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Commentary on facility location in the presence of congested regions with the rectilinear distance metric

Avijit Sarkar, Rajan Batta*, Rakesh Nagi

Department of Industrial Engineering, 342 Bell Hall, University at Buffalo (SUNY), Buffalo, NY 14260, USA

Abstract

This paper is a commentary on the work of Butt and Cavalier (Socio-Econ. Plann. Sci. 31(2) (1997) 103), a paper that was published in an earlier issue of this journal. With the aid of an example problem, we demonstrate that the set of gridlines proposed by them to find the rectilinear least cost path between two points in the presence of convex polygonal congested regions is inadequate. We proceed to prove its adequacy for the case of rectangular congested regions in which the edges of the rectangles are parallel to the travel directions. In wake of the difficulties of the general problem, we consider a specific example of a convex quadrilateral congestion region and a pair of external origin and destination points. Finally, we revisit the example shown in Butt and Cavalier's paper and present a mixed integer linear programming formulation that determines the optimal locations of the entry and exit points for this example.

1. Introduction

Location problems that impose restrictions on locating new facilities and/or travel through are typically referred to as *constrained* or *restricted*. Such problems have the following two topographical properties: (1) The new facilities cannot be located within certain predescribed restricted areas in the plane. (2) It is not always necessary that any two points in the plane would be "simply communicating", i.e., the minimum travel distance between any two points in the plane may be made longer by the presence of the restricted regions.

Restricted location problems have been studied by Larson and Sadiq [1] and Batta et al. [2]. Larson and Sadiq examine the rectilinear *p*-median problem with arbitrarily shaped barriers (bounded areas in $\Re^2$ which allow neither location nor travel through). Batta et al. examine the *p*-median problem in the presence of arbitrarily shaped barriers and convex forbidden regions

*Corresponding author. Tel.: +1-716-645-2357; fax: +1-716-645-3302.
E-mail address: batta@buffalo.edu (Rajan Batta).

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(bounded areas in $\Re^2$ that do not allow location but allow travel through at no extra cost) under the rectilinear distance metric.

Butt and Cavalier [3] consider such a restricted location problem in which the restriction comes in the form of a *congested region*. Congested regions are defined in [3] as closed and bounded areas in $\Re^2$ in which facility location is prohibited but traveling through is allowed at a possible additional cost. The authors introduce the concept of *least cost paths* and conclude that a rectilinear least cost path between two points in the presence of congested regions may not necessarily be the path of shortest length. They formulate the problem of calculating a least cost path as a linear program.

Based on the results obtained, the authors propose an extension of the grid construction procedure for the corresponding barrier problem considered by Larson and Sadiq. They claim that at least one least cost path will always coincide with segments of the grid obtained by drawing horizontal and vertical lines through the existing facilities and the vertices of the congested region. Based on such a grid construction procedure, the authors transform the constrained form of the planar p-median problem to an unconstrained p-median problem on a network where an optimal set of new facility locations is chosen from a finite set of candidate points.

The remainder of this paper is organized as follows. In Section 2, we critique the work of Butt and Cavalier and demonstrate that their proposed grid is not correct under certain conditions. In Section 3, we consider rectangular congested regions whose edges are parallel to the travel axes and prove the optimality of the Butt and Cavalier grid structure for this special case. In Section 4, we revisit the example presented in the Butt and Cavalier paper and present a mixed integer linear programming (MILP) formulation that determines the least cost path for this example. Finally, in Section 5, we present our conclusions and directions for further research.

2. Critique of Butt and Cavalier's paper

2.1. Some definitions and assumptions

Butt and Cavalier define a congested region as a closed and bounded area in $\Re^2$ in which a new facility cannot be located but traveling through is allowed at an additional cost per unit distance. This additional cost per unit distance is called the congestion factor of the congested region and is denoted by α, $0 \leqslant \alpha < \infty$. Thus, if w is the cost of travel per unit distance between two points lying outside a congested region, then the cost of travel between the same points when lying inside the congested region would be $(1+\alpha)w$.

The authors assume the following in their work:

- A congested region is the interior of a convex polygon that is defined by a finite number of vertices. This implies that there is no congestion along the boundary of the congested region. Thus, traveling along the boundary of a congested region would not result in an increase in the cost per unit distance.
- The congested regions are non-intersecting and share no common boundaries.
- No existing facility is located inside a congested region.

Note that barriers and forbidden regions can be considered special cases of congested regions. Barriers can be considered congested regions with $\alpha = \infty$ (because they do not allow through travel) whereas forbidden regions can be considered congested regions with $\alpha = 0$ (because they allow through travel at no additional cost).

2.2. *Least cost paths in the presence of congested regions*

Larson and Sadiq proposed a grid structure to solve the rectilinear *p*-median problem in the presence of arbitrarily shaped barriers to travel. The grid consists of tangential X and Y lines drawn through the existing facilities and the vertices of the barriers (barrier vertices are points of tangency lying on the barrier boundary through which horizontal or vertical line segments can be passed). The resulting set of lines, called node traversal lines, are terminated when they intersect barriers. With congested regions, one may wish to travel through (or slip out as in the case of barriers) depending on the location of the origin and destination points with respect to both the congested region and the congestion factor. Hence, Butt and Cavalier extend the node traversal lines of Larson and Sadiq to pass through congested regions. However, it is not necessarily the case that the rectilinear least cost path between two points in the presence of a congested region is the shortest rectilinear path between two points. This is evident from the four scenarios depicted in Fig. 1.

