STATE UNIVERSITY OF NEW YORK AT BUFFALO
Mechanical and Aerospace Engineering Department.

MAE 569 SYSTEM ANALYSIS

FINAL PROJECT

Controllability Analysis of an Underactuated Three links Manipulator

NAME: LENG-FENG LEE

ADVISOR: DR. JOHN L. CRASSIDIS

DATE: 8 Dec 2005.

Table of Contents

TITLE: **CONTROLLABILITY ANALYSIS OF AN UNDERACTUATED THREE LINKS MANIPULATOR.**

ABSTRACT:

In this project, the controllability of an under-actuated three links manipulator is being analyzed. First, the dynamic equation of a fully-actuated three links manipulator is formulated using Lagrangian formulation. Then, we linearized the nonlinear dynamic equation about its equilibrium positions, and formulated the linearized dynamic equation in state-space form. By taking away the actuator on one or more joints, we are able to obtain the dynamic equations of an underactuated system. We studied the controllability of two cases in this project. We first studied the controllability matrix of the fully-actuated system and then we look at the controllability matrix of an underactuated system-where two motors at the joints are being turned off. The controllability of the system is analyzed by examine the controllability matrix obtained from the linearized state-space model.

INTRODUCTION:

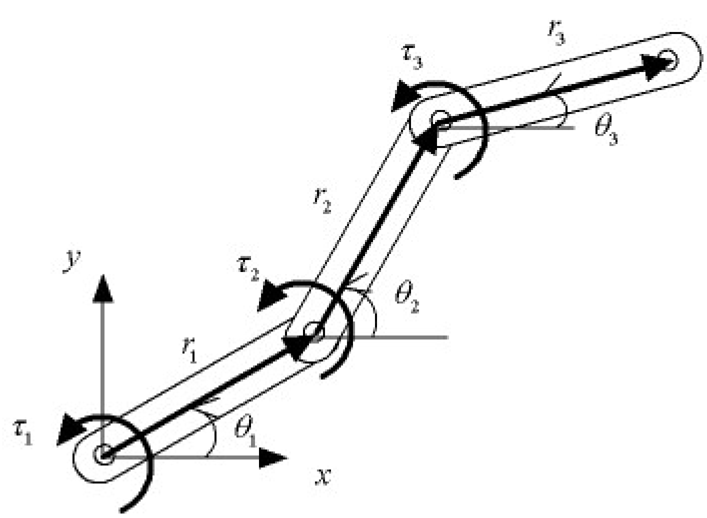


Figure 1: A schematic diagram of a fully actuated three links manipulator.

Underactuated mechanical systems are mechanical system with fewer actuators than its degree-of-freedom. There are board range of system that can be classify as underactuated system, however, in this project we will only consider the underactuated system of a planar three links manipulator compose of active and passive joints. As shown in Figure 1, a manipulator with all joints actuated is controllable, however, when one or more of the actuator failed, the system becomes underactuated and thus the controllability of this system is unknown. On the other hand, an underactuated system is sometime desirable, such as the case of a snake-like robot. Further, an actuator of a particular joint can be turned off or locked (to save energy), to create an underactuated system.

LAGRANGIAN FORMULATION:

To formulate the equations of motion for the three link manipulator, we adopted the Lagrangian Method. In this formulation, the dynamic equations of a mechanical system are modeled by energy method. Lagrangian of a system is defined as:

$$L\left(\underset{\sim}{q},\underset{\sim}{\dot{q}}\right)=T\left(\underset{\sim}{q},\underset{\sim}{\dot{q}}\right)-V\left(\underset{\sim}{q}\right) \tag{1}$$

Where T is the total kinetic energy and V is the total potential energy in the system, $\underset{\sim}{q}$ and $\underset{\sim}{\dot{q}}$ are the generalize coordinates and generalize velocities of the system. The equation of motion is then given by:

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \underset{\sim}{\dot{q}}}\right)-\frac{\partial L}{\partial q}=\frac{\partial \prod}{\partial \underset{\sim}{\dot{q}}}-\frac{\partial \Delta}{\partial \underset{\sim}{\dot{q}}} \tag{2}$$

Where $\prod\left(\underset{\sim}{q},\underset{\sim}{\dot{q}}\right)$ is the power supplied to the system and $\Delta\left(\underset{\sim}{q},\underset{\sim}{\dot{q}}\right)$ is the dissipation function.

In the subsequent paragraphs, the formulation of the dynamic equation of the three links manipulator using this method is presented in detail.

Generalized coordinates and generalized velocities.

A three links mechanism has three degree of freedom (DOF), and hence three generalized coordinates are needed to describe it in an arbitrary configuration. We can define the generalized coordinates as $\theta_n, n=1,2,3$, as shown.

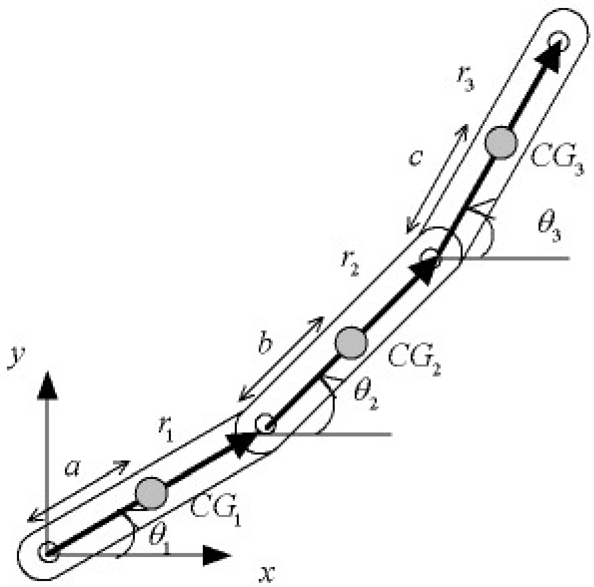


Figure 2: The schematic diagram of a three links manipulator used in the study.

