

STATE UNIVERSITY OF NEW YORK AT BUFFALO
Mechanical and Aerospace Engineering Department.

MAE 529 Finite Element Method

Homework 04

NAME: LENG-FENG LEE

INSTRUCTOR: DR. PATRA.

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Student: Leng-Feng Lee, Professor: Dr. Patra.
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Note: In order to view all the result plotted in color, please download the pdf color version of this document from [here](http://www.eng.buffalo.edu/llee3/FEM_HW4.pdf). (http://www.eng.buffalo.edu/llee3/FEM_HW4.pdf)

Question 1:

Q1. Consider a steady state heat conduction problem in an isotropic rectangular region of dimensions 3x1.

$$\begin{aligned} -\nabla \cdot k \nabla T &= 0 & 0 < x < 3, 0 < y < 1 \\ x=0 & \quad k \frac{\partial T}{\partial y} = 0 & y=0 & \quad k \frac{\partial T}{\partial x} = 0 \\ x=3 & \quad T = 125 & y=1 & \quad T = 100 \end{aligned}$$

Determine the temperature distribution using the finite element method inside the region and the heat flux at the boundaries $x=3$ and $y=1$. Assume $k_{ij} = 0.5(x^2+y^2)\delta_{ij}$

Write a FEM code using a uniform mesh of Quadrilateral elements for discretization of the domain. Use bilinear elements, obtain the solutions and study the convergence behaviour with respect to mesh refinement ($h \rightarrow 0$) in the

norm $\|T\| = \sqrt{\int_{\Omega} T^2 dx}$ using the error indicator $E = 0.5 \sqrt{\int_{\Gamma} \left[\frac{\partial T}{\partial n} \right]^2 ds}$, $\Gamma = \bigcup \partial\Omega_x / \partial\Omega$

where $\left[\frac{\partial T}{\partial n} \right] = \left| \begin{array}{cc} \frac{\partial T}{\partial n_{Left}} & - \frac{\partial T}{\partial n_{Right}} \end{array} \right|$ where n is the outward pointing unit normal.

Solution: Given following differential equation:

$$\begin{aligned} \nabla \cdot k \nabla T &= 0 \\ \Rightarrow \left[\frac{\partial}{\partial x} + \frac{\partial}{\partial y} \right] \cdot k \left[\frac{\partial T}{\partial x} + \frac{\partial T}{\partial y} \right] &= 0 \\ \Rightarrow k \frac{\partial^2 T}{\partial x^2} + k \frac{\partial^2 T}{\partial y^2} &= 0 \end{aligned} \tag{1}$$

So the steady state heat conduction problem is given as:

$$\begin{aligned} k \frac{\partial^2 T}{\partial x^2} + k \frac{\partial^2 T}{\partial y^2} &= 0, & 0 < x < 3, & \quad 0 < y < 1 \\ x=0, & \quad k \frac{\partial T}{\partial y} = 0; & y=0, & \quad k \frac{\partial T}{\partial x} = 0 & \quad \text{Natural B.C} \\ x=3, & \quad T = 125; & x=3, & \quad T = 100 & \quad \text{Essential B.C} \end{aligned} \tag{2}$$

To simplify the notation used, Eq(2) can be expressed as the following strong form:

$$(S) \left\{ \begin{array}{l} \text{Given } f = 0 \rightarrow \mathbb{R}, g : \Gamma_g \rightarrow \mathbb{R} \text{ and } h : 0 \rightarrow \mathbb{R}, \text{ find } u : \bar{\Omega} \rightarrow \mathbb{R} \text{ such that:} \\ q_{i,i} = f, \quad \text{in } \Omega \\ T = g \quad \text{on } \Gamma_g \\ -q_i n_i = 0 \quad \text{on } \Gamma_h \end{array} \right. \quad (3)$$

Where $q_i = -\kappa_{ij} T_{,j}$.

To get the weak form of Eq(3), let ℓ denote the trial solution space and ξ the variation space. Wherein ℓ and ξ consist of real-valued functions defined on $\bar{\Omega}$ satisfying certain smoothness requirements, such that all members of ℓ satisfy, whereas if $w \in \xi$, then $w = 0$ on Γ_g . Multiply Eq(3) by smooth function w , to get the following weak form:

$$(W) \left\{ \begin{array}{l} \text{Given } f = 0 \rightarrow \mathbb{R}, g : \Gamma_g \rightarrow \mathbb{R} \text{ and } h : 0 \rightarrow \mathbb{R}, \text{ find } u \in \ell \text{ such that for all } w \in \xi: \\ -\int_{\Omega} w_{,i} q_i \, d\Omega = 0 \end{array} \right. \quad (4)$$

Let $a(w, u) = \int_{\Omega} w_{,i} \kappa_{ij} u_{,j} \, d\Omega = 0$, and in two space dimensions and isotropic case, the conductivity matrix can be written as:

$$\kappa = \kappa_{ij} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad (5)$$

Where we can use the following expression:

$$a(w, u) = \int_{\Omega} (\nabla w)^T \kappa (\nabla u) \, d\Omega = 0 \quad (6)$$

To construct the Galerkin-form of the problem, construct the finite-dimensional approximations of ℓ and ξ , these collections of functions are denoted by ℓ^h and ξ^h , where ℓ^h and ξ^h as being subset of ℓ and ξ . i.e. $\ell^h \subset \ell$ and $\xi^h \subset \xi$.

Assuming the collection ξ^h is given, to each member $v^h \in \xi^h$, construct a function $u^h \in \ell^h$ by $u^h = v^h + g^h$.

Substitute $u^h = v^h + g^h$ into equation(6), the Galerkin form becomes:

$$(G) \left\{ \begin{array}{l} \text{find } u^h = v^h + g^h, \text{ where } v^h \in \xi^h, \text{ such that for all } w^h \in \xi^h \\ a(w^h, v^h) = -a(w^h, g^h) \end{array} \right. \quad (7)$$

To write the matrix form of Eq(7):

Let ξ^h consist of all linear combinations of given functions denoted by

$N_A : \Omega \rightarrow \mathbb{R}, A = 1, 2, \dots, n$.

$$w^h(x) = \sum_{A \in \eta - \eta_g}^n N_A(x) c_A \quad (8)$$

Each N_A are basis functions, and required that:

$$N_A(\pi) = 0, \quad A = 1, 2, \dots, n \quad (9)$$

And similarly,

$$v^h = \sum_{A \in \eta - \eta_g}^n N_A(x) d_A \quad (10)$$

Eq(7) can then be expressed as:

$$\sum_{B \in \eta - \eta_g}^n a(N_A, N_B) d_A = - \sum_{B \in \eta - \eta_g}^n a(N_A, N_B) g_B, \quad A \in \eta - \eta_g \quad (11)$$

In matrix form:

$$Kd = F \quad (12)$$

Where $K = \{K_{PQ}\} = a(N_A, N_B)$, $d = \{d_Q\}$, $F = \{F_P\} = - \sum_{B \in \eta - \eta_g}^n a(N_A, N_B) g_B$.

The bilinear quadrilateral shape functions for this problem are given as:

$$\begin{aligned} N_1(\xi, \eta) &= 1/4(1-\xi)(1-\eta) \\ N_2(\xi, \eta) &= 1/4(1+\xi)(1-\eta) \\ N_3(\xi, \eta) &= 1/4(1+\xi)(1+\eta) \\ N_4(\xi, \eta) &= 1/4(1-\xi)(1+\eta) \end{aligned} \quad (13)$$

To get the stiffness matrix, we have to perform numerical integration over the domain, using Gaussian Quadrature, in two dimensions, this can be expressed as:

$$\begin{aligned} \int_{-1}^1 \int_{-1}^1 g(\xi, \eta) d\xi d\eta &= \int_{-1}^1 \left\{ \sum_{l^{(1)}=1}^{n_{\text{int}}^{(1)}} g(\bar{\xi}_{l^{(1)}}^{(1)}, \eta) W_{l^{(1)}}^{(1)} \right\} d\eta \\ &= \sum_{l^{(1)}=1}^{n_{\text{int}}^{(1)}} \sum_{l^{(2)}=1}^{n_{\text{int}}^{(2)}} g(\bar{\xi}_{l^{(1)}}^{(1)}, \bar{\eta}_{l^{(2)}}^{(2)}) W_{l^{(1)}}^{(1)} W_{l^{(2)}}^{(2)} \end{aligned} \quad (14)$$

If we use two points rule in each direction, we will evaluate the integral in the following points and weights:

$$\begin{aligned} \bar{\xi}_1 &= \frac{-1}{\sqrt{3}}, & \bar{\xi}_2 &= \frac{1}{\sqrt{3}}, & \bar{\xi}_3 &= \frac{1}{\sqrt{3}}, & \bar{\xi}_4 &= \frac{-1}{\sqrt{3}} \\ \bar{\eta}_1 &= \frac{-1}{\sqrt{3}}, & \bar{\eta}_2 &= \frac{-1}{\sqrt{3}}, & \bar{\eta}_3 &= \frac{1}{\sqrt{3}}, & \bar{\eta}_4 &= \frac{1}{\sqrt{3}} \\ W_1 &= W_2 = W_3 = W_4 = 1 \end{aligned} \quad (15)$$

The stiffness matrix K:

Note that to facilitate the coding, ID of -1 value represent node on the $x=3$ boundary (the 'right wall'), and ID of 0 means that the node is on the $y=1$ boundary (the 'top wall').