Fig. 1 considers a congested region $ABCD$ with congestion factor α, an origin X and a destination P. We assume without loss of generality that $w = 1$. In Fig. 1a, X and P are "simply communicating". Hence, the least cost path between X and P will never enter $ABCD$. Infinitely many such paths can be conceived. In Fig. 1b, X and P are not visible in the rectilinear sense. A possible path between them (as shown by the continuous line) has a cost $d(X,K) + (1+\alpha)d(K,M) + d(M,P)$. However, the rectilinear least cost path between X and P could be a path $XLMP$ (shown by dotted line) with a cost $d(X,L) + (1+\alpha)d(L,M) + d(M,P)$ if

$$d(X,L) + (1+\alpha)d(L,M) + d(M,P) < d(X,K) + (1+\alpha)d(K,M) + d(M,P),$$

where $d(A,B)$ denotes the length of the shortest rectilinear path between points A and B.

Fig. 1c shows two possible paths between X and P. However, the costs of the two paths are not necessarily equal even though both are shortest rectilinear distance paths between X and P. The cost of the two paths would depend on the distance traveled outside and inside the congested region, and α. Finally, Fig. 1d emphasizes that a least cost path between X and P can enter and exit a congested region more than once, thereby incurring savings in cost. However, as $\alpha \to \infty$, the least cost path will be gradually forced out of the congested region. We call the threshold value of α for which a least cost path bypasses a congested region the "*break-point*" of α.

Butt and Cavalier define any point where a path enters and leaves a congested region as an *entry point*, and an *exit point*, respectively. They formulate the problem of calculating the cost of a least cost path in the presence of congested regions as a linear programming (LP) problem. The LP determines the optimal location of a single entry and a single exit point of a least cost path. Based on the LP solution, Butt and Cavalier conclude that at least one optimal least cost path between two points will coincide with segments of the horizontal and vertical lines drawn through the two points and the vertices of the congested region.

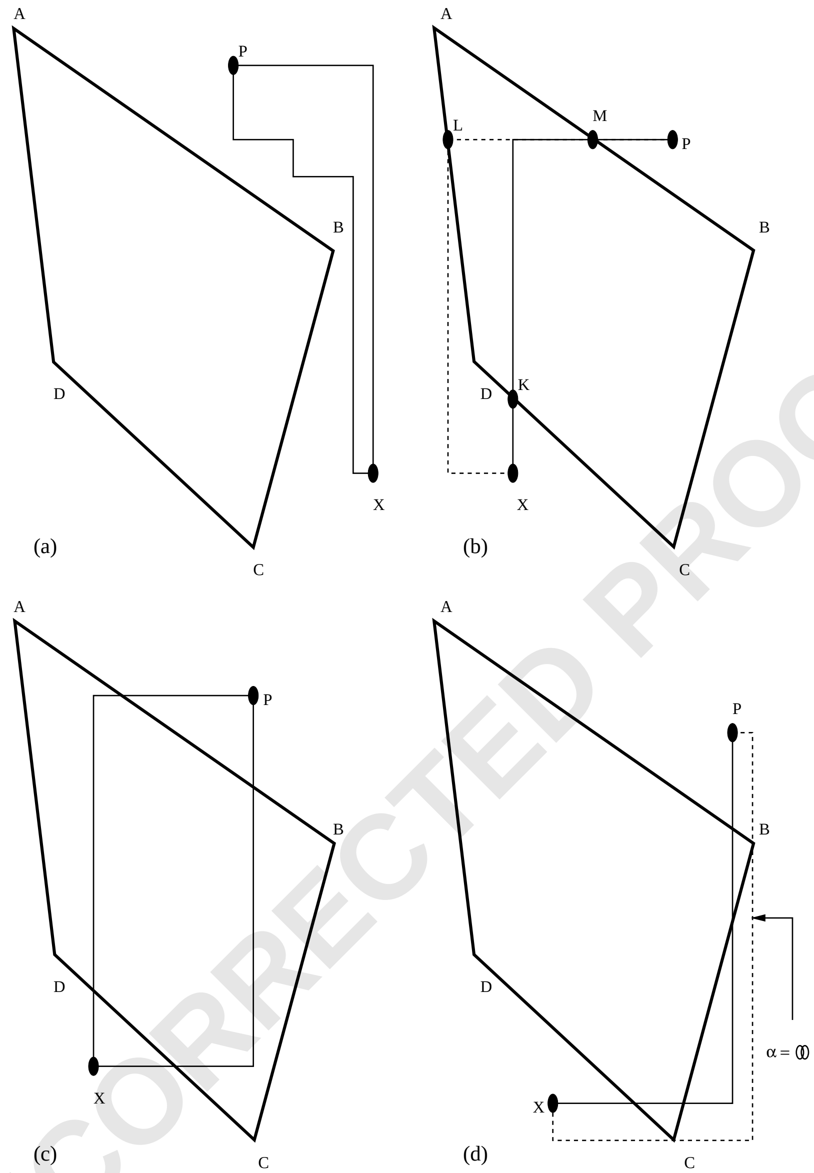

Fig. 1. Least cost paths in the presence of a congested regions: different scenarios adapted from Butt and Cavalier.

To solve the *p*-median problem in the presence of congested regions, the authors devise a grid construction procedure in which they pass horizontal and vertical lines through each congested region vertex and each existing facility location. The resulting grid divides the feasible region into cells. The main results claimed by Butt and Cavalier are:

1. The optimal 1-median in a given cell must coincide with a cell corner.

2. Based on the proposed grid construction procedure, there is at least one optimal solution to the rectilinear p-median problem in the presence of congested regions, where each new facility location coincides with a cell corner of the grid.

2.3. A contradictory example

Fig. 2a shows a four-sided congested region $ABCD$ with vertices $A(1, 11)$, $B(13, 8)$, $C(11, 2)$ and $D(2, 5)$ and two existing facility locations $X(4, 3)$ and $P(9, 10)$. According to the formulation presented in Butt and Cavalier, the rectilinear least cost path from X to P should enter $ABCD$ at $E_1(4, 4.33)$ and exit $ABCD$ at $E_2(5, 10)$. The least cost path, shown by a bold line, coincides with the grid obtained by passing horizontal and vertical lines through X, P, A, B, C and D. The cost incurred by traveling on this path for a congestion factor $\alpha = 0.3$ is 14.0 units.

