EE 459/611: Smart Grid Economics, Policy, and Engineering

Lecture 3: Linear Algebra and Multi-Variable Calculus Review

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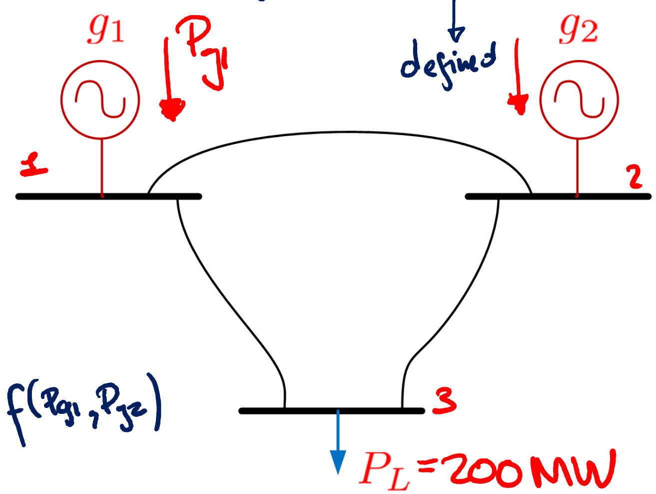
Motivating Example

$\underline{x} = \begin{pmatrix} P_{g1} \\ P_{g2} \end{pmatrix} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \in \mathbb{R}^2$



- Two generator power system:

- Cost of generator 1 per MW is:

  $\rightarrow C_{g1} = \underbrace{3P_{g1}^2 + 1P_{g1}}$ (\$) $\quad \underline{10} \leq \underline{P_{g1}} \leq \underline{90}$

- Cost of generator 2 per MW is:

  $C_{g2} = \underline{2P_{g2}^2 + 3P_{g2}} \qquad \underline{5} \leq \underline{P_{g2}} \leq \underline{150}$

  $f(x) = f(P_{g1}, P_{g2})$

- Load is 200 MW

  $P_L = 200$

- What is the total cost? $= C_{g1}(P_{g1}) + C_{g2}(P_{g2})$
- What is the cheapest way to operate the generators? $\rightarrow$

- Constraints:

  $* \quad \underset{\text{(generation)}}{P_g} = \underset{\text{(load)}}{P_L} \Rightarrow \quad P_{g1} + P_{g2} = 200$

  $* \; 10 \leq P_{g1} \leq 90$

  $* \quad 5 \leq P_{g2} \leq 150$

best/optimal way P_{g1}^*, P_{g2}^* $\quad x \in \mathbb{R}^2$

$$\boxed{\begin{aligned} &\min_{P_{g1}, P_{g2}} \left[\underline{C_{g1}(P_{g1}) + C_{g2}(P_{g2})} \right] = f(x) \\ &\text{s.t.} \\ &\qquad P_{g1} + P_{g2} = 200 \Rightarrow g(x) = 200 \\ &\qquad 10 \leq P_{g1} \leq 90 \\ &\qquad 5 \leq P_{g2} \leq 150 \end{aligned}}$$

Motivating Example

- In **general**, an optimization problem has the following form:

unknowns

$$\left\{\begin{array}{l} \min\limits_{x} f(x) \\ \text{subject to:} \\ \quad h_i(x) = 0 \quad i = 1,\ 2,\ \ldots,\ \underbrace{N_e}_{\text{Total \#}} \\ \quad g_j(x) \le 0 \quad j = 1,\ 2,\ \ldots,\ N_I \end{array}\right.$$

$f(x)$: Cost function

$h_i(x)=0 \rightarrow$ Equality constraints *

$g_j(x)\le 0 \rightarrow$ Inequality constraints *

- The functions $f(x),\ h_i(x),\ g_j(x)$ can be functions of several variables!

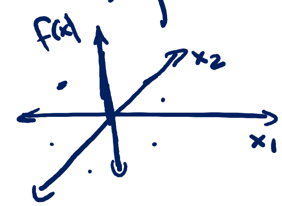
dimension

$$\text{for } x \in \mathbb{R}^n, \quad f(x) : \underbrace{\mathbb{R}^n}_{\text{domain}} \to \underbrace{\mathbb{R}}_{\text{range}}$$

$$x = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}$$

$x \in \mathbb{R}^n$, $f(x) \in \mathbb{R}$

$x \in \mathbb{R}^2$



Example

$x \in \mathbb{R}^3$ $\quad x = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$