Consider a fully actuated system, where power input to the system is the torqueses exert by motors located at each joints, another form of power input to the system is the gravitational force from the weight of the links. Before we proceed to formulate the dynamic equations, we listed all the naming conventions used in the formulation as follow:

r_i = Length of link i, where $i = 1, 2, 3$

CG_i = Locations of the center of gravity of each links, where $i = 1, 2, 3$

a = Distance from joint 1 to CG_1.

b = Distance from joint 1 to CG_2.

c = Distance from joint 1 to CG_3.

m_i = Mass of each links, where $i = 1, 2, 3$

I_i = Moment of Inertia of link i about CG_i, where $i = 1, 2, 3$

k_i = Friction factor of link i, where $i = 1, 2, 3$

Assuming that the center of gravity ($CG_i, i = 1,2,3$) of each link is at the middle of its link, the displacements of the CG_i with respect to the fixed frame are:

Link 1:

$$\vec{r}_{CG_1} = a\cos\theta_1 \underset{\sim}{i} + a\sin\theta_1 \underset{\sim}{j} \tag{3}$$

Link 2:

$$\vec{r}_{CG_2} = \left(r_1\cos\theta_1 + b\cos\theta_2\right)\underset{\sim}{i} + \left(r_2\sin\theta_1 + b\sin\theta_2\right)\underset{\sim}{j} \tag{4}$$

Link 3:

$$\vec{r}_{CG_2} = \left(r_1\cos\theta_1 + r_2\cos\theta_2 + c\cos\theta_3\right)\underset{\sim}{i} + \left(r_2\sin\theta_1 + r_2\sin\theta_2 + c\sin\theta_3\right)\underset{\sim}{j} \tag{5}$$

Differentiate Equation (3), (4), and (5) give the velocities of the CGs:

Link 1:

$$\dot{\vec{r}}_{CG_1} = -\left(a\sin\theta_1\dot{\theta}_1\right)\underset{\sim}{i} + \left(a\cos\theta_1\dot{\theta}_1\right)\underset{\sim}{j} \tag{6}$$

Link 2:

$$\dot{\vec{r}}_{CG_2} = \left(-r_2\sin\theta_1\dot{\theta}_1 - b\sin\theta_2\dot{\theta}_2\right)\underset{\sim}{i} + \left(r_1\cos\theta_1\dot{\theta}_1 + b\cos\theta_2\dot{\theta}_2\right)\underset{\sim}{j} \tag{7}$$

Link 3:

$$\dot{\vec{r}}_{CG_2} = \left(-r_2\sin\theta_1\dot{\theta}_1 - r_2\sin\theta_2\dot{\theta}_2 - c\sin\theta_3\dot{\theta}_3\right)\underset{\sim}{i} + \left(r_1\cos\theta_1\dot{\theta}_1 + r_2\cos\theta_2\dot{\theta}_2 + c\cos\theta_3\dot{\theta}_3\right)\underset{\sim}{j} \tag{8}$$

Kinetic energy of the system:

The total kinetic energy of the whole system is given by:

$$T\left(\underset{\sim}{q}, \underset{\sim}{\dot{q}}\right) = \frac{1}{2}\left(m_1v_1^2 + m_2v_2^2 + m_3v_3^2 + I_1\dot{\theta}_1^2 + I_2\dot{\theta}_2^2 + I_3\dot{\theta}_3^2\right) \tag{9}$$

where v_1, v_2, v_3 are the absolute velocities of CG1 and CG2 respectively, they can be found by taking the absolute value of Equation (6), (7) and (8). After simplification, we have the following equations:

$$v_1 = \sqrt{a^2\dot{\theta}_1^2} \tag{10}$$

$$v_2 = \sqrt{r_1^2\dot{\theta}_1^2 + b^2\dot{\theta}_2^2 + 2r_1b\dot{\theta}_1\dot{\theta}_2\cos\left(\theta_2 - \theta_1\right)} \tag{11}$$

$$v_3 = \sqrt{r_1^2\dot{\theta}_1^2 + r_2^2\dot{\theta}_2^2 + c^2\dot{\theta}_3^2 + 2r_1r_2\dot{\theta}_1\dot{\theta}_2\cos(\theta_2 - \theta_1)} + \sqrt{2r_1c\dot{\theta}_1\dot{\theta}_3\cos(\theta_1 + \theta_3) + 2r_2c\dot{\theta}_2\dot{\theta}_3\cos(\theta_2 + \theta_3)} \tag{12}$$

Substitute Equation (10), (11), and (12) into Equation (9), we have the following equation for the total kinetic energy of the system:

$$\begin{aligned} T\left(\underset{\sim}{q}, \underset{\sim}{\dot{q}}\right) = {} & \frac{1}{2}\left(m_1a^2 + m_2r_1^2 + m_3r_1^2 + I_1\right)\dot{\theta}_1^2 + \frac{1}{2}\left(m_2b^2 + m_3r_2^2 + I_2\right)\dot{\theta}_2^2 + \frac{1}{2}\left(m_3c^2 + I_3\right)\dot{\theta}_3^2 \\ & + r_1\left(m_2b + m_3r_2\right)\cos(\theta_2 - \theta_1)\dot{\theta}_1\dot{\theta}_2 + m_3r_1c\cos(\theta_1 + \theta_3)\dot{\theta}_1\dot{\theta}_3 + m_3r_2c\cos(\theta_2 + \theta_3)\dot{\theta}_2\dot{\theta}_3 \end{aligned} \tag{13}$$

Potential Energy of the system:

The total potential energy of the system is given by:

$$V\left(\underset{\sim}{q}\right) = m_1gh_{CG_1} + m_2gh_{CG_2} + m_3gh_{CG_3} \tag{14}$$

Where $h_{CG_i}, \forall i = 1,2,3$, is the height of each of the center of gravity of each link, and their value are:

$$h_{CG_1} = a\sin\theta_1 \tag{15}$$

$$h_{CG_2} = r_1\sin\theta_1 + b\sin\theta_2 \tag{16}$$

$$h_{CG_2} = r_1\sin\theta_1 + r_2\sin\theta_2 + c\sin\theta_3 \tag{17}$$

Substituting Equations (15), (16), (17) into Equation (14), we have the total potential energy of the system as:

$$V\left(\underset{\sim}{q}\right) = \left(m_1a + m_2r_1 + m_3r_1\right)g\sin\theta_1 + \left(m_2b + m_3r_2\right)g\sin\theta_2 + m_3gc\sin\theta_3 \tag{18}$$