The IEN Array is shown as follow-it associate the element number with node number, i.e. the four node number corresponding to a particular element. The node numbering is CCW (Counter Clock Wise), starting from the lower left corner:

IEN Array:

<i>Element numbers</i>		→ 1	2	3	4	5	6	7	8	$n_{el} = 8$
<i>Local node numbers</i>	1	1	2	3	4	6	7	8	9	
	2	2	3	4	5	7	8	9	10	
	3	7	8	9	10	12	13	14	15	
	4	6	7	8	9	11	12	13	14	
$n_{en} = 4$										

Finally, the LM array that combine ID array and IEN array is obtained by replacing the node number with their corresponding ID value, for this example, the LM array is shown as the following:

LM Array:

<i>Element numbers</i>		→ 1	2	3	4	5	6	7	8	$n_{el} = 8$
<i>Local node numbers</i>	1	1	2	3	4	5	6	7	8	
	2	2	3	4	-1	6	7	8	-1	
	3	6	7	8	-1	0	0	0	-1	
	4	5	6	7	8	0	0	0	0	
$n_{en} = 4$										

Thermal conductivity:

The thermal conductivity is given as:

$$\kappa_{ij} = 0.5(x^2 + y^2) \tag{17}$$

Since we are evaluating the stiffness matrix in the local domain, the corresponding thermal conductivity value are obtained by the following relationship (page 147 of the textbook):

$$x(\xi, \eta) = \sum_{a=1}^{n_{en}} N_a(\xi, \eta)x_a^e, \quad y(\xi, \eta) = \sum_{a=1}^{n_{en}} N_a(\xi, \eta)y_a^e \tag{18}$$

Local Stiffness matrix:

To evaluate the $N_{a,x}, N_{a,y}$, the method in page 148 of the textbook is used:

For $l = 1, \dots, n_{\text{int}}$:

$$\begin{aligned}
 x_{,\xi}(\bar{\xi}_l, \bar{\eta}_l) &= \sum_{a=1}^{n_{\text{en}}} N_{a,\xi}(\bar{\xi}_l, \bar{\eta}_l) x_a^e \\
 x_{,\eta}(\bar{\xi}_l, \bar{\eta}_l) &= \sum_{a=1}^{n_{\text{en}}} N_{a,\eta}(\bar{\xi}_l, \bar{\eta}_l) x_a^e \\
 y_{,\xi}(\bar{\xi}_l, \bar{\eta}_l) &= \sum_{a=1}^{n_{\text{en}}} N_{a,\xi}(\bar{\xi}_l, \bar{\eta}_l) y_a^e \\
 y_{,\eta}(\bar{\xi}_l, \bar{\eta}_l) &= \sum_{a=1}^{n_{\text{en}}} N_{a,\eta}(\bar{\xi}_l, \bar{\eta}_l) y_a^e \\
 j(\bar{\xi}_l, \bar{\eta}_l) &= x_{,\xi}(\bar{\xi}_l, \bar{\eta}_l) y_{,\eta}(\bar{\xi}_l, \bar{\eta}_l) - x_{,\eta}(\bar{\xi}_l, \bar{\eta}_l) y_{,\xi}(\bar{\xi}_l, \bar{\eta}_l)
 \end{aligned} \tag{19}$$

Within the same loop,

$$\begin{aligned}
 \text{For } a = 1, \dots, n_{\text{en}} : \\
 N_{a,x}(\bar{\xi}_l, \bar{\eta}_l) &= \frac{N_{a,\xi}(\bar{\xi}_l, \bar{\eta}_l) y_{,\eta}(\bar{\xi}_l, \bar{\eta}_l) - N_{a,\eta}(\bar{\xi}_l, \bar{\eta}_l) y_{,\xi}(\bar{\xi}_l, \bar{\eta}_l)}{j(\bar{\xi}_l, \bar{\eta}_l)} \\
 N_{a,y}(\bar{\xi}_l, \bar{\eta}_l) &= \frac{-[N_{a,\xi}(\bar{\xi}_l, \bar{\eta}_l) x_{,\eta}(\bar{\xi}_l, \bar{\eta}_l) - N_{a,\eta}(\bar{\xi}_l, \bar{\eta}_l) x_{,\xi}(\bar{\xi}_l, \bar{\eta}_l)]}{j(\bar{\xi}_l, \bar{\eta}_l)}
 \end{aligned} \tag{20}$$

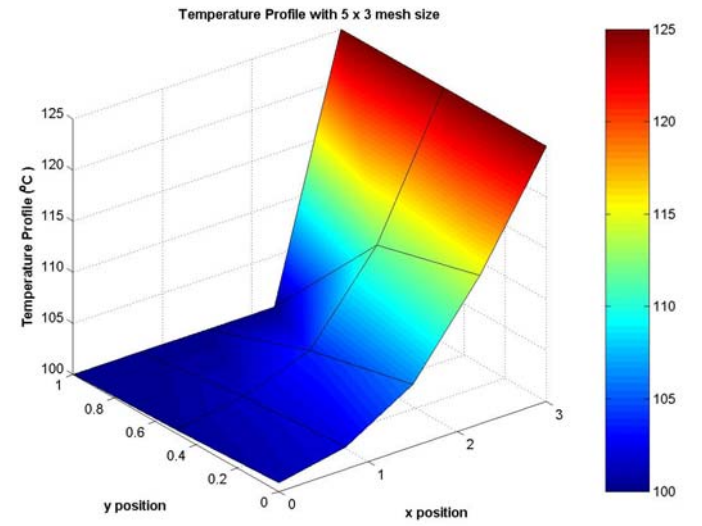
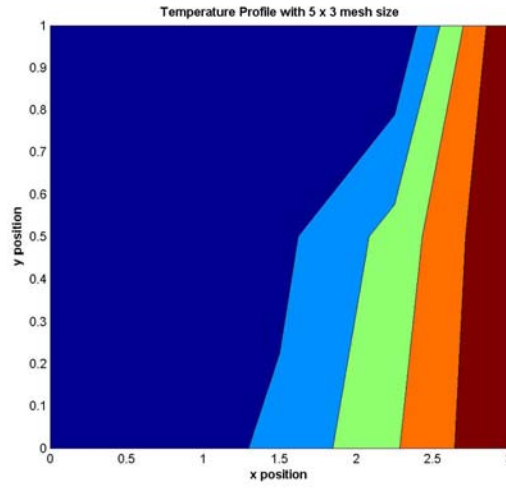
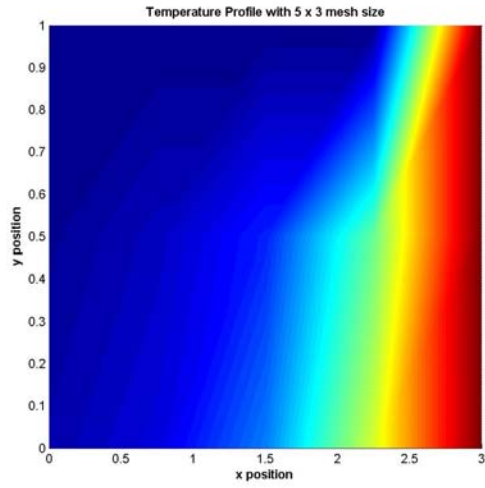
The construction of the local stiffness matrix to the corresponding global stiffness matrix is done by following the rules given in page 74 of the textbook (example 2). After solving the resulting linear system, the temperature value at each node are putting back to their corresponding location in the x, y location. The temperature profile is then plotted.