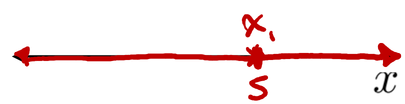
$f(x) : \mathbb{R}^3 \to \mathbb{R}$

$$\boxed{f(x) = \cos(x_1 x_2) + 3x_3^2 + x_1^2 x_2^3}$$

Real Vector Spaces

- Consider some examples of vector spaces and their notations:

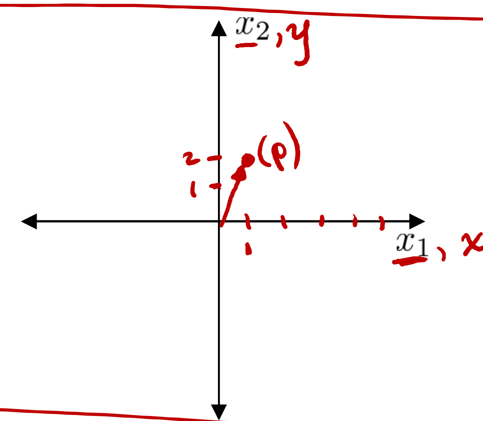
1. The real numbers $\mathbb{R}^1$



$x \in \mathbb{R}^1 \Rightarrow x = x_1 \qquad x_1 = 5 \in \mathbb{R}^1$

2. The 2 dimensional space, $\mathbb{R}^2$

How to identify an element in $\mathbb{R}^2$?

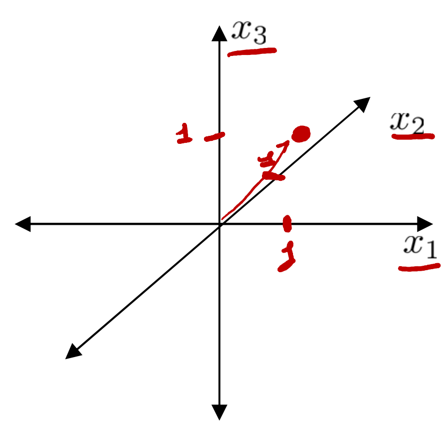


$x \in \mathbb{R}^2 \Rightarrow \quad x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$

$p = \begin{pmatrix} 1 \\ 2 \end{pmatrix} \in \mathbb{R}^2$

3. The 3 dimensional space, $\mathbb{R}^3$

How to identify an element in $\mathbb{R}^3$?



$\mathbb{R}^4$

$\vdots$

$x \in \mathbb{R}^3 \Rightarrow \quad x = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$

$q \in \mathbb{R}^3 \quad q = \begin{pmatrix} \cdot \\ \cdot \\ \cdot \end{pmatrix}$

Addition, Subtraction, and Scalar Multiplication

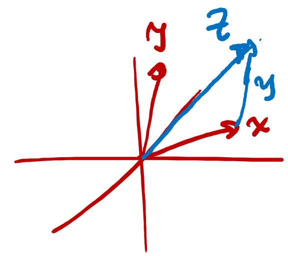
- Assume that we are in $\mathbb{R}^3$, consider the following operations:

$x, y \in \mathbb{R}^3$

$$x = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}, \qquad y = \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix}$$

- **Addition:**

$$x,\ y \in \mathbb{R}^3 \qquad x + y = \begin{pmatrix} x_1 + y_1 \\ x_2 + y_2 \\ x_3 + y_3 \end{pmatrix} = \begin{pmatrix} z_1 \\ z_2 \\ z_3 \end{pmatrix} = z \in \mathbb{R}^3$$



- **Scalar Mult.:**

$$\alpha \in \mathbb{R},\ \ x \in \mathbb{R}^3 \qquad \alpha x = \alpha \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} \alpha x_1 \\ \alpha x_2 \\ \alpha x_3 \end{pmatrix}$$

- **Subtraction:**

$$x,\ y \in \mathbb{R}^3 \qquad x - y = \begin{pmatrix} x_1 - y_1 \\ x_2 - y_2 \\ x_3 - y_3 \end{pmatrix} = x + (-1y) = z \in \mathbb{R}^3$$

