL as defined in Section 3.3. Therefore the assignment of the I/O points may vary inside the cell.

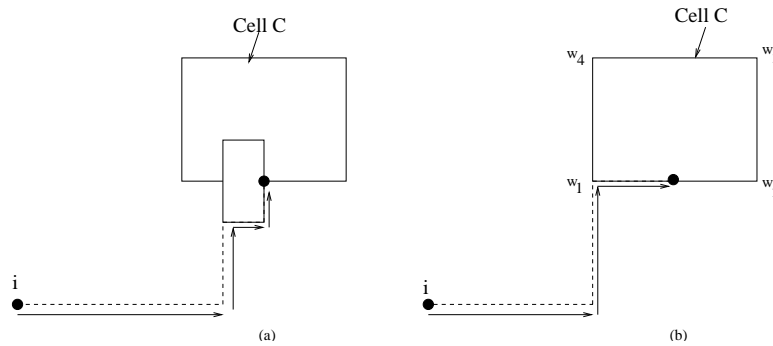


Figure 10: Cell with no interferences

Let  $w_1, w_2, w_3, w_4$  be the weights of the I/O points assigned to the corresponding corner of a cell/subcell for the purpose of shortest distance measurement. Note that corners are numbered from the lower left corner and moving counterclockwise. The objective function may be written as:

$$f(x, y) = w_1(x - x_{min}) + w_2(x_{max} - x) + w_3(x_{max} - x) + w_4(x - x_{min}) + w_1(y - y_{min}) + w_2(y - y_{min}) + w_3(y_{max} - y) + w_4(y_{max} - y) + const,$$

where  $(x_{min}, y_{min}), (x_{max}, y_{min}), (x_{max}, y_{max}), (x_{min}, y_{max})$  are the corners of the cell/subcell. As can be observed the objective function is linear in  $x$  and  $y$ . Therefore the part of the contour line inside this cell/subcell will be a line segment of the following functional form:

$$y = x \left\{ -\frac{w_1+w_4-w_3-w_2}{w_1+w_2-w_3-w_4} \right\} + const.$$

The coefficient of  $x$  yields the slope of the contour line where the numerator is the difference between the sum of the weights of the I/O points served through corners 1 and 4, and the sum of the weights of the I/O points served through corners 2 and 3; and the denominator is the difference between the sum of the weights of the I/O points served through corners 1 and 2, and the sum of the weights of the I/O points served through corners 3 and 4. This is similar to the results for point-sized NF and EFs as defined in Francis [7].

In the numerical example, cells 1, 2, 3, 11, 12, 13 and 14 of the set  $S_{35}$  have no interferences due to the NF. Inside cell 14, I/O points  $i$  and  $j$  generate ETTLs for flow to  $X$ . Therefore the assignment of the I/O points  $i$  and  $j$  to cell corners varies inside cell 14. However the NF may be considered as a point inside these and the slope may be found using the results for a point NF inside a subcell.

### 5.2.2 Inside $\mathcal{Q}$

Consider cells 9 and 10 in the numerical example where the NF interferes with the traversal path  $h_i$  of the flow between  $i - X$  and  $i - j$ .

**1. Flow between an EF I/O point and  $X$ :** Figure 11 shows the  $\mathcal{Q}$  associated with the traversal path  $h_i$ . Let the four corners of  $\mathcal{Q}$  be  $a, b, c$  and  $d$ . The ETTL for flow between  $i - X$  divides the region  $\mathcal{Q}$  into  $\mathcal{Q}1$  and  $\mathcal{Q}2$ , referred to as subcells. As shown in Fig. 11a, inside  $\mathcal{Q}1$  a shortest rectilinear path from the I/O point  $i$  to  $X$  is through  $i - 4 - 3 - X$ . Therefore the extra distance traveled due to the NF is  $Y_{h'_2} - Y_{h_i} + Y_{h'_2} - Y_X$ . The distance  $Y_{h'_2} - Y_X$  related to path  $3 - X$  is a constant. The distance that varies as we move the NF inside  $\mathcal{Q}1$  is  $Y_{h'_2} - Y_{h_i}$ . Further,  $Y_{h'_2} - Y_{h_i}$  is equal to  $Y_X - Y_{ab}$  and  $Y_{h'_2} - Y_X$  is equal to  $Y_{h_i} - Y_{ab}$ . Therefore the path  $i - 4 - 3 - X$  may be considered equivalent to the path from  $i$  through corner  $a$  to a point facility at  $X$ . Hence inside  $\mathcal{Q}1$ , the rectilinear path from the I/O point  $i$  to  $X$  can be interpreted as a path to a point-sized facility through the corner of the region  $\mathcal{Q}1$ . Similarly in  $\mathcal{Q}2$ , a shortest rectilinear path from the I/O point  $i$  to  $X$  is through  $i - 1 - 2 - X$  as shown in Fig. 11b, which is equivalent to the path from  $i$  through corner  $d$  to a point-sized facility

at  $X$ .