Lagrangian of the system:

Substituting Equations (13) and (18) into Equation (1), the Lagrangian of the system is thus:

$$
\begin{aligned}
L\left(\underset{\sim}{q},\underset{\sim}{\dot{q}}\right) &= T\left(\underset{\sim}{q},\underset{\sim}{\dot{q}}\right) - V\left(\underset{\sim}{q}\right) \\
&= \frac{1}{2}\left(m_1 a^2 + m_2 r_1^2 + m_3 r_1^2 + I_1\right)\dot{\theta}_1^2 + \frac{1}{2}\left(m_2 b^2 + m_3 r_2^2 + I_2\right)\dot{\theta}_2^2 + \frac{1}{2}\left(m_3 c^2 + I_3\right)\dot{\theta}_3^2 \\
&\quad + r_1\left(m_2 b + m_3 r_2\right)\cos\left(\theta_2 - \theta_1\right)\dot{\theta}_1\dot{\theta}_2 + m_3 r_1 c\cos\left(\theta_1 + \theta_3\right)\dot{\theta}_1\dot{\theta}_3 + m_3 r_2 c\cos\left(\theta_2 + \theta_3\right)\dot{\theta}_2\dot{\theta}_3 \\
&\quad - \left(m_1 a + m_2 r_1 + m_3 r_1\right) g\sin\theta_1 - \left(m_2 b + m_3 r_2\right) g\sin\theta_2 - m_3 g c\sin\theta_3
\end{aligned}
\tag{19}
$$

To find the dynamic equations of the system, we need the following informations:

$$
\begin{aligned}
\frac{\partial L}{\partial \theta_1} &= r_1\left(m_2 b + m_3 r_2\right)\sin\left(\theta_2 - \theta_1\right)\dot{\theta}_1\dot{\theta}_2 - m_3 r_1 c\sin\left(\theta_1 + \theta_3\right)\dot{\theta}_1\dot{\theta}_3 \\
&\quad - \left(m_1 a + m_2 r_1 + m_3 r_1\right) g\cos\theta_1
\end{aligned}
\tag{20}
$$

$$
\frac{\partial L}{\partial \theta_2} = -r_1\left(m_2 b + m_3 r_2\right)\sin\left(\theta_2 - \theta_1\right)\dot{\theta}_1\dot{\theta}_2 - m_3 r_2 c\sin\left(\theta_2 + \theta_3\right)\dot{\theta}_2\dot{\theta}_3 - \left(m_2 b + m_3 r_2\right) g\cos\theta_2 \tag{21}
$$

$$
\frac{\partial L}{\partial \theta_3} = -m_3 r_1 c\sin\left(\theta_1 + \theta_3\right)\dot{\theta}_1\dot{\theta}_3 - m_3 r_2 c\sin\left(\theta_2 + \theta_3\right)\dot{\theta}_2\dot{\theta}_3 - m_3 g c\cos\theta_3 \tag{22}
$$

$$
\frac{\partial L}{\partial \dot{\theta}_1} = \left(m_1 a^2 + m_2 r_1^2 + m_3 r_1^2 + I_1\right)\dot{\theta}_1 + r_1\left(m_2 b + m_3 r_2\right)\cos\left(\theta_2 - \theta_1\right)\dot{\theta}_2 + m_3 r_1 c\cos\left(\theta_1 + \theta_3\right)\dot{\theta}_3 \tag{23}
$$

$$
\frac{\partial L}{\partial \dot{\theta}_2} = \left(m_2 b^2 + m_3 r_2^2 + I_2\right)\dot{\theta}_2 + r_1\left(m_2 b + m_3 r_2\right)\cos\left(\theta_2 - \theta_1\right)\dot{\theta}_1 + m_3 r_2 c\cos\left(\theta_2 + \theta_3\right)\dot{\theta}_3 \tag{24}
$$

$$
\frac{\partial L}{\partial \dot{\theta}_3} = \left(m_3 c^2 + I_3\right)\dot{\theta}_3 + m_3 r_1 c\cos\left(\theta_1 + \theta_3\right)\dot{\theta}_1 + m_3 r_2 c\cos\left(\theta_2 + \theta_3\right)\dot{\theta}_2 \tag{25}
$$

$$
\begin{aligned}
\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}_1}\right) &= \left(m_1 a^2 + m_2 r_1^2 + m_3 r_1^2 + I_1\right)\ddot{\theta}_1 + r_1\left(m_2 b + m_3 r_2\right)\cos\left(\theta_2 - \theta_1\right)\ddot{\theta}_2 \\
&\quad - r_1\left(m_2 b + m_3 r_2\right)\sin\left(\theta_2 - \theta_1\right)\left(\dot{\theta}_2 - \dot{\theta}_1\right)\dot{\theta}_2 + m_3 r_1 c\cos\left(\theta_1 + \theta_3\right)\ddot{\theta}_3 \\
&\quad - m_3 r_1 c\sin\left(\theta_1 + \theta_3\right)\left(\dot{\theta}_1 + \dot{\theta}_3\right)\dot{\theta}_3
\end{aligned}
\tag{26}
$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}_2}\right)=\left(m_2b^2+m_3r_2^2+I_2\right)\ddot{\theta}_2+r_1\left(m_2b+m_3r_2\right)\cos\left(\theta_2-\theta_1\right)\ddot{\theta}_1$$
$$-r_1\left(m_2b+m_3r_2\right)\sin\left(\theta_2-\theta_1\right)\left(\dot{\theta}_2-\dot{\theta}_1\right)\dot{\theta}_1+m_3r_2c\cos\left(\theta_2+\theta_3\right)\ddot{\theta}_3 \quad (27)$$
$$-m_3r_2c\sin\left(\theta_2+\theta_3\right)\left(\dot{\theta}_2+\dot{\theta}_3\right)\dot{\theta}_3$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}_3}\right)=\left(m_3c^2+I_3\right)\ddot{\theta}_3+m_3r_1c\cos\left(\theta_1+\theta_3\right)\ddot{\theta}_1-m_3r_1c\sin\left(\theta_1+\theta_3\right)\left(\dot{\theta}_1+\dot{\theta}_3\right)\dot{\theta}_1$$
$$+m_3r_2c\cos\left(\theta_2+\theta_3\right)\ddot{\theta}_2-m_3r_2c\sin\left(\theta_2+\theta_3\right)\left(\dot{\theta}_2+\dot{\theta}_3\right)\dot{\theta}_2 \quad (28)$$