However, in Fig. 2b, using the entry point $E_1'(5, 4)$ and the exit point $E_2(5, 10)$, the cost is 13.8 units for the same congestion factor. This counterexample allows us to conclude that construction of a grid as proposed in Butt and Cavalier is indeed inadequate to determine the least cost path between two existing facilities in the presence of a general set of convex congested regions. In fact, some other gridlines, as shown in Fig. 2b, are necessary for completion of the grid. The precise set of gridlines that need to be drawn is not immediately obvious and is suggested as a direction for future research.

3. Analysis for rectangular congested regions

When a congested region is a convex polygon, the locations of the entry and exit points determine the distance traversed inside (and, also, outside) the region. Entering the congested region at some point (viz. point E_1' in Fig. 2b) rather than another (viz. point E_1 in Fig. 2b) may result in a reduction of the total cost. However, this issue will not arise if the distance traversed inside a congested region is unaffected by the location of the entry and exit points of a rectilinear least cost path. This is possible if the congested regions are squares or rectangles with their edges parallel to the travel axes. This observation motivates our developing a precise grid construction procedure for rectangular congested regions.

To begin, assume that a congested region is a closed region in $\Re^2$ with a finite area and a continuous closed boundary. Let C (an open set) denote the set of points $(x, y) \in \Re^2$ contained strictly within the congested region. We also define $\bar{C} = C \cup \{\text{boundary of congested region}\}$, a closed set. Thus, $\bar{C}$ is a congested region (viz. region $ABCD$ in Fig. 3). Furthermore, we assume here that all congested regions are disjoint and have rectangular shapes, with their sides parallel to the travel axes.

We first prove (Lemma 3.1) that the Butt and Cavalier grid structure works for the case of a single rectangular congested region. We then demonstrate (Theorem 3.1) that this result also holds for multiple rectangular congested regions.

To facilitate our analysis, we define two points to be *simply communicating* if the presence of the congested regions causes no net increase in the minimum travel distance between two points. If it does cause an increase, the points are not simply communicating. We also assume, without loss of generality, that $w = 1$.

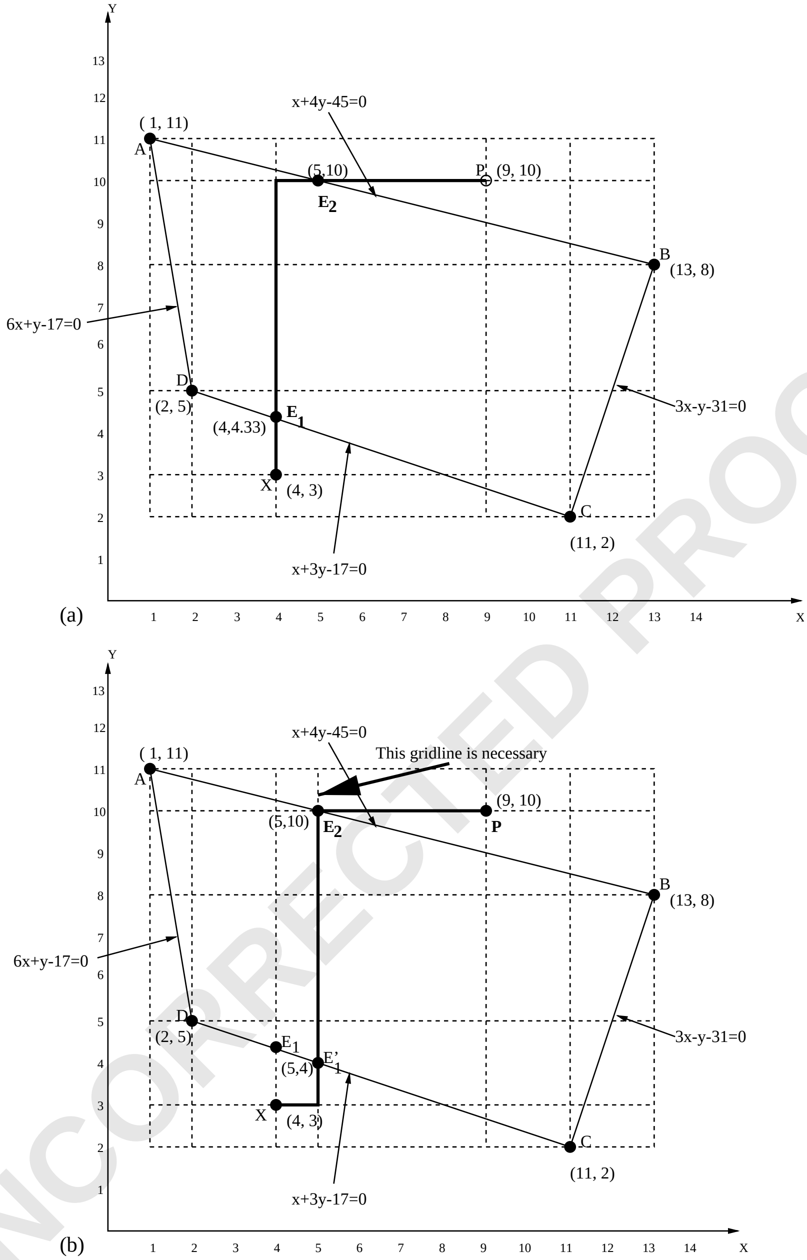

Fig. 2. Contradictory example.