Results (of different mesh size – given in A x B, where A is the number of node used in the x-direction and B is the number of node used in the y-direction) are shown in the following figures.

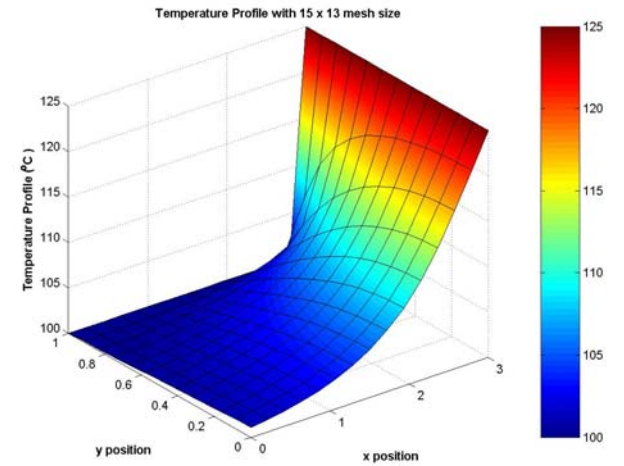
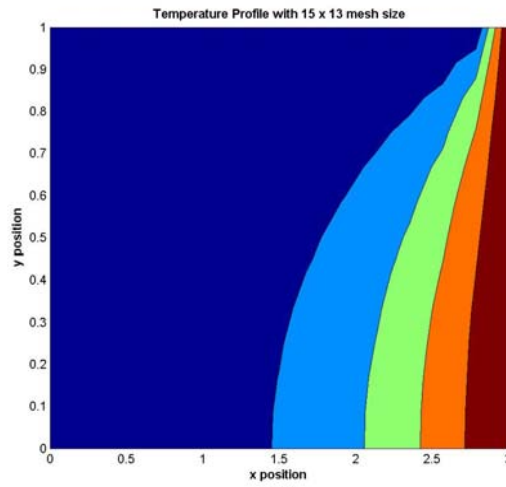
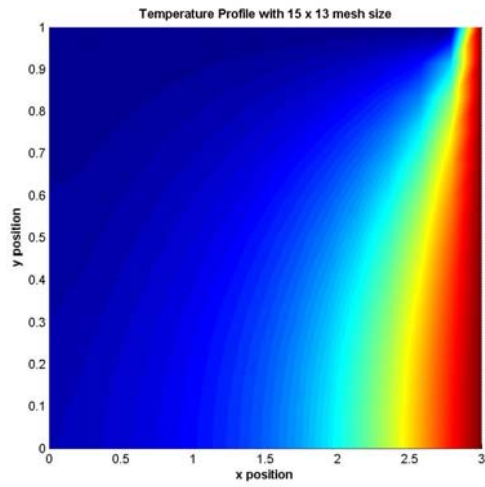
Each of the results are shown in three different plots:

- A. a contour plot that interpolate the temperature profile;
- B. a filled contour plot that do not interpolate the temperature profile; and
- C. a surface plot of the temperature profile.

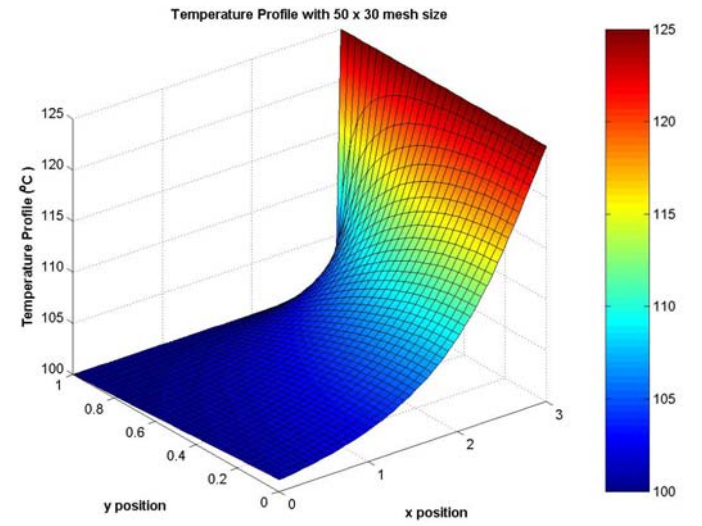
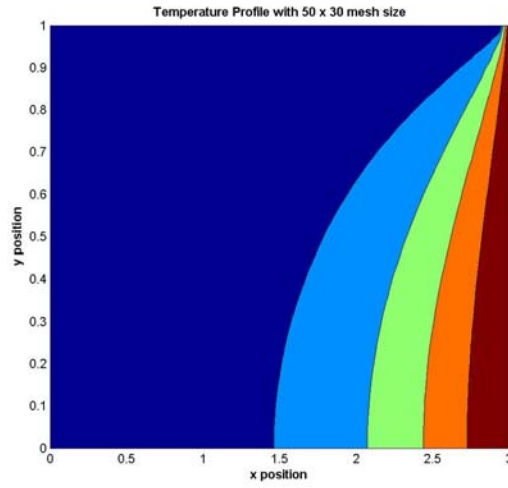
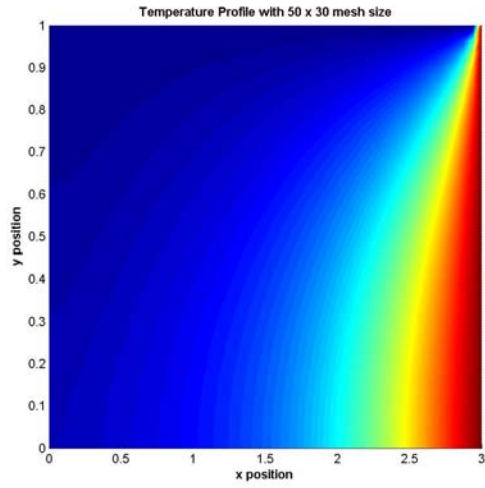
Mesh size: 5 x 3



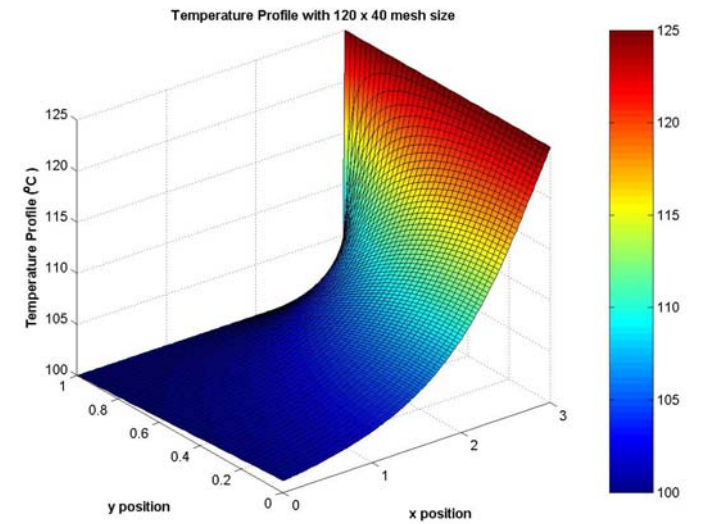
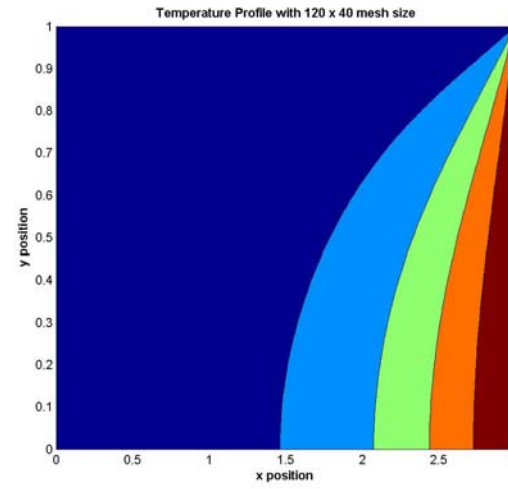
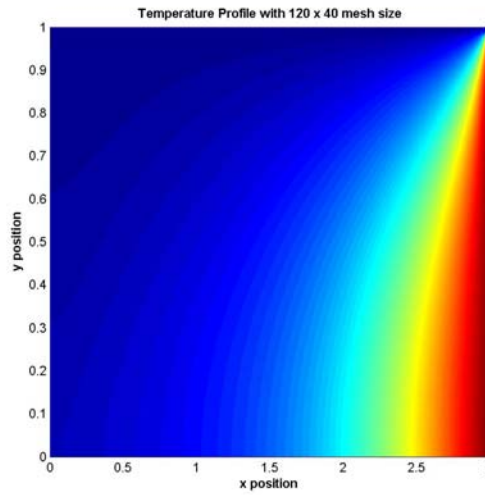
Mesh size: 15 x 13