Inner Product in $\mathbf{R}^n$

- Consider two vectors $x,\ y \in \mathbb{R}^n$, the inner product is:

$$x = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} \quad y = \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{pmatrix} \qquad \langle x, y \rangle = x^T y = \begin{pmatrix} x_1 & x_2 & \cdots & x_n \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{pmatrix}$$

$$x^T y = x_1 y_1 + x_2 y_2 + \cdots + x_n y_n \in \mathbb{R}$$

- Example

$$x = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} \quad y = \begin{pmatrix} 4 \\ 5 \\ 6 \end{pmatrix} \Rightarrow \quad \langle x, y \rangle = x^T y = \begin{pmatrix} 1 & 2 & 3 \end{pmatrix} \begin{pmatrix} 4 \\ 5 \\ 6 \end{pmatrix}$$

$$\langle x, y \rangle = x^T y = 1(4) + 2(5) + 3(6) = \ ?\ \boxed{32}$$

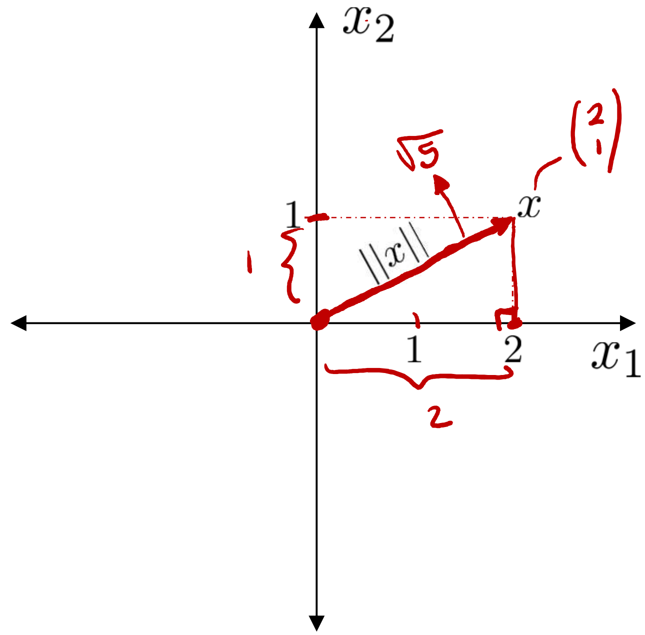
2-Norm in R^n

- The inner product in $\mathbb{R}^n$ can be used to define "length" (Norm)
- (2-norm)
  The **length** of a vector is defined as:

$$\|x\|_2 = \sqrt{\langle x, x\rangle} = \sqrt{x^T x} \qquad x \in \mathbb{R}^n \Rightarrow x = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}$$

$$\|x\|_2 = \sqrt{x^T x} = \sqrt{\sum_{i=1}^{n} x_i^2} = \sqrt{x_1^2 + x_2^2 + \dots + x_n^2}$$

- For example, in $\mathbb{R}^2$, consider $x = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$



$$\langle x, x\rangle = x^T x = \begin{pmatrix} 2 & 1 \end{pmatrix} \begin{pmatrix} 2 \\ 1 \end{pmatrix} = 2^2 + 1^2 = 5$$

$$\Rightarrow \quad ||x||_2 = \sqrt{\langle x, x\rangle} = \sqrt{2^2 + 1^2} = \sqrt{5}$$

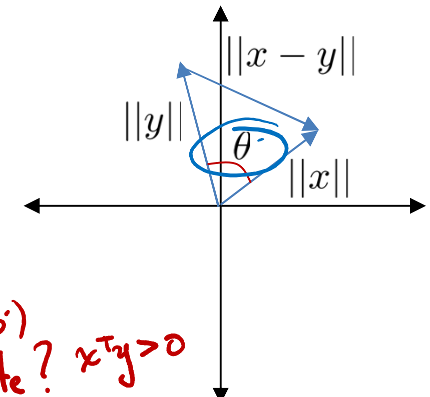
$$\|x\|_2^2 = x^T x = \langle x, x\rangle$$

Geometry/Angle Between Vectors

- We can also compute the "**angle**" between two vectors $x, y \in \mathbb{R}^n$

$$\boxed{x^T y} = \langle \mathbf{x}, \mathbf{y} \rangle = \underbrace{\|\mathbf{x}\|_2 \|\mathbf{y}\|_2 \cos(\theta)}\ = \boxed{\alpha} = x^T y$$

$$\Rightarrow \quad \frac{\langle \mathbf{x}, \mathbf{y} \rangle}{\|\mathbf{x}\|_2 \|\mathbf{y}\|_2} = \cos(\theta) \quad \Rightarrow \quad \cos^{-1}\left(\frac{\mathbf{x}^T\mathbf{y}}{\|\mathbf{x}\|_2\|\mathbf{y}\|_2}\right) = \theta \; *$$



angles are: acute? $x^T y > 0$ (<90°)

obtuse? $x^T y < 0$ (>90°)