Lemma 3.1. *The grid structure proposed in the Butt and Cavalier paper works for the case of a single rectangular congested region, when the edges of the rectangle are parallel to the travel axes.*

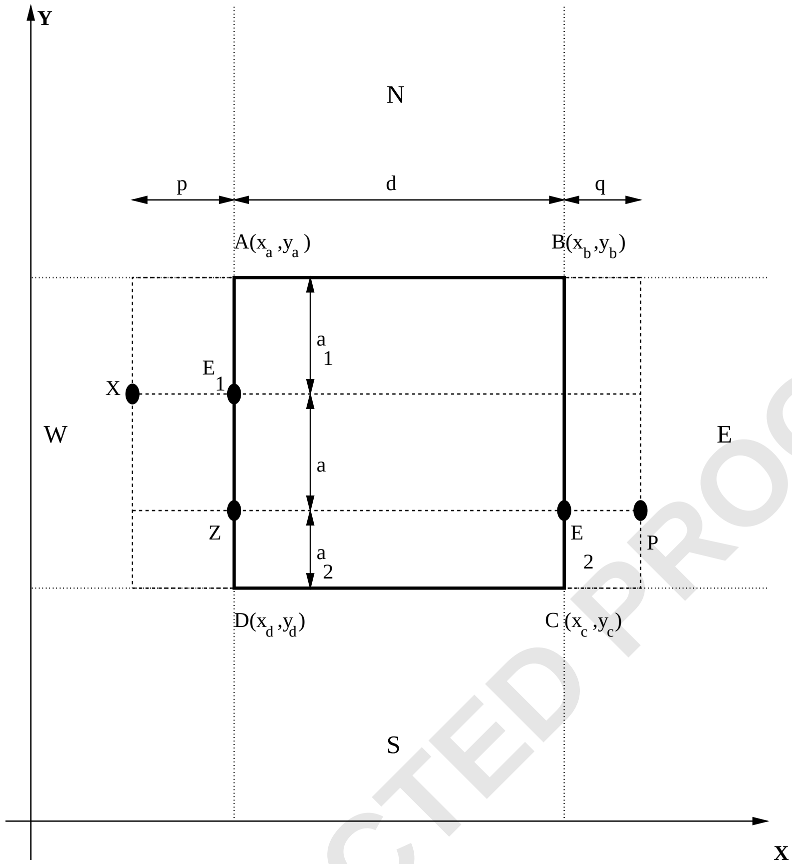

Fig. 3. A rectangular congested region $\bar{C}$, used for proof of Lemma 3.1.

Proof. We consider the following two cases with points X and P in the presence of a rectangular congested region $ABCD$ as shown in Fig. 3.

Case 1: *X and P are simply communicating*. In this case, a least cost path cannot penetrate a congested region (otherwise, its cost would increase). Thus, the congested region can be thought of as a barrier to travel and the grid structure of Larson and Sadiq would apply. However, the grid structure of Larson and Sadiq is a subset of the grid structure of Butt and Cavalier (because their grid lines terminate when they intersect a barrier). The Butt and Cavalier grid structure thus suffices.

Case 2: *X and P are not simply communicating*. Consider the congested region $\bar{C}$ shown in Fig. 3. We divide the region $\Re^2 - \bar{C}$ into regions $E, W, N, S \in \Re^2$ as shown in Fig. 3 and note the following for a point (x, y):

- $(x, y) \in E$ if $x > x_c$ and $y_c < y < y_b$,
- $(x, y) \in W$ if $x < x_d$ and $y_d < y < y_a$,
- $(x, y) \in N$ if $x_a < x < x_b$ and $y > y_a$,
- $(x, y) \in S$ if $x_d < x < x_c$ and $y < y_d$.

We consider the sample case where $P \in E$ and $X \in W$. Other situations for X and P can be analyzed in a similar manner.

In moving from X to P through $ABCD$, the total distance traveled along the path XE_1ZE_2P is $(p + d(1+\alpha) + q + a)$, as is evident from Fig. 3. In moving from X to P bypassing $ABCD$ but along the edge DC, the total distance traveled is $(a + a_2 + p + d + q + a_2)$. We would thus travel through $ABCD$ as long as $\alpha < 2a_2/d$. The congested region $ABCD$ could also be bypassed by traveling along the edge AB. In that case, it can be similarly shown that $\alpha < 2a_1/d$. We conclude that $\alpha < 2\min(a_1, a_2)/d$ implies that we travel through $ABCD$, while $\alpha \geqslant 2\min(a_1, a_2)/d$ implies that we bypass $ABCD$. We now consider the following two subcases:

Subcase 2a: $\alpha \geqslant 2\min(a_1, a_2)/d$. In this situation, we can treat the congested region as a barrier. Following the reasoning in Case 1, above, we can conclude that the Butt and Cavalier grid structure suffices.

Subcase 2b: $\alpha < 2\min(a_1, a_2)/d$. Since $\alpha > 0$, we would need to minimize the length of the path that passes through the congested region. This is achieved by traveling along the path XE_1ZE_2P. The Butt and Cavalier grid structure would work as it contains this path.

The lemma follows. □

Theorem 3.1. *The grid structure proposed in the Butt and Cavalier paper works for the case of multiple rectangular congested regions, when the edges of the rectangles are parallel to the travel axes.*

Proof. As in the proof of Lemma 3.1, consider points X and P as shown in Fig. 4, along with one congested region CR1. Applying Lemma 3.1 to this case would result in lines 1–8.

Now, consider adding a second congested region CR2. If the least cost path from X to P enters CR1, then, without loss of generality, we can assume that its entry point is d and its exit point is f. If the least cost path from f to P enters CR2, without loss of generality, its entry point will be b. But, this point has already been defined due to the earlier application of Lemma 3.1.

On the other hand, if the least cost path from X to P bypasses CR1, then its exit point, without loss of generality, is either e or g. Again, if the least cost path from e (or g) to P enters CR2, its entry point without loss of generality, will be a (or c). But, these points have also been defined earlier.

We conclude that the only required additional lines are those necessary to bypass CR2. These are lines 9–12.

By similar reasoning, for each additional congested region that is present, the only new lines that need to be introduced are those created by its edges.

The theorem follows. □

We conclude that, for the special case when congested regions are rectangles with edges parallel to the travel axes, the construction of a grid as proposed in Butt and Cavalier is adequate to determine the least cost path between existing facilities. Furthermore, the resulting grid can also be used to solve the rectilinear p-median problem in the presence of congested regions based on the solution process in Butt and Cavalier.