Mesh size: 50 x 30



Mesh size: 120 x 40



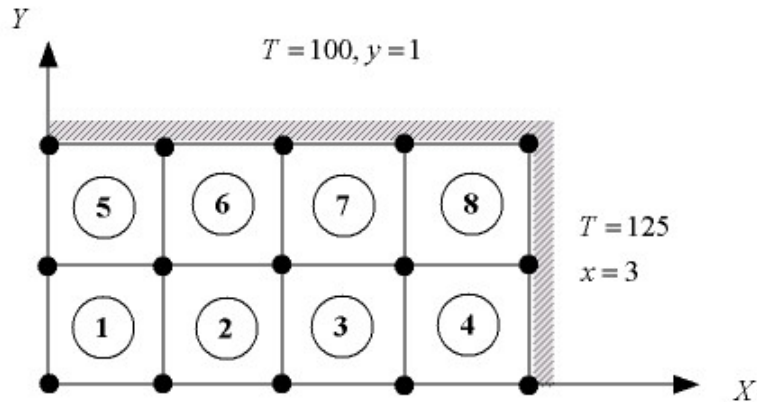
Error calculation:

The error indicator is given by:

$$E = 0.5 \sqrt{\oint_{\Gamma} \left[\frac{\partial T}{\partial n} \right]^2 ds, \Gamma = \cup \frac{\partial \Omega_k}{\partial \Omega}} \quad (21)$$

Where $\frac{\partial T}{\partial n} = \left| \frac{\partial T}{\partial n_{Left}} - \frac{\partial T}{\partial n_{Right}} \right|$, which is to calculate the slope difference of the temperature profile of every elements with respect to its surrounding elements.

I tried to code this but it required an extra array to keep track of the surrounding element, and creation of an algorithm to generate this array for any size of mesh. For example, for the example mesh size of 5x3 we used in demonstrated the construction of ID, IEN, and LM before, this array (let call it SE (Surrounding Element) array):



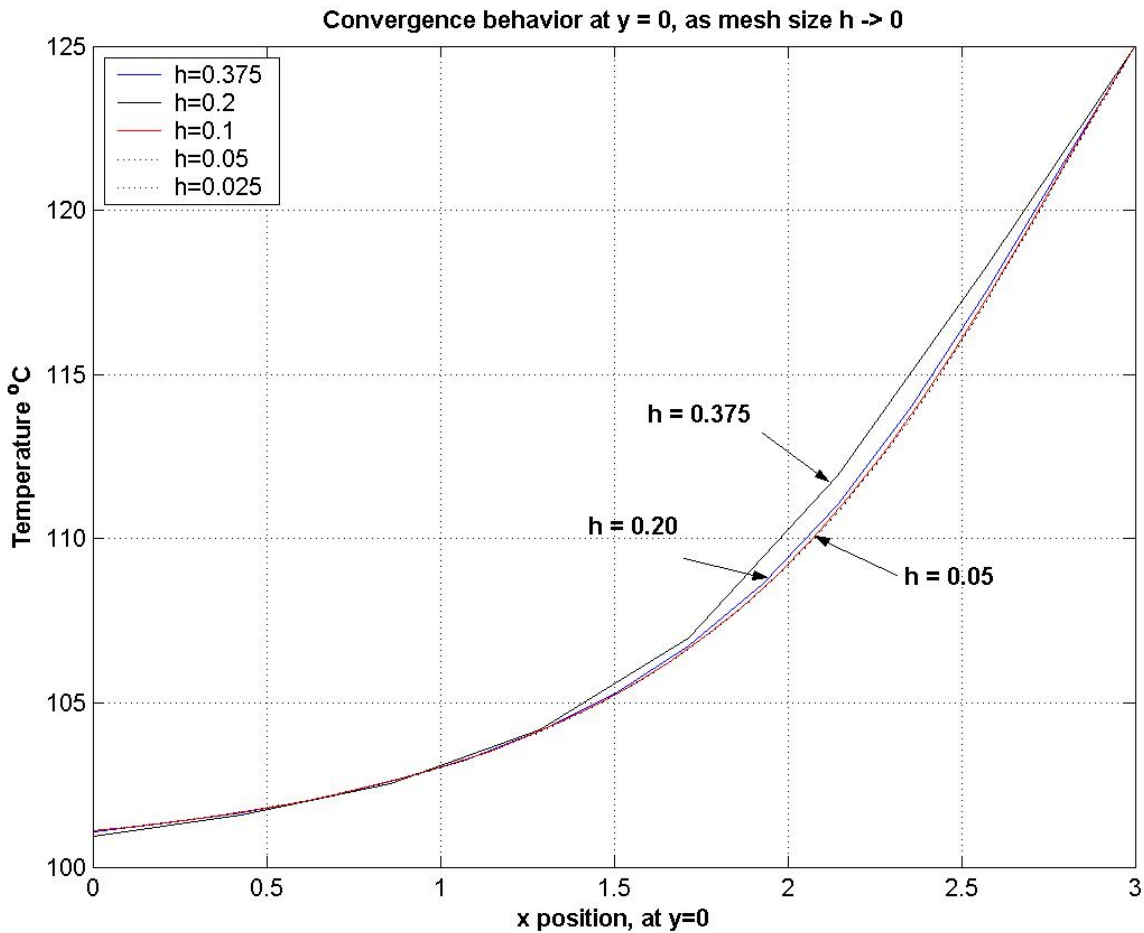
SE Array:

<i>Element numbers</i>		→ 1	2	3	4	5	6	7	8	$n_{el} = 8$
<i>Surrounding Element numbers</i>	1	0	1	2	3	0	5	6	7	
	2	0	0	0	0	1	2	3	4	
	3	2	3	4	0	6	7	8	0	
	4	5	6	7	8	0	0	0	0	

$n_{en} = 4$

I didn't successfully implement this in the code. On the other hand, since we want to study the convergence behavior as the mesh size $h \rightarrow 0$, alternately, we observe that the temperature profile at $x = 0$ could be a very good indicator of the convergence behavior for this specific case. Hence, we

implemented this ‘alternate’ way (a very crude way) of observing the convergence rate, by plotting the temperature profile at $x = 0$ with different number of mesh size. By doing this, we could a 2D plot to observe the convergence rate with respect to mesh size. This is shown in the following figures.



Discussion:

In this problem, we use FEM method to solve the steady state heat equation. It was a challenging code as we have to write the algorithm to generate the corresponding ID, IEN, LM array, mesh generation, and to evaluate the stiffness matrix in the local element. Once the coding is done, we are able to see the result with different mesh size. As we can see, as we increased the elements used, the temperature profile become very smooth, and converged (from the error study) at some point. One interesting thing to note is that at the point where $x=3$ and $y=1$, it is the point where the boundary conditions were conflicting each other, there is a huge discontinuity at that point and the point beside it. However, we can see that FEM method was able to handle this situation pretty well and does not cause any instability in the solution. Unfortunately, I am not able to perform the error analysis as required but only able to use a very crude way to observe the convergence rate as we refine the mesh size. In this way, it was estimated that the temperature profile converged as $h = 0.05$.