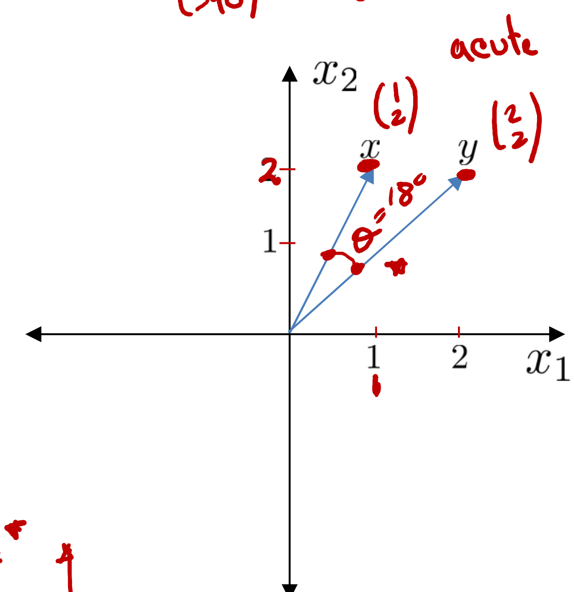
- Example, find the angle between $x = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$, $y = \begin{bmatrix} 2 \\ 2 \end{bmatrix}$

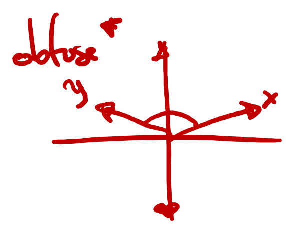
$$\langle x, y \rangle = x^T y = \begin{bmatrix} 1 & 2 \end{bmatrix} \begin{bmatrix} 2 \\ 2 \end{bmatrix} = 6$$

$$\underbrace{\|x\|_2 = \sqrt{x^T x} = \sqrt{5}} \qquad \underbrace{\|y\|_2 = \sqrt{y^T y} = \sqrt{8}}$$

$$\Rightarrow \boxed{\frac{\langle x, y \rangle}{\|x\|\|y\|}} = \boxed{\frac{6}{\sqrt{5}\sqrt{8}}} = \cos(\theta)$$

$$\Rightarrow \quad \theta = \cos^{-1}\left(\frac{6}{\sqrt{5}\sqrt{8}}\right) \approx 18^o$$





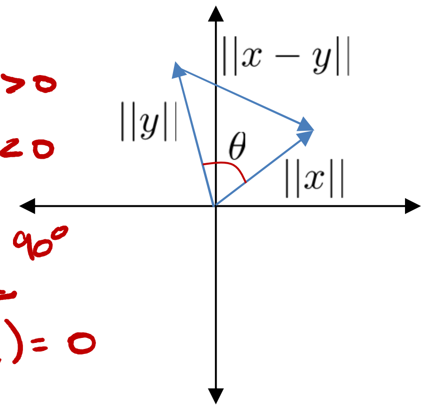
Geometry/Angle Between Vectors

- We can also compute the "**angle**" between two vectors $x, y \in \mathbb{R}^n$

  $\Rightarrow \quad \langle \mathbf{x}, \mathbf{y} \rangle = \underbrace{||\mathbf{x}||\,||\mathbf{y}|| \cos(\theta)}_{*} = x^T y$

  $\boxed{\cos(\theta) > 0 \quad 0 < \theta < 90^\circ} \Rightarrow x^T y > 0$

  $\rightarrow \cos(\theta) < 0 \quad \theta > 90^\circ \rightarrow x^T y < 0$



- Now, we can define when two vectors are "perpendicular" or $\theta = \frac{\pi}{2} = 90^\circ$

  $\cos(\frac{\pi}{2}) = 0$

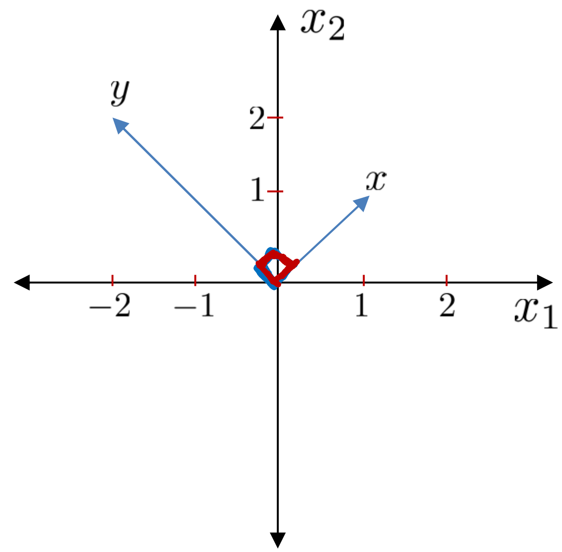
- Two vectors $x, y \in \mathbb{R}^n$ are **orthogonal** if $\langle x, y \rangle = 0$