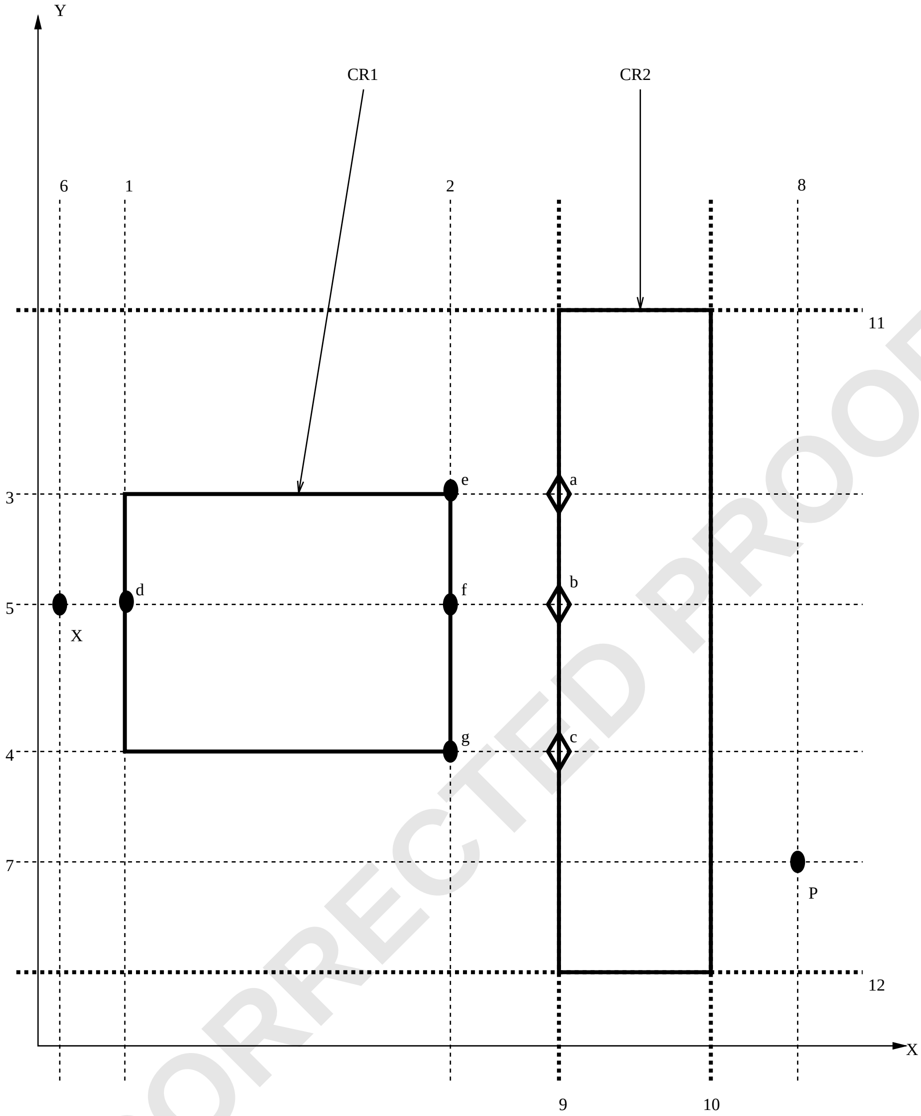

Fig. 4. Figure for proof of Theorem 3.1.

4. The Butt and Cavalier example: revisited

In this section we revisit the example in Butt and Cavalier. Our goal is to obtain a procedure that will find the least cost path from origin to destination. In order to do this, we first establish a series of results that help us limit the number of entries/exits from the congested region and also identify on which edges these entries and exits can occur. With these results in place, we then present a MILP formulation of the problem. It is not possible to formulate this as an LP since the issue of how many times we enter and exit the congested region, and on which edges these entries and exits occur needs to be explicitly considered—leading us to the need for choice (0,1) variables.

4.1. Results on entry/exit of congested region

Consider the example in Butt and Cavalier, as shown in Fig. 5. We can easily verify the following facts, some of which are presented as lemmas. Note that these lemmas may not hold for all possible shapes of the congested region, including other quadrilaterals.