Problem 2:

Q2. Let us now consider the transient heat conduction problem

$$\begin{aligned} \frac{\partial T}{\partial t} - \nabla \cdot k \nabla T &= 0 & 0 < x < 3, 0 < y < 1 \\ x=0 \quad k \frac{\partial T}{\partial y} &= 0 & y=0 \quad k \frac{\partial T}{\partial x} &= 0 \\ x=3 \quad T &= 100 \cos t & y=1 \quad T &= 100 \\ t=0 \quad T &= 100 \end{aligned}$$

Please modify the code developed in Q1 using the Generalized trapezoidal algorithm for $\alpha=0, 0.5$ and $\alpha=1$ to compute the temperature profiles as a function of time.

Solution:

The strong form of the problem is given by:

$$(S) \left\{ \begin{aligned} &\text{Given } f = 0 \rightarrow \mathbb{R}, g : \Gamma_g \rightarrow \mathbb{R} \text{ and } h : 0 \rightarrow \mathbb{R}, \text{ find } u : \bar{\Omega} \times]0, T[\rightarrow \mathbb{R} \text{ such that:} \\ &T_{,t} + q_{i,i} = f, \quad \text{in } \Omega \times]0, T[\\ &T = g \quad \text{on } \Gamma_g \times]0, T[\\ &-q_i n_i = 0 \quad \text{on } \Gamma_h \times]0, T[\\ &T(x, 0) = T_0(x) \quad x \in \Omega \end{aligned} \right. \quad (22)$$

Where $q_i = -\kappa_{ij} T_{,j}$; $T_{,t} = \partial T / \partial t$.

To get the weak form of Eq(22), let ℓ denote the trial solution space and ξ the variation space. Wherein ℓ and ξ consist of real-valued functions defined on $\bar{\Omega}$ satisfying certain smoothness requirements, such that all members of ℓ satisfy, whereas if $w \in \xi$, then $w = 0$ on Γ_g . Multiply Eq(22) by smooth function w , to get the following weak form:

$$(W) \left\{ \begin{aligned} &\text{Given } f = 0 \rightarrow \mathbb{R}, g : \Gamma_g \rightarrow \mathbb{R} \text{ and } h : 0 \rightarrow \mathbb{R}, \text{ find } u(t) \in \ell, t \in [0, T] \text{ such that for all } w \in \xi: \\ &a(w, \dot{u}) + a(w^h, u^h) = 0 \\ &(w, T(0)) = (w, T_0) \end{aligned} \right. \quad (23)$$

To construct the Galerkin-form of the problem, construct the finite-dimensional approximations of ℓ and ξ , these collections of functions are denoted by ℓ^h and ξ^h , where ℓ^h and ξ^h as being subset of ℓ and ξ . i.e. $\ell^h \subset \ell$ and $\xi^h \subset \xi$.

Assuming the collection ξ^h is given, to each member $v^h(t) \in \xi^h$, construct a function $u^h(t) \in \ell^h$ by $u^h(x, t) = v^h(x, t) + g^h(x, t)$.

Substitute $u^h = v^h + g^h$ into equation(6), the Galerkin form becomes:

$$(G) \begin{cases} \text{find } u^h(x,t) = v^h(x,t) + g^h(x,t), \text{ where } u^h(t) \in \xi_t^h, \text{ such that for all } w^h \in \xi^h \\ a(w^h, \dot{v}^h) + a(w^h, v^h) = -a(w^h, \dot{g}^h) - a(w^h, g^h) \\ (w^h, v^h(0)) = (w^h, u_0) - (w^h, g^h(0)) \end{cases} \quad (24)$$

To write the matrix form of Eq(24):

Let ξ^h consist of all linear combinations of given functions denoted by $N_A : \Omega \rightarrow \mathbb{R}$, $A = 1, 2, \dots, n$.

$$w^h(x) = \sum_{A \in \eta - \eta_g}^n N_A(x) c_A(t) \quad (25)$$

Each N_A are basis functions, and required that:

$$N_A(\pi) = 0, \quad A = 1, 2, \dots, n \quad (26)$$

And similarly,

$$v^h = \sum_{A \in \eta - \eta_g}^n N_A(x) d_A(t); \quad g^h = \sum_{A \in \eta - \eta_g}^n N_A(x) g_A(t) \quad (27)$$

We can write into the following matrix form:

$$\begin{aligned} M\dot{d} + Kd &= F \quad t \in [0, T[\\ d(0) &= d_0 \end{aligned} \quad (28)$$

Where:

$$M = \underset{e=1}{\overset{n_e}{A}}(m^e); \quad m^e = [m_{ab}^e]; \quad m_{ab}^e = \int_{\Omega_e} N_a N_b d\Omega$$

$$K = \underset{e=1}{\overset{n_e}{A}}(k^e); \quad k^e = [k_{ab}^e]; \quad k_{ab}^e = \int_{\Omega_e} B_a^T D B_b d\Omega$$

$$F(t) = \underset{e=1}{\overset{n_e}{A}}(f^e(t)); \quad f^e = [f_a^e]; \quad f_a^e = -\sum_{b=1}^{n_e} (k_{ab}^e g_b^e + m_{ab}^e \dot{g}_b^e)$$

The construction of $K, M, F(t)$ are similar to the rules shown in problem 1. However, to solve this semidiscrete heat equation, we adopt the generalized trapezoidal method as follow:

$$\begin{aligned} Mv_{n+1} + Kd_{n+1} &= F_{n+1} \\ d_{n+1} &= d_n + \Delta t v_{n+\alpha} \\ v_{n+\alpha} &= (1 - \alpha)v_n + \alpha v_{n+1} \end{aligned} \quad (29)$$

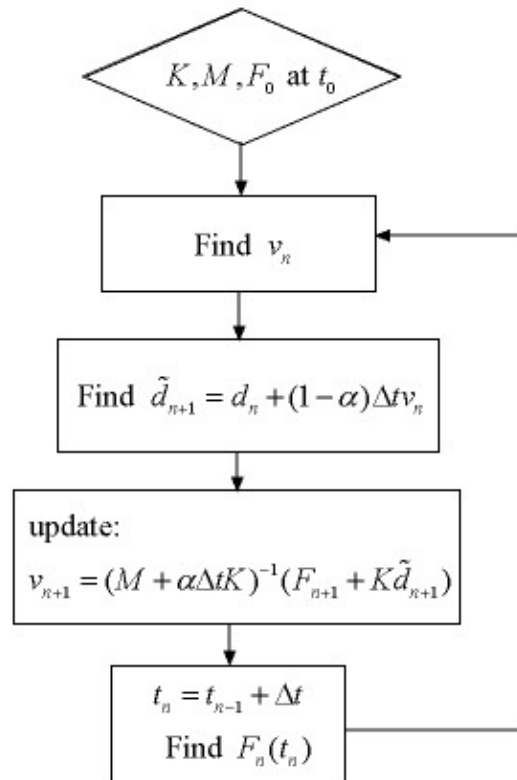
Where d_n, v_n are the approximation to $d(t_n), \dot{d}(t_n)$, and

$\alpha = 0$ – Forward difference; forward Euler,

$\alpha = 0.5$ – Trapezoidal rule; midpoint rule; crank-Nicolson,

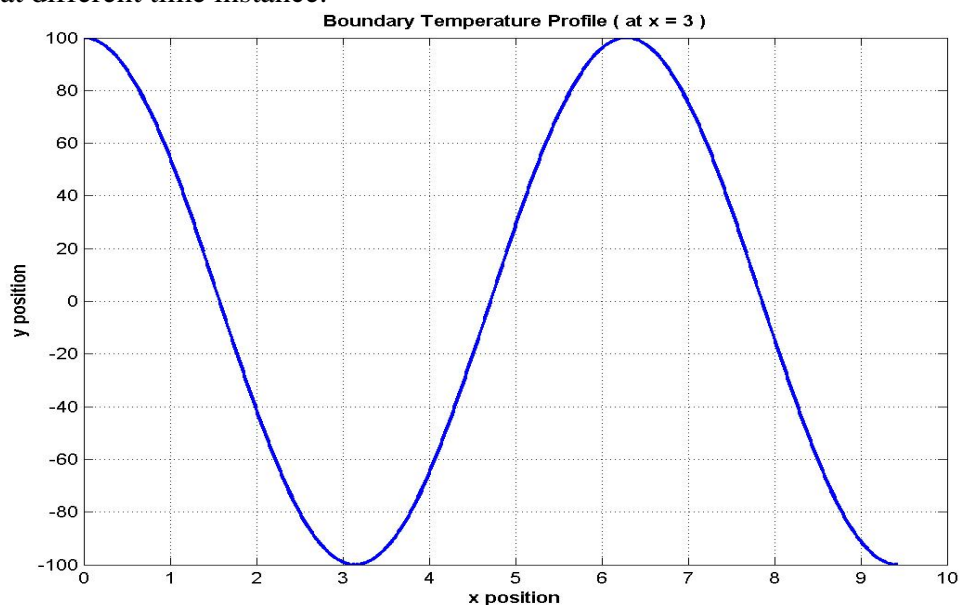
$\alpha = 1.0$ – Backward differences; backward Euler.