Power supplied to the system:

If we consider the three links mechanism is fully actuated, we have:

$$\Pi\left(\dot{\underset{\sim}{q}}\right)=\tau_1\dot{\theta}_1+\tau_2\dot{\theta}_2+\tau_3\dot{\theta}_3 \quad (29)$$

and:

$$\frac{\partial \Pi}{\partial \dot{\theta}_1}=\tau_1,\qquad \frac{\partial \Pi}{\partial \dot{\theta}_2}=\tau_2,\qquad \frac{\partial \Pi}{\partial \dot{\theta}_3}=\tau_3 \quad (30)$$

Dissipation functions of the system:

Assume the dissipation of the system comes from friction in the joints, we have:

$$\Delta\left(\dot{\underset{\sim}{q}}\right)=\frac{1}{2}\left[k_1\dot{\theta}_1^2+k_2\left(\dot{\theta}_2-\dot{\theta}_1\right)^2+k_3\left(\dot{\theta}_3-\dot{\theta}_2\right)^2\right] \quad (31)$$

and:

$$\frac{\partial \Delta}{\partial \dot{\theta}_1}=\left(k_1+k_2\right)\dot{\theta}_1-k_2\dot{\theta}_2 \quad (32)$$

$$\frac{\partial \Delta}{\partial \dot{\theta}_2}=\left(k_2+k_3\right)\dot{\theta}_2-k_2\dot{\theta}_1-k_3\dot{\theta}_3 \quad (33)$$

$$\frac{\partial \Delta}{\partial \dot{\theta}_3}=k_3\left(\dot{\theta}_3-\dot{\theta}_2\right) \quad (34)$$

Dynamic equation of fully-actuated three links mechanism:

Substitute Equations (20)-(34) into Equation (2), after simplification, the dynamic equations of this three links mechanism are given by:

$$\begin{aligned}&\left(m_1a^2+m_2r_1^2+m_3r_1^2+I_1\right)\ddot{\theta}_1+r_1\left(m_2b+m_3r_2\right)\cos\left(\theta_2-\theta_1\right)\ddot{\theta}_2\\&\quad+m_3r_1c\cos\left(\theta_1+\theta_3\right)\ddot{\theta}_3-r_1\left(m_2b+m_3r_2\right)\sin\left(\theta_2-\theta_1\right)\dot{\theta}_2^2\\&\qquad-m_3r_1c\sin\left(\theta_1+\theta_3\right)\dot{\theta}_3^2+\left(m_1a+m_2r_1+m_3r_1\right)g\cos\theta_1=\tau_1-\left(k_1+k_2\right)\dot{\theta}_1+k_2\dot{\theta}_2\end{aligned} \tag{35}$$

$$\begin{aligned}&\left(m_2b^2+m_3r_2^2+I_2\right)\ddot{\theta}_2+r_1\left(m_2b+m_3r_2\right)\cos\left(\theta_2-\theta_1\right)\ddot{\theta}_1\\&+r_1\left(m_2b+m_3r_2\right)\sin\left(\theta_2-\theta_1\right)\dot{\theta}_1^2+m_3r_2c\cos\left(\theta_2+\theta_3\right)\ddot{\theta}_3\\&\qquad-m_3r_2c\sin\left(\theta_2+\theta_3\right)\dot{\theta}_3^2+\left(m_2b+m_3r_2\right)g\cos\theta_2=\tau_1-\left(k_2+k_3\right)\dot{\theta}_2+k_2\dot{\theta}_1+k_3\dot{\theta}_3\end{aligned} \tag{36}$$

$$\begin{aligned}&\left(m_3c^2+I_3\right)\ddot{\theta}_3+m_3r_1c\cos\left(\theta_1+\theta_3\right)\ddot{\theta}_1-m_3r_1c\sin\left(\theta_1+\theta_3\right)\dot{\theta}_1^2\\&+m_3r_2c\cos\left(\theta_2+\theta_3\right)\ddot{\theta}_2-m_3r_2c\sin\left(\theta_2+\theta_3\right)\dot{\theta}_2^2+m_3gc\cos\theta_3=\tau_3-k_3\left(\dot{\theta}_3-\dot{\theta}_2\right)\end{aligned} \tag{37}$$

LINEARIZATION:

The dynamic equation given in Equations (35) - (37) is highly nonlinear. To linearize the dynamic equations given in Equations (35), (36), and (37), we need to determine the equilibrium points of the system. First we assume that all the motors are turned off, such that $\tau_1=\tau_2=\tau_3=0$. Also, $\dot{\theta}_i=\ddot{\theta}_i=0,\forall i=1,2,3$. By doing so, we obtained the following properties from Equations (35), (36), and (37), respectively.

$$\left(m_1a+m_2r_1+m_3r_1\right)g\cos\theta_1=0 \tag{38}$$

$$\left(m_2b+m_3r_2\right)g\cos\theta_2=0 \tag{39}$$

$$m_3gc\cos\theta_3=0 \tag{40}$$

From these equations, we see that the equilibrium points occur at:

$$\cos\theta_1 = 0, \cos\theta_2 = 0, \cos\theta_3 = 0 \tag{41}$$

Solving Equation (41), we have:

$$\theta_1 = \frac{\pi}{2}, \frac{3}{2}\pi, \qquad \theta_2 = \frac{\pi}{2}, \frac{3}{2}\pi, \qquad \theta_3 = \frac{\pi}{2}, \frac{3}{2}\pi$$

and these give total of eight equilibrium points shown in Figure 3:

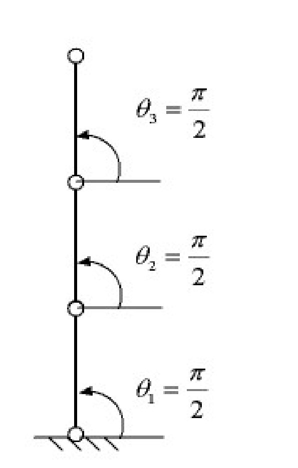
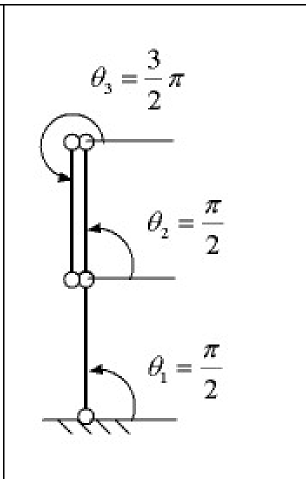
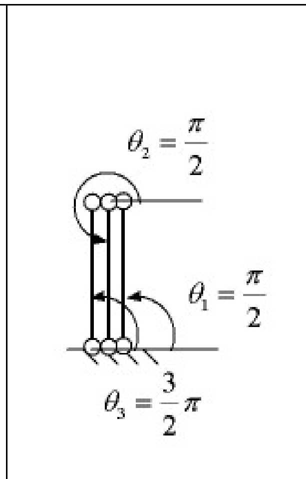
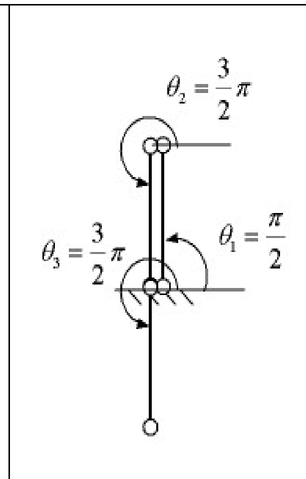
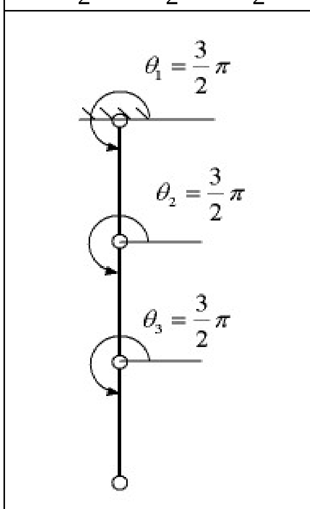
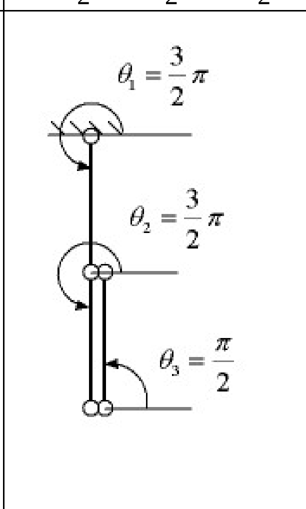
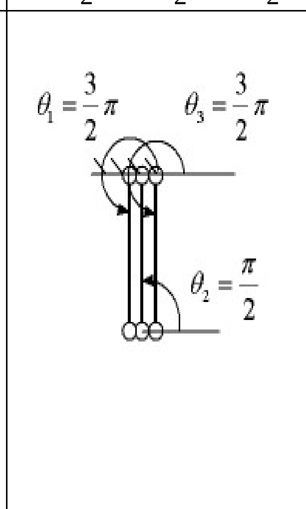
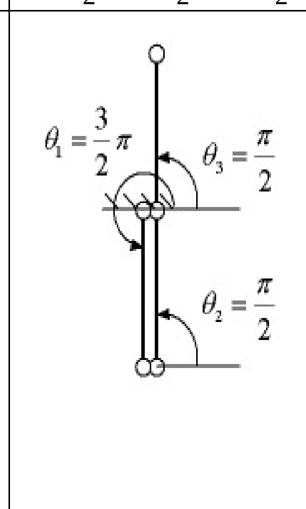
			
$\theta_1 = \frac{\pi}{2}, \theta_2 = \frac{\pi}{2}, \theta_3 = \frac{\pi}{2}$	$\theta_1 = \frac{\pi}{2}, \theta_2 = \frac{\pi}{2}, \theta_3 = \frac{3\pi}{2}$	$\theta_1 = \frac{\pi}{2}, \theta_2 = \frac{3\pi}{2}, \theta_3 = \frac{\pi}{2}$	$\theta_1 = \frac{\pi}{2}, \theta_2 = \frac{3\pi}{2}, \theta_3 = \frac{3\pi}{2}$
			
$\theta_1 = \frac{3\pi}{2}, \theta_2 = \frac{3\pi}{2}, \theta_3 = \frac{3\pi}{2}$	$\theta_1 = \frac{3\pi}{2}, \theta_2 = \frac{3\pi}{2}, \theta_3 = \frac{\pi}{2}$	$\theta_1 = \frac{3\pi}{2}, \theta_2 = \frac{\pi}{2}, \theta_3 = \frac{3\pi}{2}$	$\theta_1 = \frac{3\pi}{2}, \theta_2 = \frac{\pi}{2}, \theta_3 = \frac{\pi}{2}$

Figure 3: Eight equilibrium positions of a three links manipulator.