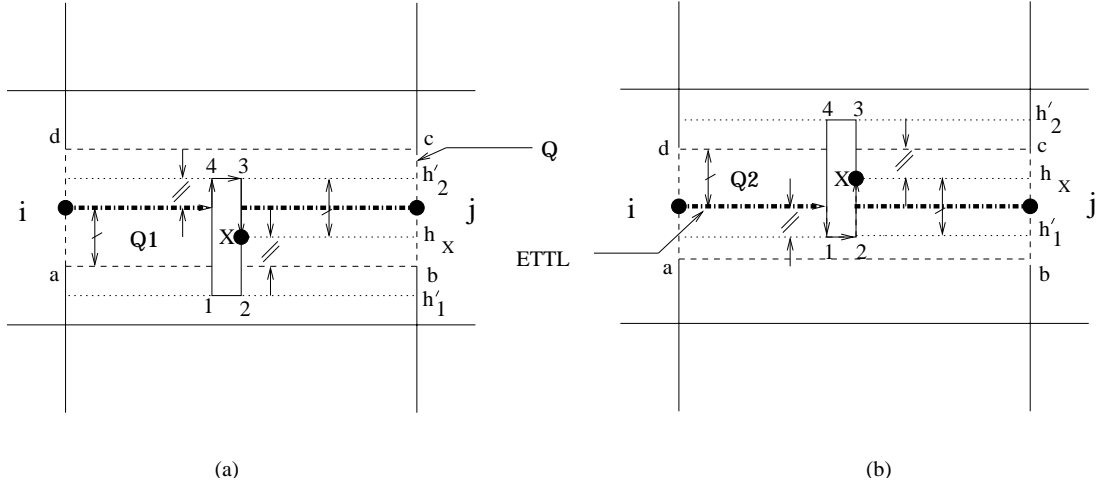


Figure 11: Shortest rectilinear paths in  $Q1$  and  $Q2$

In summary, when  $Q$  is split into subcells by an ETTL generated due to I/O point  $i$ , the shortest rectilinear distance from  $i$  to  $X$  in a subcell can be measured from the subcell corner that is closer to  $i$  and at the other side of the ETTL from  $X$ . Let  $w_1, w_2, w_3, w_4$  be the weights of the I/O points not affected by the NF and assigned to the corresponding corner of a cell/subcell for the purpose of shortest distance measurement. Then the total weighted travel distance between EF-NF inside  $Q1$  (and  $Q2$ ) is:

$$\begin{aligned}
 f_{Q1}^{EF-NF}(x, y) &= w_1(x - x_{min}) + w_1(y - y_{min}) + w_2(x_{max} - x) + w_2(y - y_{min}) + w_3(x_{max} - x) + \\
 &w_3(y_{max} - y) + w_4(x - x_{min}) + w_4(y_{max} - y) + w_i(x - x_{min}) + w_i(y - y_{min}) + const. \\
 &= x(w_1 + w_i + w_4 - w_3 - w_2) + y(w_1 + w_i + w_2 - w_3 - w_4) + const. \\
 f_{Q2}^{EF-NF}(x, y) &= w_1(x - x_{min}) + w_1(y - y_{min}) + w_2(x_{max} - x) + w_2(y - y_{min}) + w_3(x_{max} - x) + \\
 &w_3(y_{max} - y) + w_4(x - x_{min}) + w_4(y_{max} - y) + w_i(x - x_{min}) + w_i(y_{max} - y) + const. \\
 &= x(w_1 + w_i + w_4 - w_3 - w_2) + y(w_1 + w_2 - w_3 - w_4 - w_i) + const.
 \end{aligned}$$