We implemented this algorithm using the v-form given in page 460 of textbook:



The stability of the generalized trapezoidal methods are given in page 467 of the textbook in which for $\alpha \geq 0.5$ the method is unconditionally stable, whereas for $\alpha < 0.5$, the method is conditional stable, in which the time step had to be very small.

To see how the boundary temperature vary with respect to time t , we plot the temperature fluctuation of the boundary at different time instance:



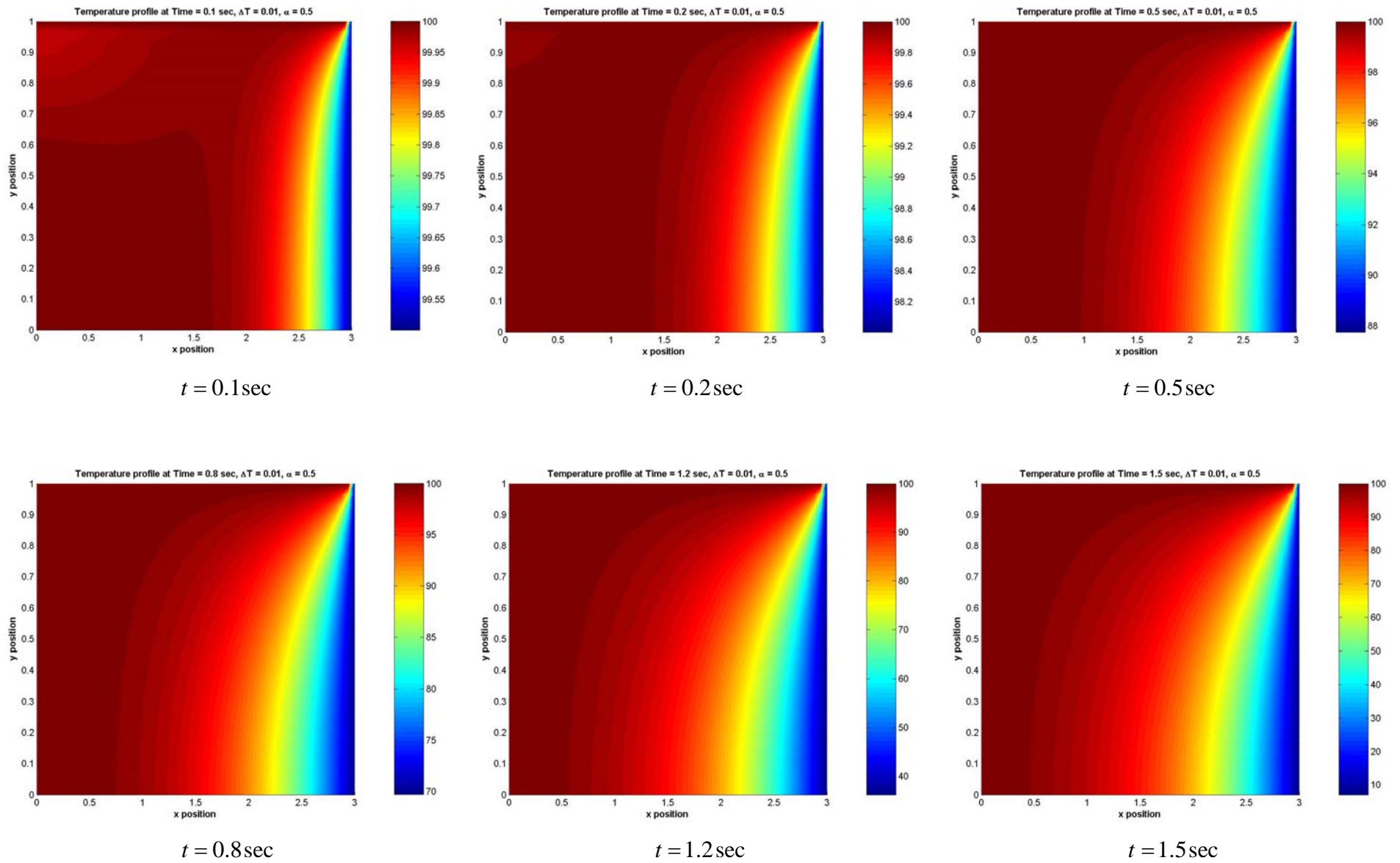
So for $\alpha = 0.5, \alpha = 1.0$ we plot the temperature profile as $t = 0 \rightarrow 2\pi$, since after $t > 2\pi$, the temperature profile will repeat itself.

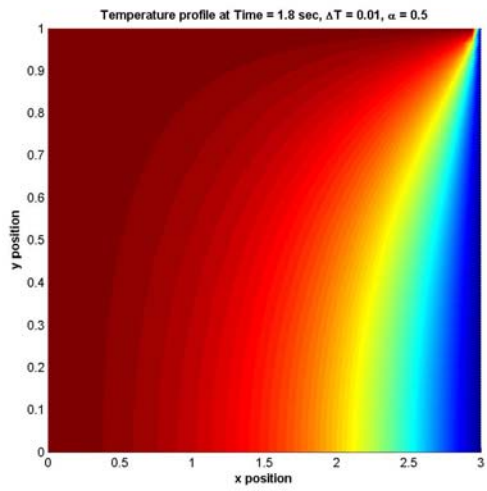
Results:

Because of stability issue of the trapezoidal rule as stated above, we are able to run the algorithm for $\alpha \geq 0.5$ with pretty large time step while maintaining stability of the algorithm. In particular we use $\alpha = 0.5, \alpha = 1.0$ with a time step of $\Delta t = 0.01$, and we can obtain pretty good result while maintain stability (However, since the result for $\alpha = 0.5, \alpha = 1.0$ are the same, we only show those results we obtained for $\alpha = 0.5$ here). For $\alpha = 0.0$, The time step need to be very small, by trial and error, for a 50x30 size mesh, it had to be $\Delta t = 0.00001$ to obtain stable solution, in this case, we only plot the solution at $t = 0.1$ sec (which is the same as the one obtained with $\alpha = 0.5$).

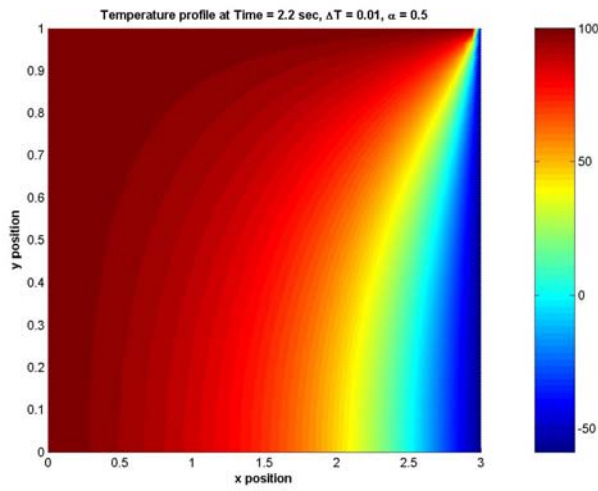
Temperature profile for mesh size 50x30, $\alpha = 0.5$, at different time t .