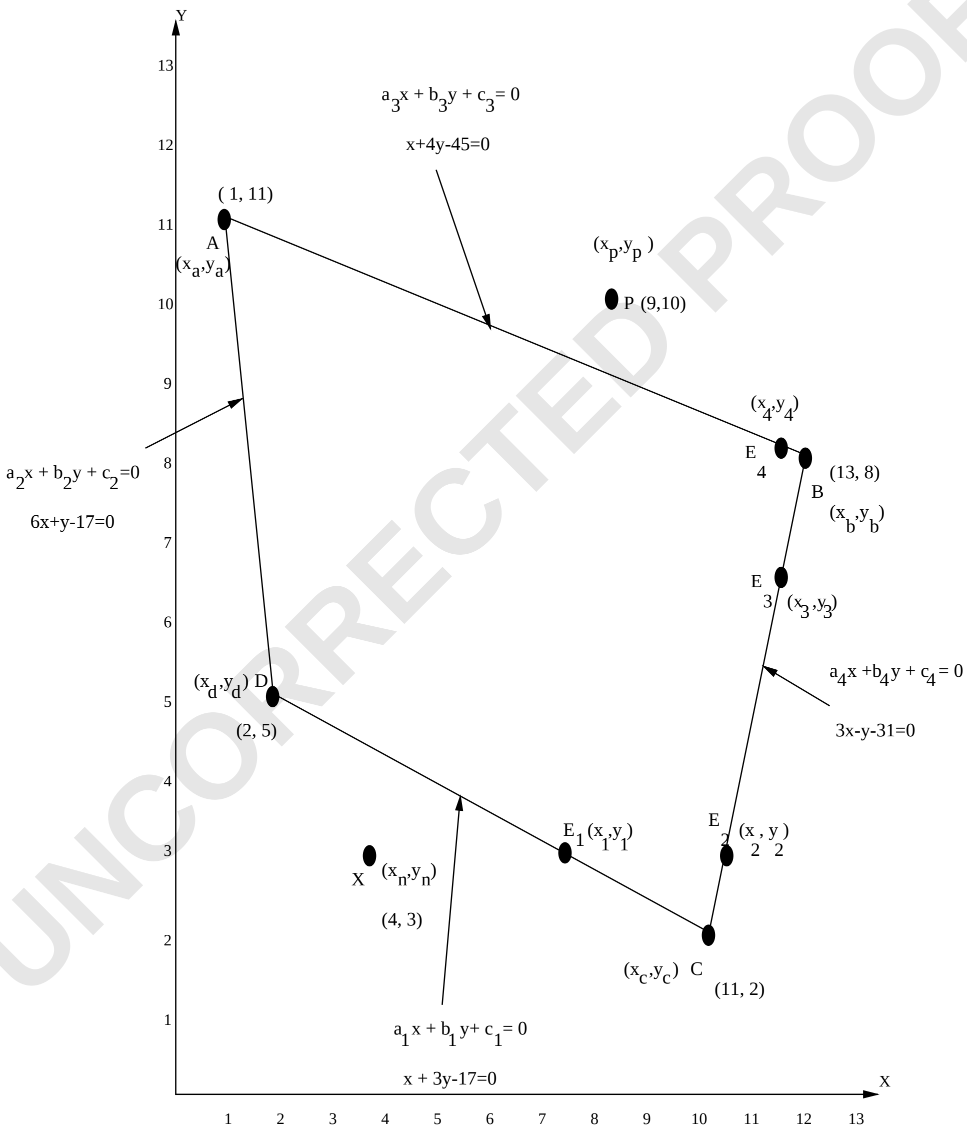

Fig. 5. Figure for MILP formulation.

1. The least cost path from X (origin) to P (destination) either bypasses the congested region $ABCD$ or enters it.
2. **Lemma 4.1**. *If the least cost path bypasses $ABCD$, the path is either $XDAP$ or $XCBP$.*

Proof. If the least cost path bypasses $ABCD$, the congested region can be thought of as a barrier to travel. In that case, the least cost path between X and P would be the shortest path between X and P that bypasses $ABCD$. The lemma follows from Theorem 2 of Larson and Li. □

3. **Lemma 4.2**. *If the least cost path enters $ABCD$, the first entry point is on edge AD, DC or BC.*

Proof. Assume the contrary, i.e. the entry point to $ABCD$ lies on edge AB. For the first entry point to be on AB, we must already have passed through either A or B (because, if we came from within the congested region, it would not be the first entry point). The lemma follows from the fact that A and B simply communicate with P. □

4. **Lemma 4.3**. *Given exactly one entry point of the least cost path, the exit point must either be any point on edge AB (excluding A, B), or the path from this exit point to P must pass through either A or B.*

Proof. If the path enters $ABCD$ exactly once, it must exit it exactly once. If this exit point lies either on edges AD, DC or CB, then the least cost path from it to P must go through either A or B. The other case is the situation where this exit point is any point on AB excluding A and B. The lemma follows. □

5. **Lemma 4.4**. *Given exactly two entry points of the least cost path*: (i) *the second exit point must be on AB (excluding A, B), or the path from this exit point to P must pass through A or B*; (ii) *the first exit point must lie on AD (excluding A, D) or on BC (excluding B, C).*

Proof. Assertion (i) follows from the arguments in Lemma 4.3. Assertion (ii) follows from the observation that if the first exit point is on AB (excluding A, B), then this point simply communicates with P. Consequently there would not be a second entry point. Also, the first exit point cannot be on edge DC, since we could directly go on a simply communicating path from X to this point and reduce the cost. The lemma follows. □

6. **Lemma 4.5**. *The least cost path from X to P will not enter the congested region $ABCD$ more than two times.*

Proof. From Lemma 4.2, we know that the first entry point is on edge AD, DC or BC. If we enter on DC, then the first exit is on edge AD, BC or AB. If we exit on AB, there are no more entries into $ABCD$, from arguments in Lemmas 4.3 and 4.4. If we exit on AD, then we may reenter on AD

at a point with a higher y-coordinate. In this case, our second exit must be on AB. This follows from the fact that edge AD is a straight line. A similar reasoning applies for edge BC. The lemma follows. □

4.2. MILP formulation

We now present an MILP formulation. The above stated lemmas allow us to conclude that this formulation is sufficient to find the optimal entry and exit points for the least cost path through the chosen congested region. Here, E_1 represents the first entry point, E_2 the first exit point, E_3 the second entry point and E_4 the second exit point. If the least cost path bypasses $ABCD$, then E_1 is coincident with C or D, E_4 is coincident with A or B, and E_2 and E_3 are both coincident with E_1 or E_4. We note here that if E_1 or E_2 or E_3 or E_4 are coincident with any vertex of $ABCD$, we no longer consider them as entry/exit points.

The MILP outputs the optimal locations of E_1, E_2, E_3 and E_4. Binary variables are needed to capture the edges on which E_1, E_2 and E_3 could lie. For simplicity in presentation, we label the edges CD, AD, AB and BC as 1, 2, 3 and 4, respectively.