Assume small perturbation is applied to the equilibrium states of the three links manipulator; the following properties are thus valid:

$$\theta_i = \delta\theta_i; \quad \dot{\theta}_i = \delta\dot{\theta}_i; \quad \ddot{\theta}_i = \delta\ddot{\theta}_i; \quad \tau_i = \delta\tau_i$$

$$\sin(\delta\theta_i) \approx \delta\theta_i, \qquad \cos(\delta\theta_i) \approx 1$$

$$\delta\theta_1\delta\theta_2 \approx 0, \qquad \delta\theta_1\delta\theta_3 \approx 0, \qquad \delta\theta_2\delta\theta_3 \approx 0, \qquad \delta\theta_i^2 \approx 0$$

$$\sin(\delta\theta_i + \delta\theta_i) \approx \sin(\delta\theta_i)\cos(\delta\theta_i) + \cos(\delta\theta_i)\sin(\delta\theta_i) \approx \delta\theta_i + \delta\theta_i$$

$$\sin(\delta\theta_i - \delta\theta_i) \approx \sin(\delta\theta_i)\cos(\delta\theta_i) - \cos(\delta\theta_i)\sin(\delta\theta_i) \approx \delta\theta_i - \delta\theta_i$$

$$\cos(\delta\theta_i + \delta\theta_i) \approx \cos(\delta\theta_i)\cos(\delta\theta_i) - \sin(\delta\theta_i)\sin(\delta\theta_i) \approx 1 - \delta\theta_i\delta\theta_i \approx 1$$

$$\cos(\delta\theta_i - \delta\theta_i) \approx \cos(\delta\theta_i)\cos(\delta\theta_i) + \sin(\delta\theta_i)\sin(\delta\theta_i) \approx 1 + \delta\theta_i\delta\theta_i \approx 1$$

Using these properties, the linearized dynamic equations are given by:

$$\begin{aligned}&\left(m_1a^2 + m_2r_1^2 + m_3r_1^2 + I_1\right)\delta\ddot{\theta}_1 + r_1\left(m_2b + m_3r_2\right)\delta\ddot{\theta}_2 \\ &\quad + m_3r_1c\delta\ddot{\theta}_3 + \left(m_1a + m_2r_1 + m_3r_1\right)g = \tau_1 - \left(k_1 + k_2\right)\delta\dot{\theta}_1 + k_2\delta\dot{\theta}_2\end{aligned} \tag{42}$$

$$\begin{aligned}&\left(m_2b^2 + m_3r_2^2 + I_2\right)\delta\ddot{\theta}_2 + r_1\left(m_2b + m_3r_2\right)\delta\ddot{\theta}_1 \\ &+ m_3r_2c\delta\ddot{\theta}_3 + \left(m_2b + m_3r_2\right)g = \tau_1 - \left(k_2 + k_3\right)\delta\dot{\theta}_2 + k_2\delta\dot{\theta}_1 + k_3\delta\dot{\theta}_3\end{aligned} \tag{43}$$

$$\left(m_3c^2 + I_3\right)\delta\ddot{\theta}_3 + m_3r_1c\delta\ddot{\theta}_1 + m_3r_2c\delta\ddot{\theta}_2 + m_3gc = \tau_3 - k_3\left(\delta\dot{\theta}_3 - \delta\dot{\theta}_2\right) \tag{44}$$

Equations (42), (43), and (44) can be written in the following matrix form:

$$\begin{aligned}&\begin{bmatrix} \left(m_1a^2 + m_2r_1^2 + m_3r_1^2 + I_1\right) & r_1\left(m_2b + m_3r_2\right) & m_3r_1c \\ r_1\left(m_2b + m_3r_2\right) & \left(m_2b^2 + m_3r_2^2 + I_2\right) & m_3r_2c \\ m_3r_1c & m_3r_2c & \left(m_3c^2 + I_3\right) \end{bmatrix} \begin{bmatrix} \delta\ddot{\theta}_1 \\ \delta\ddot{\theta}_2 \\ \delta\ddot{\theta}_3 \end{bmatrix} = \\ &\begin{bmatrix} -\left(k_1 + k_2\right) & k_2 & 0 \\ k_2 & -\left(k_2 + k_3\right) & k_3 \\ 0 & k_3 & -k_3 \end{bmatrix} \begin{bmatrix} \delta\dot{\theta}_1 \\ \delta\dot{\theta}_2 \\ \delta\dot{\theta}_3 \end{bmatrix} + \begin{bmatrix} 1 & 0 & 0 & -\left(m_1a + m_2r_1 + m_3r_1\right) \\ 0 & 1 & 0 & -\left(m_2b + m_3r_2\right) \\ 0 & 0 & 1 & m_3c \end{bmatrix} \begin{bmatrix} \tau_1 \\ \tau_2 \\ \tau_3 \\ g \end{bmatrix}\end{aligned} \tag{45}$$

Let

$$M = \begin{bmatrix} \left(m_1a^2 + m_2r_1^2 + m_3r_1^2 + I_1\right) & r_1\left(m_2b + m_3r_2\right) & m_3r_1c \\ r_1\left(m_2b + m_3r_2\right) & \left(m_2b^2 + m_3r_2^2 + I_2\right) & m_3r_2c \\ m_3r_1c & m_3r_2c & \left(m_3c^2 + I_3\right) \end{bmatrix} \tag{46}$$

Hence, the dynamic equation of the fully actuated three links manipulator (Equation (45)) can be expressed in the following state-space form as:

$$\begin{bmatrix} \delta\ddot{\theta}_1 \\ \delta\ddot{\theta}_2 \\ \delta\ddot{\theta}_3 \end{bmatrix} = M^{-1} \begin{bmatrix} -(k_1+k_2) & k_2 & 0 \\ k_2 & -(k_2+k_3) & k_3 \\ 0 & k_3 & -k_3 \end{bmatrix} \begin{bmatrix} \delta\dot{\theta}_1 \\ \delta\dot{\theta}_2 \\ \delta\dot{\theta}_3 \end{bmatrix} + M^{-1} \begin{bmatrix} 1 & 0 & 0 & -(m_1a+m_2r_1+m_3r_1) \\ 0 & 1 & 0 & -(m_2b+m_3r_2) \\ 0 & 0 & 1 & m_3c \end{bmatrix} \begin{bmatrix} \tau_1 \\ \tau_2 \\ \tau_3 \\ g \end{bmatrix} \quad (47)$$

Expressed in the standard form:

$$\dot{\underset{\sim}{x}} = A\underset{\sim}{x} + B\underset{\sim}{u} \quad (48)$$