(Note that the lowest value is changing at each time instant, the value can be determine from the colorbar)

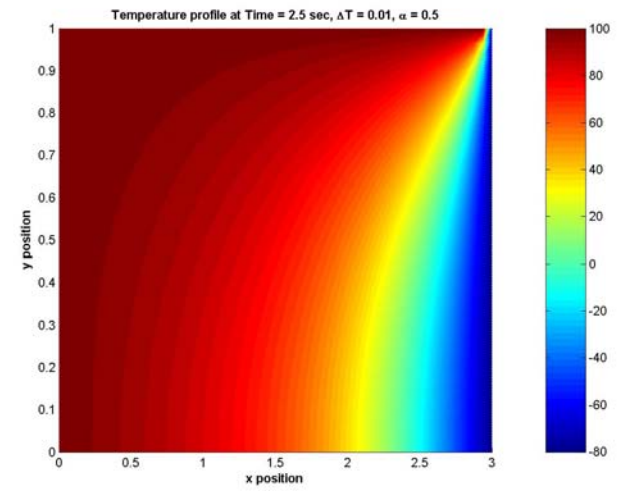




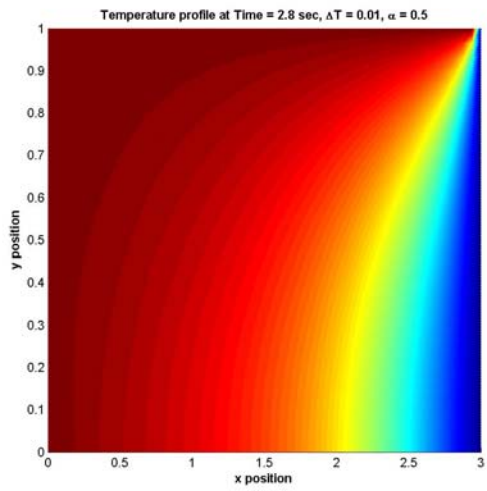
$t = 1.8 \text{ sec}$



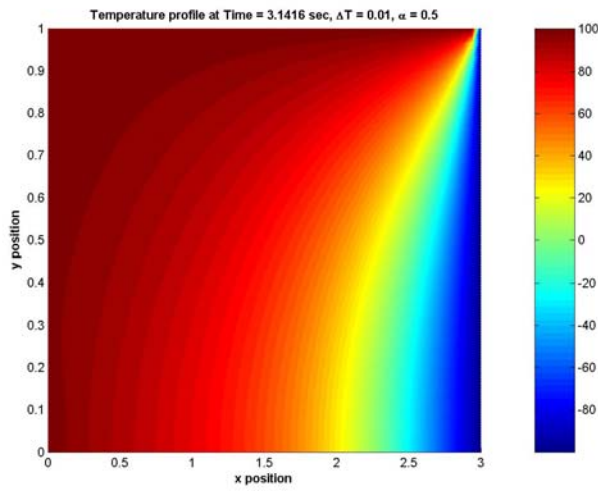
$t = 2.2 \text{ sec}$



$t = 2.5 \text{ sec}$

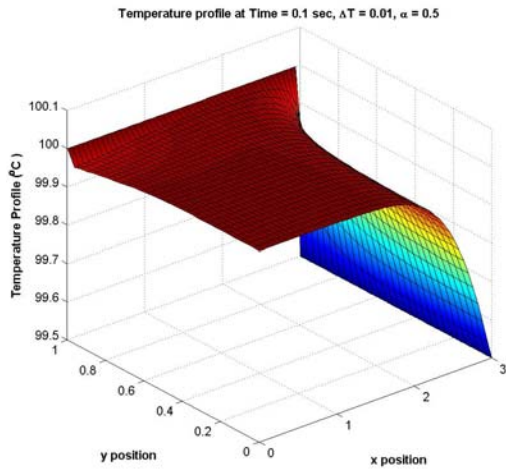


$t = 2.8 \text{ sec}$

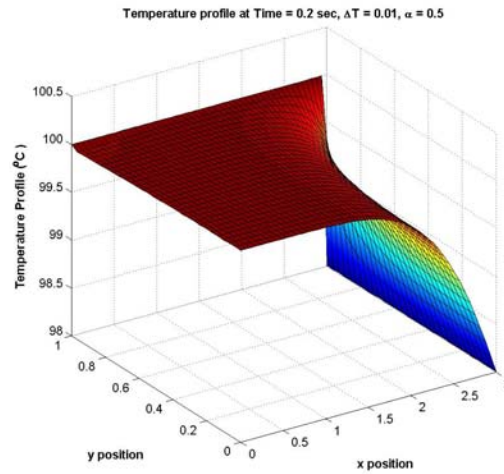


$t = 3.14 \text{ sec}$

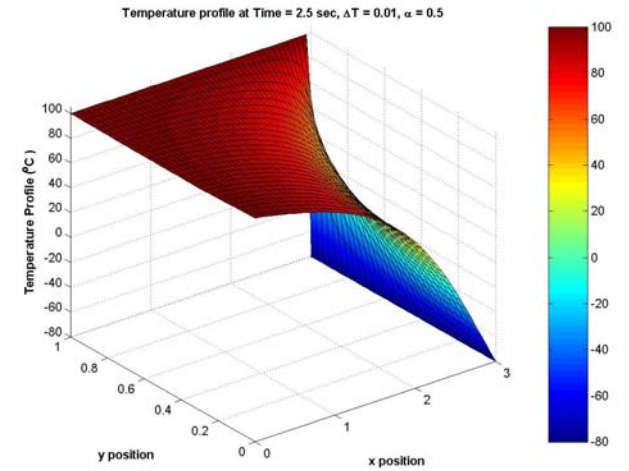
Temperature profile (3D plot for better viewing in black & white printing) for mesh size 50x30, $\alpha = 0.5$, at different time t .
(Note that the lowest value is changing at each time instant, the value can be determine from the colorbar)