**2. Flow between a pair of EF I/O points:** Figure 12 shows the ETTL for flow between  $i - j$  in the region  $Q$  associated with the traversal path  $h_i$ . It divides the region  $Q$  into  $Q1$  and  $Q2$ . Inside  $Q1$  a shortest rectilinear path from the I/O point  $i$  to the point  $j$  is through  $i - 4 - 3 - j$ . Therefore the extra distance traveled due to the NF is  $2(Y_{h'_2} - Y_{h_i})$ . As shown in Fig. 12a,  $Y_{h'_2} - Y_{h_i}$  is equal to  $Y_X - Y_{ab}$ . Similarly in  $Q2$ , the shortest rectilinear path from the I/O point  $i$  to  $j$  is through



$i - 1 - 2 - j$ . Therefore the extra distance travelled due to the NF is  $2(Y_{h_i} - Y_{h'_1})$ . As shown in Fig. 12b,  $Y_{h_i} - Y_{h'_1}$  is equal to  $Y_{cd} - Y_X$ .

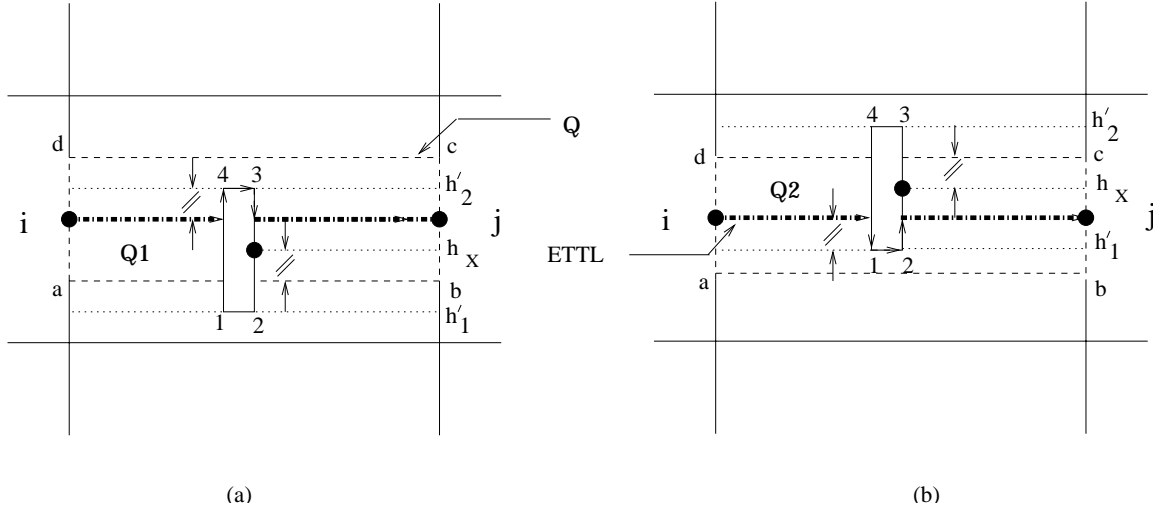


Figure 12: Shortest rectilinear paths in  $Q_1$  and  $Q_2$

Then the total weighted travel distance between EF-EF inside  $Q_1$  (and  $Q_2$ ) is:

$$\begin{aligned}
 f_{Q_1}^{EF-EF}(x, y) &= 2u_{ij}(y - y_{min}) + 2u_{ji}(y - y_{min}) + const \\
 &= y(2u_{ij} + 2u_{ji}) + const. \\
 f_{Q_2}^{EF-EF}(x, y) &= 2u_{ij}(y_{max} - y) + 2u_{ji}(y_{max} - y) + const \\
 &= y(-2u_{ij} - 2u_{ji}) + const.
 \end{aligned}$$

**3. Objective function:** Inside  $Q$  the ETTLs for flow between and  $i - X$  and  $i - j$  are at the same location as shown in Fig. 13. ETTLs divide  $Q$  into  $Q_1$  and  $Q_2$ . Considering  $Q_1$ , the objective function may be written as:

$$f_{Q_1}(x, y) = x(w_1 + w_i + w_4 - w_3 - w_2) + y(w_1 + w_i + w_2 - w_3 - w_4) + y(2u_{ij} + 2u_{ji}) + const.$$

Once again, the objective function is linear in  $x$  and  $y$ . Therefore the part of the contour line inside subcell  $Q_1$  will be a line segment of the following functional form:

$$y = x \left\{ -\frac{w_1 + w_i + w_4 - w_3 - w_2}{w_1 + w_i + 2u_{ij} + 2u_{ji} - w_3 - w_4} \right\} + const.$$

The coefficient of  $x$  gives the slope of the contour line inside  $Q_1$ . The slope of the line inside  $Q_2$  may be found in a similar manner.