  It will be denoted as $x \perp y$ if $\langle x, y \rangle = x^T y = 0$

- Show that $x = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$, $y = \begin{bmatrix} -2 \\ 2 \end{bmatrix}$ are orthogonal

$$x^T y = \begin{pmatrix} 1 & 1 \end{pmatrix} \begin{pmatrix} -2 \\ 2 \end{pmatrix} = -2 + 2 = 0$$

$$\Rightarrow \quad x \perp y$$



Matrices

- **Definition:** A *matrix* is an *m* by *n* array of scalars from a field **F** $(\text{e.g. } \mathbb{R})$

columns

$$\begin{array}{cc} & \begin{array}{cccc} 1 & 2 & \cdots & n \end{array} \\ \text{rows} \quad A = \begin{array}{c} 1 \\ 2 \\ \vdots \\ m \end{array} & \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix} \end{array}$$

$$A \in \mathbb{F}^{m \times n}$$

columns

$A \in \mathbb{R}^{m \times n}$

rows

- *A* can also be written as $\underline{A = [a_{ij}]}$
  - The element a_{ij} is located in the i^{th} row and j^{th} column

column

a_{12}

first row

- If *m=n* the matrix is said to be ***square***
- That is, the number of rows and columns are equal
- Example, a 3x3 matrix

$$A = \begin{pmatrix} 2 & 3 & 9 \\ -9 & 6 & 1 \\ 2 & 5 & 3 \end{pmatrix} \qquad A \in \mathbb{R}^{3 \times 3}$$

Matrix Transpose

- **Transpose:** The transpose of a matrix is obtained by interchanging the rows and columns of it.
  - If matrix $A \in \mathbb{R}^{\overset{\text{rows}}{m} \times \overset{\text{columns}}{n}}$, the transpose $A^T \in \mathbb{R}^{\overset{\text{rows}}{n} \times \overset{\text{columns}}{m}}$

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix} \quad \Rightarrow \quad A^T = \begin{pmatrix} a_{11} & a_{21} & \cdots & a_{m1} \\ a_{12} & a_{22} & \cdots & a_{m2} \\ \vdots & \vdots & \ddots & \vdots \\ a_{1n} & a_{2n} & \cdots & a_{mn} \end{pmatrix}$$

- **Examples:**
  - Let $A_1 = \begin{pmatrix} 7 & 2 \\ -1 & 4 \end{pmatrix}$, $\quad A_1^T = \begin{pmatrix} 7 & -1 \\ 2 & 4 \end{pmatrix}$
  - Let $B = \begin{pmatrix} 7 \\ 2 \\ 3 \\ 4 \\ -1 \end{pmatrix}$, $\quad B^T = (7 \quad 2 \quad \cdots \quad \cdots)$

$B^T \in \mathbb{R}^{1\times 5}$

$B \in \mathbb{R}^{5\times 1}$

Matrix Algebra: Addition and Subtraction

- **Matrix addition and subtraction:** let $A = [a_{ij}]\,,\ \ B = [b_{ij}]$ be matrices of the <u>same size</u>

$$C = A \pm B \ \ \Rightarrow \ \ c_{ij} = a_{ij} \pm b_{ij}$$

- **Properties**
  1. $A + B = B + A$
  2. $(A + B) + C = A + (B + C)$
  3. $A - B = -B + A$
  4. $(A - B) - C = A - (B + C)$

- **Example:** $A = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix},\quad B = \begin{pmatrix} 8 & 9 & 10 \\ 11 & 12 & 13 \end{pmatrix}$

  - $C_1 = A + B = \begin{pmatrix} 1+8 & 2+9 & 3+10 \\ 4+11 & 5+12 & 6+13 \end{pmatrix} = \begin{pmatrix} 9 & 11 & 13 \\ 15 & 17 & 19 \end{pmatrix}$