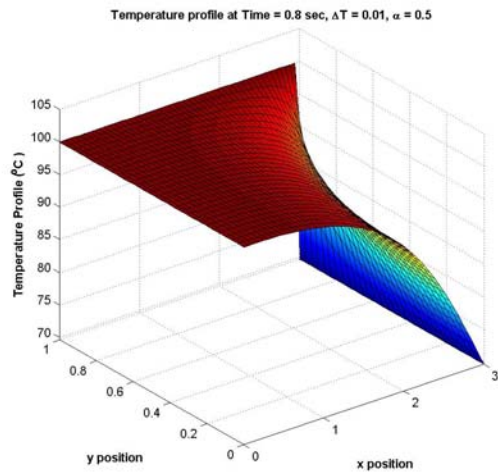
$t = 0.1$ sec



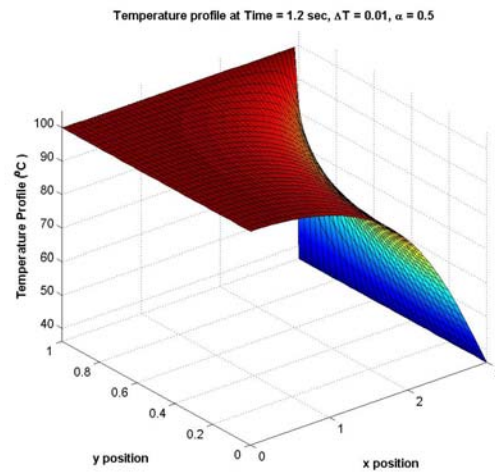
$t = 0.2$ sec



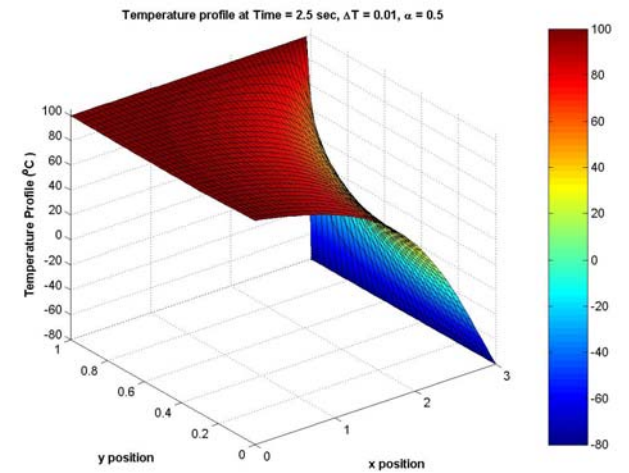
$t = 0.5$ sec



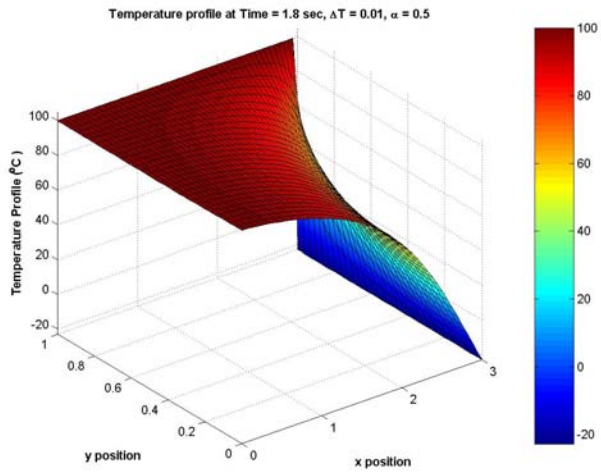
$t = 0.8$ sec



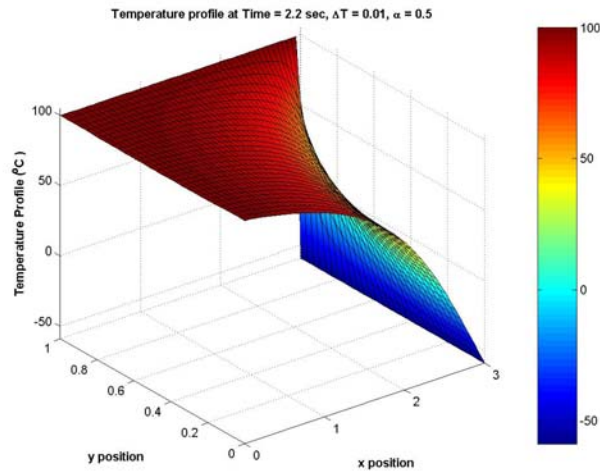
$t = 1.2$ sec



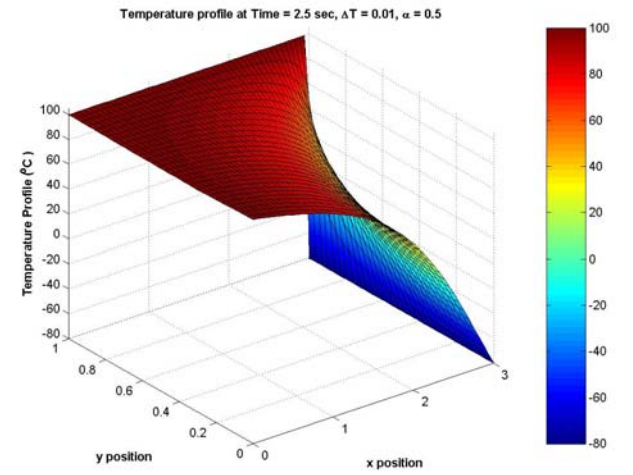
$t = 1.5$ sec



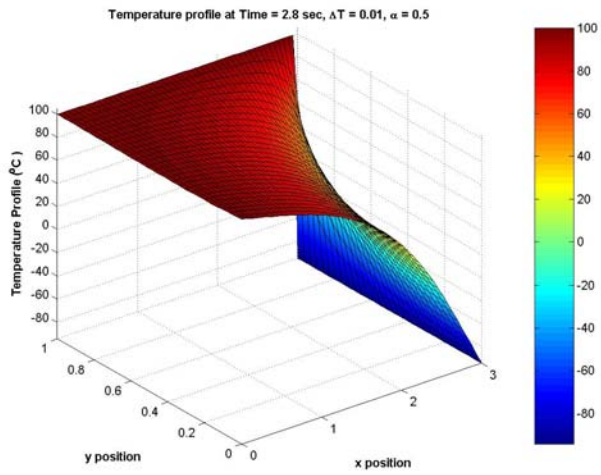
$t = 1.8 \text{ sec}$



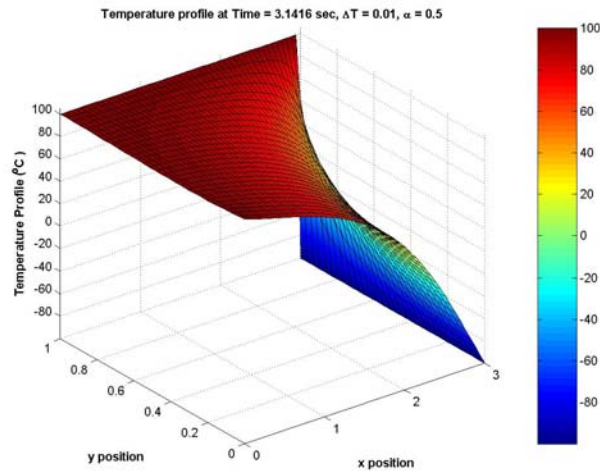
$t = 2.2 \text{ sec}$



$t = 2.5 \text{ sec}$



$t = 2.8 \text{ sec}$



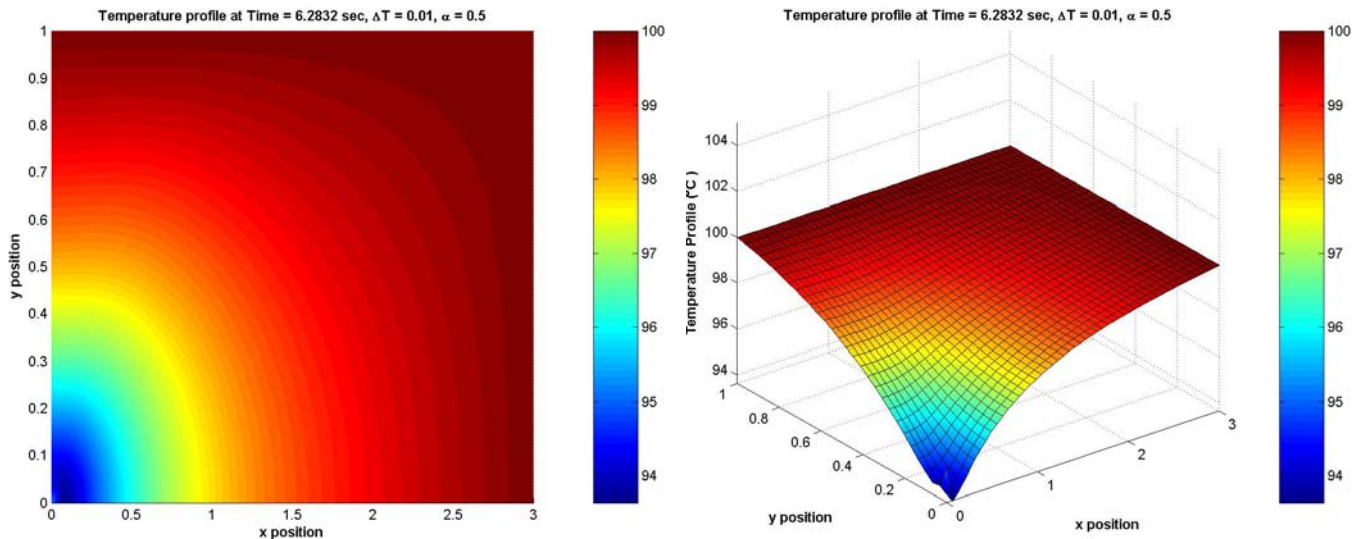
$t = 3.14 \text{ sec}$

Discussion:

In this problem, we applied different trapezoidal rule to solve the semi discrete time dependent heat equation. However, once problem 1 was done, it is pretty easy to code for this problem. As we mentioned in the formulation, the stability of using trapezoidal rule is very important. We experienced this while using different value of α . We observe that as $\alpha \geq 0.5$, the time step that we can use in this method can be pretty large ($\Delta t = 0.01$) and still have a stable result. Since we observe that the temperature profile for $\alpha = 0.5$ and $\alpha = 1.0$ are the same, we only plot the one for $\alpha = 0.5$.

As we can see, the boundary temperature fluctuated between 100 and -100 degree as t goes from 0 to 3.1425 (2π) sec. Hence, we only plot the temperature profile from $t = 0$ to $t = \pi/2$, since after that, the temperature profile just repeating itself.

However one interesting phenomena occurred as $t = 2\pi$, at this time, the boundary temperature are all 100 degree, we obtain the following temperature distribution at $t = 2\pi$ (mesh size 50x30):



I then decided to increase the mesh size to 140x40 but similar result is obtained. These graphs are included here for reference.