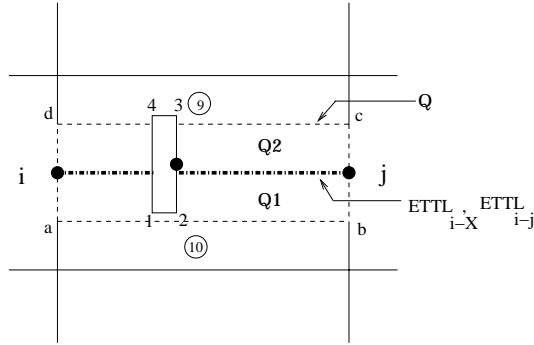


Figure 13: Subcells

### 5.2.3 Outside $Q$

Consider the region outside  $Q$  in cell 9. As illustrated in Fig. 14, the NF does not interfere with any of the material flow in the region outside  $Q$  and hence it may be considered as a point inside the region. For a point-sized facility, irrespective of its position inside the region, the total weighted travel distance between a pair of EF I/O points is a constant. However the total weighted travel distance between an EF I/O point and  $X$  will vary and the weights of the I/O points may be assigned to the corners of the region. Region outside  $Q$  may also be considered as a subcell. In this example,  $i$  may be assigned to the lower left corner and  $j$  to the lower right corner of the subcell. The slope may then be found by applying the results discussed in Section 5.2.1 for a point-sized NF.

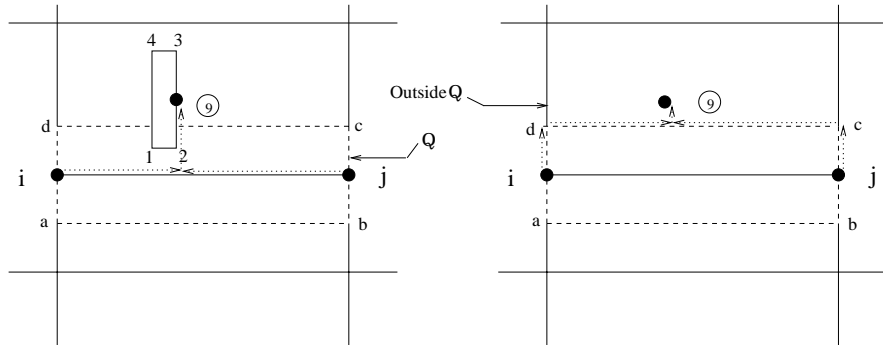


Figure 14: Outside  $Q$

## 5.3 Contour line construction method

We summarize the contour line construction procedure as follows:

Consider starting at a feasible point  $X \in F$  which has an objective function value of  $z$ .

1. Draw the EF traversal lines and form the grid.
2. Identify the cell  $C$  which contains  $X$ .
3. Identify the subcells in cell  $C$ .
  - (a) Identify ETTLs formed due to EF I/O points, for flow between EF I/O points and  $X$  in cell  $C$  and determine its subcells. Note that the NF is finite-sized and could cause the ETTL to move from the point NF case (Result 3).
  - (b) Identify the interference sets  $\mathcal{Q}_s$ , where the NF interferes with any of the material flows. Identify the ETTLs formed, due to the NF for material flow between a pair of EF I/O points or an EF I/O point and  $X$ , inside cell  $C$  and hence determine the subcells.
4. For cell  $C$  and its subcells, assign weights of the unaffected EF I/O point to  $X$  material flow to the appropriate cell corner. For affected material flow between a pair of EF I/O points and between an EF I/O point and  $X$ , assign weights to the corners as described in Section 5.2.2.
5. For cell  $C$  and its subcells, the slope of the contour line is determined as follows:
  - (a) From step 4, let,
 

$w_i$  be the sum of the weights of the material flow between EF I/O points and  $X$  assigned to corner  $i$ ,