The MILP follows. We note that the values for a_i, b_i, c_i, x_i^{ℓ}, x_i^{r}, x_n, y_n, x_p and y_p are obtained from Fig. 5.

$$\text{minimize} |x_n - x_1| + |y_n - y_1| + (1+\alpha)(|x_1 - x_2| + |y_1 - y_2|) + |x_2 - x_3| + |y_2 - y_3| + (1+\alpha)(|x_3 - x_4| + |y_3 - y_4|) + |x_4 - x_p| + |y_4 - y_p| + \theta$$

subject to

$$a_i x_1 + b_i y_1 + c_i + (1 - z_i)M \geqslant 0, \quad i = 1, 2, 4, \tag{1}$$

$$a_i x_1 + b_i y_1 + c_i + (z_i - 1)M \leqslant 0, \quad i = 1, 2, 4, \tag{2}$$

$$z_1 + z_2 + z_4 = 1, \tag{3}$$

$$a_i x_2 + b_i y_2 + c_i + (1 - u_i)M \geqslant 0, \quad i = 1, 2, 4, \tag{4}$$

$$a_i x_2 + b_i y_2 + c_i + (u_i - 1)M \leqslant 0, \quad i = 1, 2, 4, \tag{5}$$

$$u_1 + u_2 + u_4 = 1, \tag{6}$$

$$a_i x_3 + b_i y_3 + c_i + (1 - w_i)M \geqslant 0, \quad i = 1, 2, 4, \tag{7}$$

$$a_i x_3 + b_i y_3 + c_i + (w_i - 1)M \leqslant 0, \quad i = 1, 2, 4, \tag{8}$$

$$w_1 + w_2 + w_4 = 1, \tag{9}$$

$$x_{\ell}^{i} \leqslant x_1 + (1 - z_i)M, \quad i = 1, 2, 4, \tag{10}$$

$$x_1 + (z_i - 1)M \leqslant x_{\mathrm{r}}^{i}, \quad i = 1, 2, 4, \tag{11}$$

$$x_\ell^i \leqslant x_2 + (1 - u_i)M, \quad i = 1, 2, 4, \tag{12}$$

$$x_2 + (u_i - 1)M \leqslant x_r^i, \quad i = 1, 2, 4, \tag{13}$$

$$x_\ell^i \leqslant x_3 + (1 - w_i)M, \quad i = 1, 2, 4, \tag{14}$$

$$x_3 + (w_i - 1)M \leqslant x_r^i, \quad i = 1, 2, 4, \tag{15}$$

$$w_i \geqslant z_i + u_i - 1, \quad i = 1, 2, 4, \tag{16}$$

$$a_2x_4 + b_2y_4 + c_2 = 0, \tag{17}$$

$$x_a \leqslant x_4 \leqslant x_b, \tag{18}$$

$$\theta \geqslant |x_n - x_c| + |y_n - y_c| + |x_c - x_2| + |y_c - y_2| - (1 - z_4)M, \tag{19}$$

$$z_i, u_i, w_i \in \{0, 1\}. \tag{20}$$

z_i, u_i and w_i are the binary variables associated with the entry/exit points (x_1, y_1), (x_2, y_2) and (x_3, y_3), respectively, for sides 1, 2 and 4.

$$z_i = \begin{cases} 1 & \text{if } E_1 \text{ lies on edge } i \text{ of } ABCD, \ i = 1, 2, 4, \\ 0 & \text{otherwise} \end{cases}$$

$$u_i = \begin{cases} 1 & \text{if } E_2 \text{ lies on edge } i \text{ of } ABCD, \ i = 1, 2, 4, \\ 0 & \text{otherwise} \end{cases}$$

$$w_i = \begin{cases} 1 & \text{if } E_3 \text{ lies on edge } i \text{ of } ABCD, \ i = 1, 2, 4, \\ 0 & \text{otherwise.} \end{cases}$$

$x_\ell^i \leqslant x_r^i \ \forall i = 1, 2, 4$ denote the x-coordinates of the left and right vertices for any side i of a congested region.

For a single entry/single exit case, E_2 and E_3 will be coincident with either E_1 or E_4. Hence, we allow E_2 and E_3 to lie on CD. Constraints (1)–(3) ensure that E_1 lies on exactly one of the sides 1, 2 or 4. Here, M is a large scalar. Constraints (4)–(6) ensure that E_2 lies on exactly one of the sides 1, 2 or 4. Similarly, constraints (7)–(9) ensure that E_3 lies on exactly one of the sides 1, 2 or 4. Constraints (10) and (11) provide the bounds on the x-coordinates of E_1. Similarly, constraints (12)–(13) and (14)–(15) provide the bounds on the x-coordinates of E_2 and E_3, respectively. Constraint (16) ensures that, if the points E_1 and E_2 lie on the same edge of $ABCD$, then E_3 must also lie on that same edge. Constraints (17) and (18) ensure that the final exit point E_4 lies on edge AB of $ABCD$. Constraint (19) takes care of the extra distance that is traversed if the rectilinear least cost path goes through the vertex C of $ABCD$. Constraint (20) represents the binary variables.

We used the LP solver LINDO 6.1 to obtain solutions for different values of α (the solution times in all cases were less than a second). Our results show two possible paths as illustrated in Fig. 6a by bold lines. For $0 < \alpha \leqslant 1.33$, the path is XE_1E_4P [$E_1 = (5, 4)$, $E_2 = (5, 4)$, $E_3 =$