  - $C_2 = B - A = \begin{pmatrix} 7 & & \\ & & \end{pmatrix}$

Matrix Algebra: Multiplication

- **Matrix multiplication:** let $A \in \mathbb{R}^{n\times m}$, $B \in \mathbb{R}^{m\times k}$ columns rows
  - Then $C = AB$ is a $n \times k$ matrix defined as

$$C = \underbrace{\overbrace{\begin{pmatrix} \quad & \quad & \quad \\ & & \\ & & \end{pmatrix}}^{A}}_{n\times m} \underbrace{\overbrace{\begin{pmatrix} \quad & \quad & \quad \\ & & \\ & & \end{pmatrix}}^{B}}_{m\times k} \qquad C_{n\times k} = A_{n\times m}B_{m\times k}$$

  - column, rows: $c_{ij} = \sum_{l=1}^{m} a_{il} b_{lj}$ fix column, fix row (multiplying the i^{th} row of A by the j^{th} column of B)
- In general $AB \neq BA$ (Not commutative)
- **Properties**
  1. Distributed: $A(B + C) = AB + AC$
  2. Associative: $A(BC) = (AB)C$ *
  3. $A = AI = IA$, I is the identity matrix
  4. $A^{m+n} = A^m A^n = A^n A^m$

$$I = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{pmatrix}$$

$$I \in \mathbb{R}^{2\times 2} \Rightarrow I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

Matrix Algebra: Multiplication

- Example

$$A = \begin{pmatrix} 1 & 2 & -1 \\ 2 & 1 & 10 \end{pmatrix} \qquad B = \begin{pmatrix} -1 & 4 \\ 9 & 1 \\ 1 & -1 \end{pmatrix}$$

2×3 $\qquad$ 3×2

$\overset{2\times 2}{C} = AB =$

$(2\times 3)(3\times 2)$

$$C = AB = \begin{pmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{pmatrix}$$

$C_{11} = -1 + 18 - 1 = 16$

$C_{12} = 4 + 2 + 1 = 7$

$C_{21} = -2 + 9 + 10 = 17$

$C_{22} = 8 + 1 - 10 = -1$

$$C = AB = \begin{pmatrix} 16 & 7 \\ 17 & -1 \end{pmatrix}$$

Matrix Determinant

- It is useful to characterize matrices by a single number such as the **determinant**

- Denote A_{ij} the submatrix defined by eliminating the i row and j column

$$\det(A) = \overbrace{\sum_{j=1}^{n}(-1)^{i+j}a_{ij}\det(A_{ij})}^{\text{Fix row}} = \overbrace{\sum_{j=1}^{n}(-1)^{i+j}a_{ji}\det(A_{ji})}^{\text{fix column}} \quad \forall i \quad \text{(pick any row or column!)}$$

- $\det[a_{11}] = \sum_{j=1}^{1}(-1)^{1+j}a_{1j} = (-1)^{1+1}a_{11} = a_{11}$

- $\det\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} = a_{11}a_{22} - a_{21}a =$ (pick 1st row) $i=1$
  $\sum_{j=1}^{2}(-1)^{1+j}a_{ij}\det(A_{ij}) = (-1)^{2}a_{11}a_{22} + (-1)^{1+2}a_{12}a_{21}$
  $= a_{11}a_{22} - a_{12}a_{21}$

- $\det\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} = \sum_{j=1}^{3}(-1)^{1+j}a_{ij}\det(A_{ij})$ ($i=1$)

$A_{12} = \begin{bmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{bmatrix}$

Nonsingular Matrices

- A matrix is said to be ***nonsingular*** if it produces the output 0, only for the input 0
  - if A is nonsingular $Ax = 0$ iff $x = 0$

$A \in \mathbb{R}^{n \times n}$ $x \in \mathbb{R}^{n}$ $Ax = 0$ iff $x = 0$

- If a matrix is *nonsingular* then it is ***invertible***
  - For $A \in \mathbb{R}^{n\times n}$, the inverse is defined as $A^{-1} \Rightarrow A^{-1}A = AA^{-1} = I$ (identity matrix)

- Equivalent conditions for non-singularity:
  - A is nonsingular
  - $Ax = 0$ iff $x = 0$
  - A^{-1} exists
  - $\boxed{\det(A) \neq 0}$ *

System of Linear Equations

$A^{-1}(A\underset{\text{unknowns}}{x} = b)$

if A is nonsingular - $\det(A) \neq 0$

$x = \underbrace{A^{-1}A}_{I}x = A^{-1}b$

→ ○ Let $A \in \mathbb{F}^{n\times n}$ (square matrix). If $\det(A) = 0$ then A is singular $\Rightarrow A^{-1}$ doesn't exist!