$u_{v_j}$  and  $u_{h_k}$  be the sum of the weights of the material flow between pairs of EF I/O points affected along the vertical and horizontal path(s),

let 1, 2, 3, 4 be the corners of the cell/subcell starting from the lower left corner and moving anticlockwise, then,

$$a = (w_1 + w_4 + 2u_{v_1} + 2u_{v_4}) - (w_2 + w_3 + 2u_{v_2} + 2u_{v_3}), \text{ and}$$

$$b = (w_1 + w_2 + 2u_{h_1} + 2u_{h_2}) - (w_3 + w_4 + 2u_{h_3} + 2u_{h_4})$$
  - (b) Slope of the contour line =  $-\frac{a}{b}$
6. Proceed to construct the contour line that passes through  $X$ , using the slope computed in Step 5. If this line terminates (in either direction) at the boundary of an adjacent cell/subcell, it identifies a point in this adjacent cell/subcell that has an objective function value of  $z$ . In this

case, recompute the slope using Step 5 and then proceed to draw the continuation of this contour line. It is also possible that the line terminates at the boundary of the feasible region  $F$ . In this case, examine cells in set  $S_z$  (determined using the algorithm of Section 5.1) through which this contour line has not yet passed, and attempt to find a new starting point that has an objective function value  $z$ . To do this one can take advantage of the fact that inside a cell/subcell the assignment of EF I/O points to the corners of the cell/subcell remains unchanged. Hence the determination of a point of objective function value  $z$  can be set up as a linear program with no objective, and with constraints that limits the location of  $X$  to be within the cell/subcell as well as a constraint that specifies that the objective function value for the chosen  $X$  is  $z$ . If this linear program is infeasible, the cell/subcell contains no point with objective function value  $z$ . Otherwise, it returns one such point. The procedure is then restarted with this new location and its associated cell  $C'$ .

## 6 Numerical Example

Figure 15 shows the contour lines constructed for  $z = 31, 35, 41$  and  $43$  using the above procedure. As can be observed the contour lines for  $z = 31$  and  $35$  are intercepted by the EFs making them disconnected.

To see the application of contour line construction to a layout context, consider the situation where  $EF_1$  and  $EF_2$  are existing departments and that a new department indicated by NF needs to be placed. The difficulty is that the NF is a machine that produces significant vibration and has to be set into a concrete pad that needs to be built into the floor. The floor areas where this concrete pad can be placed is indicated in Figure 15. (In general, there might be other such disjoint areas available throughout the facility.) We immediately recognize that given this practical constraint on the NF placement, the optimum solution given by the contour line labeled “31” will not work. The solution to this constrained problem is to locate the NF at E (as shown in the figure) yielding an objective value of “35”. Aside from solving the constrained problem, contour lines dramatically improve our intuition regarding the objective space. The visual picture which is akin to an elevation map allows us to readily compare solution alternatives. Having the contour lines in the entire feasible region can give the designer a holistic perspective for evaluating construction costs and material handling costs along with other practical issues that the formulation does not address directly.