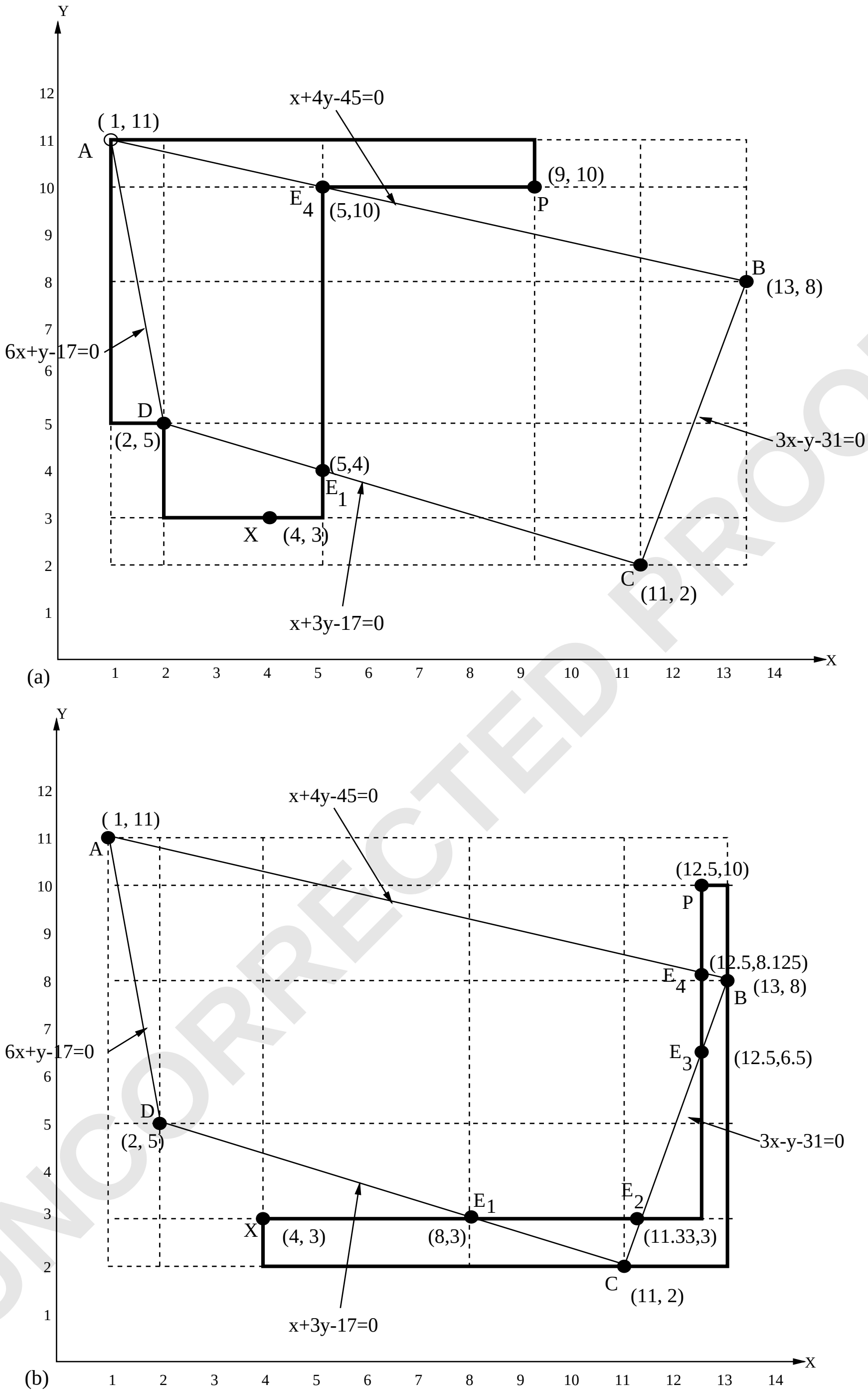

Fig. 6. Example for MILP formulation.

$(5,4)$, $E_4 = (5,10)$]. For $\alpha > 1.33$, path is $XDAP$ [$E_1 = (2,5)$, $E_2 = (2,5)$, $E_3 = (1,11)$ and $E_4 = (1,11)$] with an objective function value of 20.

The example in Fig. 6a shows that when the congestion factor, α, is reasonable, the least cost path goes through the congested region with one entry and exit. The path found is superior to that obtained by using the Butt and Cavalier grid structure—see Section 2.3.

To demonstrate the case for two entry and exit points, we consider a new example wherein the only change is that $P = (12.5, 10)$ as illustrated in Fig. 6b. We find now that for $0 < \alpha \leqslant 0.59$, the least cost path between X and P is $XE_1E_2E_3E_4P$ [$E_1 = (8,3)$, $E_2 = (11.33,3)$, $E_3 = (12.5,6.5)$, $E_4 = (12.5,8.125)$]. For $\alpha = 0.60$ and $\alpha = 0.61$, the path is XCE_3E_4P [$E_1 = (11,2)$, $E_2 = (11,2)$, $E_3 = (12.5,6.5)$, $E_4 = (12.5,8.125)$]. For $\alpha > 0.61$, the rectilinear least cost path is $XCBP$ [$E_1 = (11,2)$, $E_2 = (11,2)$, $E_3 = (13,8)$, $E_4 = (13,8)$] with an objective function value of 18.5.

The example in Fig. 6b demonstrates the need to enter and exit the congested region two times. A different choice of origin and destination points from those in the Butt and Cavalier example are needed to show this case. The intuition is that the double entry and exit reduces our travel through the congested region and hence the cost of the path.

As illustrated by the situations in Figs. 6a and b, the MILP formulation can also act as a useful tool to determine the break-point of α for any congested region.

5. Conclusion and future work

Based on the contradictory example of Section 2.3, we conclude that Butt and Cavalier's contention that at least one rectilinear least cost path will always coincide with segments of the grid formed by drawing horizontal and vertical lines through each existing facility and the vertices of a convex polygonal congested region is not correct under certain conditions. Stated in another way, a straightforward "barrier" extension of the grid structure proposed by Larson and Sadiq [1] is inadequate. Nevertheless, as shown in Section 3, the grid suffices for rectangular congested regions since, through the point of intersection of a node traversal line with the edge of a rectangle, no additional X or Y node traversal line can be drawn. This is the case as the edge of the rectangle is already perpendicular at that point.

Finally, as shown in Section 4, an MILP approach can be used to find the least cost path for the example in the Butt and Cavalier paper. When the congested regions are convex polyhedra, we conjecture that such a grid would have an extended set of node traversal lines that are perpendicular to a traditional node traversal line at its point of incidence to a congested region with $\alpha < \infty$. The completeness and optimality of such a grid structure needs to be proven. If this conjecture is true, it would imply that the methodology proposed by Butt and Cavalier could be used to solve the rectilinear p-median problem in the presence of convex polygonal congested regions—based on the modified grid.

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