→ $Ax = b$

but $\det(A) = 0 \Rightarrow A^{-1}$ doesn't exist $\Rightarrow$?

Matrix Inversion

- Let A be an $n \times n$ matrix. The inverse of A denoted as A^{-1} also $n \times n$ satisfies:

$$\underline{A^{-1}A} = \underline{AA^{-1}} = I$$

- If A^{-1} exists, then A is nonsingular and $\underline{\det(A) \neq 0}$ adjoint Matrix

- A^{-1} can be computed as follows: $A^{-1} = \dfrac{\text{adj}\,(A)}{(\underline{\det(A)})}$ Matlab: inv(A)

$$\text{adj}\,(A) = \begin{pmatrix} \underline{C_{11}} & \underline{C_{12}} & \cdots & C_{1n} \\ C_{21} & C_{22} & \cdots & C_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ C_{n1} & C_{n2} & \cdots & C_{nn} \end{pmatrix}^T \quad \text{where } C_{ij} = (-1)^{i+j}\,\underline{\det(A_{ij})}$$

row column matrix by elim. i^{th} row and j^{th} column

- Note that $(\underline{\text{adj}\,(A)})\underline{A} = A(\underline{\text{adj}}\,(A)) = \underline{\det(A)}I$

$A^{-1} = \dfrac{\text{adj}(A)}{\det(A)}$

Matrix Inversion

$$\mathrm{adj}(A) = \begin{pmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{pmatrix}$$

$$C_{ij} = (-1)^{i+j} \det(A_{ij})$$

$$C_{11} = (-1)^2 \det(A_{11})$$

- If A is a 2×2 matrix then $A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$

$$A^{-1} = \frac{\mathrm{adj}\,(A)}{\det(A)} = \frac{1}{[a_{11}a_{22} - a_{21}a_{22}]} \overbrace{\begin{pmatrix} \boxed{a_{22}} & -a_{12} \\ -a_{21} & a_{11} \end{pmatrix}}^{\mathrm{adj}(A)}$$

$\to \det(A)$

- If A is an $n \times n$ $\boxed{\text{diagonal matrix}}$ then

$$A = \begin{pmatrix} d_{11} & 0 & \cdots & 0 \\ 0 & d_{22} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & d_{nn} \end{pmatrix} \Rightarrow \underline{A^{-1}} = \begin{pmatrix} \frac{1}{d_{11}} & 0 & \cdots & 0 \\ 0 & \frac{1}{d_{22}} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \frac{1}{d_{nn}} \end{pmatrix}$$

Eigenvalues and Eigenvectors

- Eigenvalues are very important in many applications
  - Optimization (positive definite, semi-definite matrices)
  - Linear differential equations

can be a complex number. $x \neq 0$

- Let $A \in \mathbb{R}^{n\times n}$. If a scalar $\lambda \in \mathbb{C}$ and a nonzero vector $x \in \mathbb{R}^n$ satisfy the equation:

$$Ax = \lambda x \; .$$

then λ is called an **eigenvalue** and x an **eigenvector** associated with λ

- How can we find the eigenvalue/eigenvector pair?

$$\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \lambda \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

- for an $n\times n$ matrix, you have at most n distinct eigenvalues/eigenvectors.

$x \neq 0$

$$Ax = \lambda x \rightarrow Ax - \lambda x = 0 \Rightarrow \underbrace{(A - \lambda I)}_{\text{singular}} x = 0$$

singular $\Rightarrow \det(A - \lambda I) = 0$

- To find eigvalues

$\det(A - \lambda I) = 0$

characteristic ~~polynomial~~ of A find roots

To find eigenvectors λ_1

$(A - \lambda_1 I)x = 0$ solve for x

Eigenvalues and Eigenvectors

- Find the eigenvalues and eigenvectors for
  - $A = \begin{pmatrix} 3.5 & 0.5 \\ 0.5 & 3.5 \end{pmatrix} = A^T$ (symmetric)

Sym. Matrix: · the eigenvalues are always real
· eigenvectors are orthogonal
x_1, x_2 $\quad x_1^T x_2 = 0$

$\det(A - \lambda I) = 0$

$\det\left(\begin{bmatrix} 3.5 & 0.5 \\ 0.5 & 3.5 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right) = 0 \Rightarrow \det\left(\begin{bmatrix} 3.5-\lambda & 0.5 \\ 0.5 & 3.5-\lambda \end{bmatrix}\right) = 0$