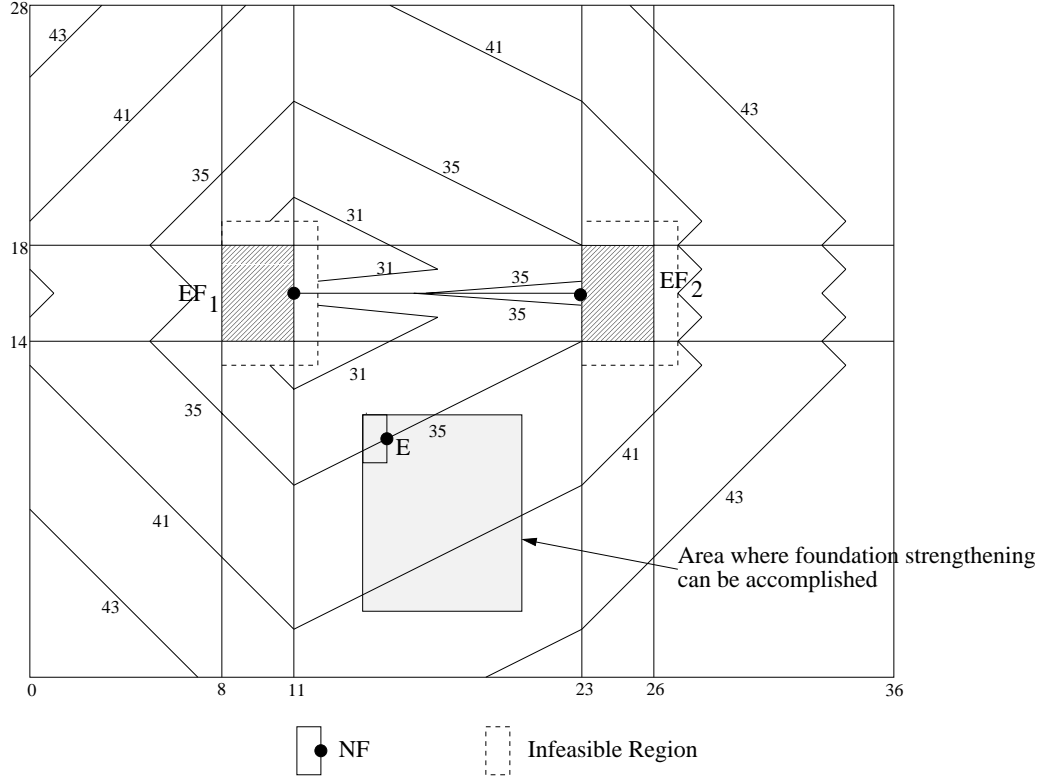


Figure 15: Contour lines

## 7 Solution Complexity

Our solution methodology proceeds in two steps. First the number of cells in which objective function value  $z$  is potentially present has to be identified using the algorithm of Section 5.1. Each EF generates 2 horizontal and 2 vertical traversal lines from its boundaries. Hence if there are  $m$  EFs,  $2m$  horizontal and  $2m$  vertical lines will be produced. Similarly  $n$  I/O points can generate at most  $n$  horizontal and  $n$  vertical lines. Hence the maximum number of cells generated is  $O((m + n)^2)$ .

The contour line is then constructed by calculating the slope of the line in each of the identified cells. Here the complexity depends on the number of subcells generated inside each cell. Result 2 states that ETTLs between diagonally-opposite corners for rectangular EFs are not possible. Nandikonda *et al.* [12] have shown that an I/O point of an EF will generate at most one ETTL. Hence  $n$  I/O points of EFs can generate  $n$  ETTLs, in the worst case generating  $O(n^2)$  subcells per cell. In addition to this, subcells may be generated due to ETTLs for material flow between a pair of EF I/O points and between an EF I/O point and  $X$ . If, in a cell, the NF interferes the material flow with  $k$  ( $k \leq n$ ) EF I/O points, then the maximum number of ETTLs formed inside the cell for flow between EF I/O

points and  $X$  will be  $k$ . Additionally the flow between  $k$  I/O points and the rest of the I/O points may be affected by the NF. This can generate a maximum of  $k(n - k)$  ETTLs for the flows between pairs of EF I/O points. In addition to the ETTLs the boundaries of  $\mathcal{Q}$ s will also divide the cell into subcells. Hence the number of subcells that can be generated inside a cell is  $O(n^2)$ . Therefore the total number of subcells that can be generated is  $O(m^2n^2 + n^4)$ , which is polynomially bound.

## 8 Summary

This paper addresses the contour line construction procedure for a finite-sized rectangular new facility to be placed in a layout having other existing rectangular facilities. Optimal placement of a finite-sized new facility in the presence of other facilities has been studied by Savas *et al.* [14]. However due to other considerations the optimal site may not be always suitable for placement of the new facility. This necessitates the new facility to be placed at alternate locations and provides the motivation for this paper. Contour lines helps to place the I/O point of the new facility at locations other than the optimal site with characterized increase in the objective function value. To draw the contour lines, we divide the plane into regions and formulate a method to calculate the slope of the contour line in each of these regions. Our work can be viewed as an extension to the work by Francis [7] who developed the contour line procedure for locating a point NF in the presence of point EFs. Francis [7] also use the concept of dividing the plane into regions and calculating the slope of the contour line in each of those regions. This work is applicable for a facilities location problem. To extend this to a layout context we have to assume finite size for the new facility and the presence of other finite-sized existing facilities. However, the finite size of the new facility and the presence of other facilities makes the problem complex to solve since travel is not permitted in these regions and the rectilinear metric gets destroyed. By a systematic division of the plane into subregions where the slope of the line remains constant, we develop a procedure for constructing layout contour lines. Unlike the location case, layout contour lines are non-convex and disconnected. Complexity analysis establishes that the methodology developed is polynomially bound.

It needs to be mentioned that from a facility location point of view the I/O points are indeed existing facilities and the rectangular regions of the EFs are barriers. Within this context, the paper deals with a planar single median facility location problem with the rectangular metric in the presence of rectangular barriers, where one of the barriers belongs to the new facility. Since the main purpose

of this paper was to develop contour lines for this problem and demonstrate its potential applicability in layout analysis (by generalizing Francis [7]), we used the terms “I/O points”, “existing facilities” and “new facility”.

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