Where:

$$A = M^{-1} \begin{bmatrix} -(k_1+k_2) & k_2 & 0 \\ k_2 & -(k_2+k_3) & k_3 \\ 0 & k_3 & -k_3 \end{bmatrix} \qquad B = M^{-1} \begin{bmatrix} 1 & 0 & 0 & -(m_1a+m_2r_1+m_3r_1) \\ 0 & 1 & 0 & -(m_2b+m_3r_2) \\ 0 & 0 & 1 & m_3c \end{bmatrix}$$

with the states:

$$\underset{\sim}{x} = \begin{bmatrix} \delta\dot{\theta}_1 & \delta\dot{\theta}_2 & \delta\dot{\theta}_3 \end{bmatrix}^T$$

and the inputs are:

$$\underset{\sim}{u} = \begin{bmatrix} \tau_1 & \tau_2 & \tau_3 & g \end{bmatrix}^T$$

Controllability Analysis:

The controllability of a system is analyzed using the controllability matrix formed from the state-space model of the system. Suppose the state-space model is given by the form of Equation (48), the controllability matrix is given by:

$$C_{n\times(np)} = \begin{bmatrix} B & AB & A^2B & \cdots & A^{n-1}B \end{bmatrix} \quad (49)$$

where A is an $n\times n$ matrix and B is an $n\times p$ matrix. If the system is controllable, the matrix C has no linearly dependent columns and has full rank.

In the following sections, we will study two cases of this three links manipulator, the first case is where all the joints are fully-actuated; and the second case is the underactuated system where we turned off two of the actuators on the manipulator.

Case 1: Fully-actuated three links manipulator.

From Equations (47), we know that:

$$A = M^{-1}\begin{bmatrix} -(k_1+k_2) & k_2 & 0 \\ k_2 & -(k_2+k_3) & k_3 \\ 0 & k_3 & -k_3 \end{bmatrix} \qquad B = M^{-1}\begin{bmatrix} 1 & 0 & 0 & -(m_1a+m_2r_1+m_3r_1) \\ 0 & 1 & 0 & -(m_2b+m_3r_2) \\ 0 & 0 & 1 & m_3c \end{bmatrix}$$

The controllability matrix for this case is:

$$C_I = \begin{bmatrix} B & AB & A^2B \end{bmatrix}_{3\times 12} \tag{50}$$

Substitute the corresponding values and after simplification, we have the following controllability matrix of this system.

Given:

$$A^2 = \begin{bmatrix} (k_1+k_2)^2+k_2^2 & -k_2(k_1+k_2)-k_2(k_2+k_3) & k_2k_3 \\ -k_2(k_1+k_2)-k_2(k_2+k_3) & k_2^2+(k_2+k_3)^2+k_3^2 & -k_3(k_2+k_3)-k_3^2 \\ k_2k_3 & -k_3(k_2+k_3)-k_3^2 & 2k_3^2 \end{bmatrix}$$

The resulting C_I matrix is a size 3×12 matrix; its first six columns are given by:

$$M^{-1}\begin{bmatrix} 1 & 0 & 0 & -(m_1a+m_2r_1+m_3r_1) & -(k_1+k_2) & k_2 \\ 0 & 1 & 0 & -(m_2b+m_3r_2) & k_2 & -(k_2+k_3) \\ 0 & 0 & 1 & m_3c & 0 & k_3 \end{bmatrix}$$

The following four columns are:

$$\begin{matrix} 0 & (k_1+k_2)(m_1a+m_2r_1+m_3r_1)-k_2(m_2b+m_3r_2) & (k_1+k_2)^2+k_2^2 & -k_2(k_1+k_2)-k_2(k_2+k_3) \\ k_3 & -k_2(m_1a+m_2r_1+m_3r_1)+(k_2+k_3)(m_2b+m_3r_2)+k_3m_3c & -k_2(k_1+k_2)-k_2(k_2+k_3) & k_2^2+(k_2+k_3)^2+k_3^2 \\ -k_3 & -k_3(m_2b+m_3r_2)-k_3m_3c & k_2k_3 & -k_3(k_2+k_3)-k_3^2 \end{matrix}$$

And the last two columns are:

$$\begin{matrix} k_2k_3 & -\left((k_1+k_2)^2+k_2^2\right)(m_1a+m_2r_1+m_3r_1)-(m_2b+m_3r_2)\left(-k_2(k_1+k_2)-k_2(k_2+k_3)\right)+k_2k_3m_3c \\ -k_3(k_2+k_3)-k_3^2 & -(m_1a+m_2r_1+m_3r_1)\left(-k_2(k_1+k_2)-k_2(k_2+k_3)\right)-(m_2b+m_3r_2)\left(k_2^2+(k_2+k_3)^2+k_3^2\right)+m_3c\left(-k_3(k_2+k_3)-k_3^2\right) \\ 2k_3^2 & -k_2k_3(m_1a+m_2r_1+m_3r_1)-(m_2b+m_3r_2)\left(-k_3(k_2+k_3)-k_3^2\right)+2k_3^2m_3c \end{matrix}\Bigg]$$

Observing matrix C_I, we notice that the columns vectors are linearly independent to each other. Hence, we conclude that a fully actuated three links manipulators is controllable.

Case II: underactuated three links manipulators where $\tau_2 = \tau_3 = 0$.

Now we consider a case where the motors at joint 2 and 3 are dead, the dynamic equation in state-space form is given by:

$$\begin{bmatrix} \delta\ddot{\theta}_1 \\ \delta\ddot{\theta}_2 \\ \delta\ddot{\theta}_3 \end{bmatrix} = M^{-1} \begin{bmatrix} -(k_1+k_2) & k_2 & 0 \\ k_2 & -(k_2+k_3) & k_3 \\ 0 & k_3 & -k_3 \end{bmatrix} \begin{bmatrix} \delta\dot{\theta}_1 \\ \delta\dot{\theta}_2 \\ \delta\dot{\theta}_3 \end{bmatrix} + M^{-1} \begin{bmatrix} 1 & -(m_1 a + m_2 r_1 + m_3 r_1) \\ 0 & -(m_2 b + m_3 r_2) \\ 0 & m_3 c \end{bmatrix} \begin{bmatrix} \tau_1 \\ g \end{bmatrix} \tag{51}$$

Similarly, the controllability matrix can be found by using Equation (50). The resulting controllability matrix is:

$$C_{II} = \begin{bmatrix} B & AB & A^2B \end{bmatrix}_{3\times 6} \tag{52}$$

After simplification, the resulting C_{II} matrix is given by the following equation:

$$M^{-1} \left[\begin{matrix} 1 & -(m_1 a + m_2 r_1 + m_3 r_1) & -(k_1+k_2) & (k_1+k_2)(m_1 a + m_2 r_1 + m_3 r_1) - k_2(m_2 b + m_3 r_2) \\ 0 & -(m_2 b + m_3 r_2) & k_2 & -k_2(m_1 a + m_2 r_1 + m_3 r_1) + (k_2+k_3)(m_2 b + m_3 r_2) + k_3 m_3 c \\ 0 & m_3 c & 0 & -k_3(m_1 a + m_2 r_1 + m_3 r_1) - k_3 m_3 c \end{matrix} \right. \tag{53}$$

$$\left. \begin{matrix} (k_1+k_2)^2 + k_2^2 & -(m_1 a + m_2 r_1 + m_3 r_1)\left((k_1+k_2)^2 + k_2^2\right) - (m_2 b + m_3 r_2)\left(k_2^2 + (k_2+k_3)^2 + k_3^2\right) + k_2 k_3 m_3 c \\ -k_2(k_1+k_2) - k_2(k_2+k_3) & -(m_1 a + m_2 r_1 + m_3 r_1)\left(-k_2(k_1+k_2) - k_2(k_2+k_3)\right) - (m_2 b + m_3 r_2)\left(k_2^2 + (k_2+k_3)^2 + k_3^2\right) + m_3 c\left(-k_3(k_2+k_3) - k_3^2\right) \\ k_2 k_3 & -k_2 k_3(m_1 a + m_2 r_1 + m_3 r_1) - (m_2 b + m_3 r_2)\left(-k_3(k_2+k_3) - k_3^2\right) + 2k_3^2 m_3 c \end{matrix} \right]$$

Observing matrix C_{II}, we see that the column vectors are linearly independent of each other. Hence, the system is controllable.

DISCUSSION:

The power input to the system is the actuator torque and the gravitational force due to the weight of the links. In the previous section, we have studied the controllability of a three links manipulator in two situations: (a) fully actuated; and (b) under-actuated case where motors at joint 2 and 3 are turned off. In both cases, the controllability analyses are done by using the linearized equations and thus the controllability analysis results are only valid at the equilibrium points. There are eight equilibrium points for this three link manipulator and they are shown in Figure 3. In the first case, where the three links manipulator is fully actuated, we show that the column vectors of the controllability matrix are linearly independent of each other – hence it is controllable, as we would expect.

In the second case, we look at an underactuated three link manipulator where only its base link is actuated and the remaining two joints are turned off. From the controllability matrix, we see that it is also controllable. The result is somewhat surprising because we would expect the system is uncontrollable when only one joint is actuated. There are few reasons to explain this result. In our model, we consider there is frictions at the joints, and these friction terms show up in the controllability matrix that ensure the columns of the controllability matrix are linearly independent of each others, as shown in Equation (53). If we consider there is no frictions in joints 2 and 3 as the motors are turned off, the controllability matrix will not be full-ranked, as column vector 3 is a scalar multiple of column vector 1. On the other hand, assuming there are frictions in all the joints even though it is underactuated, we note that the system is controllable *at the equilibrium points* provided the *perturbation to the system is small* enough such that the linearized dynamic equations are still valid.

Overall, this project gives the author an opportunity to study this interesting manipulator from the formulation of its dynamic equations using Lagrangian method, linearization of the nonlinear dynamic equation about its equilibrium positions, and studied its controllability matrix.

REFERENCES:

[1] Spong, M.W., “Underactuated Mechanical Systems,", *Control Problems in Robotics and Automation*, B. Siciliano and K.P. Valavanis (Eds), *Lecture Notes in Control and Information Sciences 230* Spinger-Verlag, London, UK, 1997, presented at the International Workshop on Control Problems in Robotics and Automation: Future Directions Hyatt Regency, San Diego, California, Dec. 1997.

[2] Spong, M.W., “The Control of Underactuated Mechanical System", *Plenary Address at The First International Conference on Mechatronics* , Mexico City, Jan. 26--29, 1994.

[3] Spong, M.W., “Partial Feedback Linearization of Underactuated Mechanical Systems", *IROS'94* , Munich, Germany, pp. 314-321, Sept. 1994.

[4] K. Kobayashi, J. Imura, and T. Yoshikawa “Controllability of a Class of Under-actuated Planar Manipulator”， *Proc. of the 3rd International Conference on Advanced Mechatoronics,179/184* (1998.8)

[5] K. Kobayashi, T. Yoshikawa **“**Controllability of Under-actuated Planar Manipulators with One Unactuated Joint”, *Proc. of the 2000 IEEE/RSJ International Conference on Intelligent Robots and Systems, 133/138* (2000.11)

[6] B. Massa, S. Roccella, M. C. Carrozza, P.Dario, “Design and development of an underactuated prosthetic hand”, *ICRA '02, IEEE International Conference on Robotics and Automation, Volume: 4 , 11-15* May 2002, pp. 3374 -3379.

[7] Spong, M.W., ``Underactuated Mechanical Systems," *Control Problems in Robotics and Automation*, B. Siciliano and K.P. Valavanis (Eds), *Lecture Notes in Control and Information Sciences 230* Spinger-Verlag, London, UK, 1997, presented at the International Workshop on Control Problems in Robotics and Automation: Future Directions Hyatt Regency, San Diego, California, Dec. 1997.

[8] Marcel Bergerman, “Dynamics and controls of underactuated manipulators”, *dissertation*, Carnegie Mellon University, 1996.

[9] Kövecses, J., Piedbœuf, J.-C. and Lange, C.: "Dynamics Modeling and Simulation of Constrained Robotic Systems". *IEEE/ASME Transactions on Mechatronics,* Vol. 8, No. 2, pp. 165-177, June 2003.