$\det(A - \lambda I) = \lambda^2 - 7\lambda + 12 = 0$
$= (\lambda - 3)(\lambda - 4) = 0 \Rightarrow \lambda_1 = 3 \quad \lambda_2 = 4$

$x \in \mathbb{R}^2 \quad y \in \mathbb{R}^2$

---

pick $\lambda_1 = 3$

$(A - \lambda_1 I)x = 0$

$\begin{pmatrix} 0.5 & 0.5 \\ 0.5 & 0.5 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

expand

$0.5x_1 + 0.5x_2 = 0 \longrightarrow \cancel{0.5}x_1 = -\cancel{0.5}x_2$
$0.5x_1 + 0.5x_2 = 0$

$x_1 = -x_2$

pick $x_1 = 1 \Rightarrow x_2 = -1$

$\boxed{\lambda_1 = 3 \quad x = \begin{pmatrix} 1 \\ -1 \end{pmatrix}}$

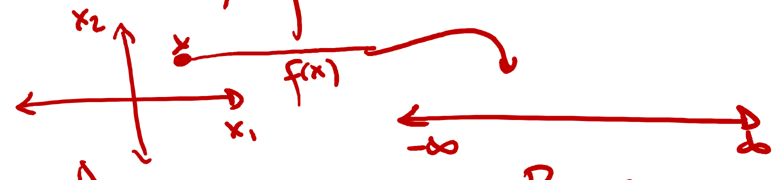
Functions of Several Variables

- One of the reasons for defining this concept of vectors spaces, is that now it makes sense to write:

domain, range

$$\underline{x \in \mathbb{R}^2}, \quad \underline{f(x)} : \underbrace{\mathbb{R}^2}_{\text{domain}} \to \underbrace{\mathbb{R}}_{\text{range}}$$

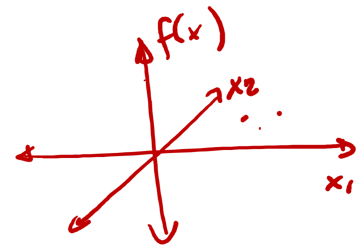
$x \in \mathbb{R}^2$, $f(x) \in \mathbb{R}$



- Think of some examples:

$$f(x) = 3x_1 + 2x_2 + 10x_1x_2$$

$$f(x) = \sqrt{\cos(x_1 x_2)}$$



- How about linear functions? Can we write them in vector form?

$x \in \mathbb{R}^3$ $\quad x = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$ $\quad f(x) : \mathbb{R}^3 \to \mathbb{R}$ $\quad f(x) = \underline{10}x_1 + \underline{5}x_2 + \underline{7}x_3 + \underline{5}$

$\underline{x \in \mathbb{R}^n}$ linear function $f(x) = \underline{c_1 x_1} + \underline{c_2 x_2} + \dots + \underline{c_n x_n} + d$

$\updownarrow$

$c \in \mathbb{R}^n$ $\quad c = \begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{pmatrix}$ $\quad f(x) = c^T x + d$

Functions of Several Variables – Quadratic Functions

- Many of the cost functions we will use are quadratic, i.e. they are polynomial functions of degree at most 2:

$f(x) : \mathbb{R}^n \to \mathbb{R}$ $\quad x \in \mathbb{R} \quad f(x) = ax^2 + bx + c$

- Example: $f(x) : \mathbb{R}^3 \to \mathbb{R}$

degree = 2

$x \in \mathbb{R}^3 \quad x = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \quad f(x) = 10x_1^2 + 5x_1x_3 + 7x_3^2 + 8x_2^2 + 9x_2x_3 + 10x_1 + 5x_3 + 8$

$\cancel{+ 8x_1x_3^2}$ (degree = 3)

- Quadratic functions can be written in the form: $\boxed{f(x) = x^T H x + c^T x + d}$

$H \in \mathbb{R}^{n \times n}$ symmetric $\quad c \in \mathbb{R}^n$

$x \in \mathbb{R}^2 \quad f(x) : \mathbb{R}^2 \to \mathbb{R}$

$f(x) = (x_1 \;\; x_2) \underbrace{\begin{pmatrix} H_{11} & H_{12} \\ H_{12} & H_{22} \end{pmatrix}}_{\text{symmetric}} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + (c_1 \;\; c_2) \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + d$

$\longrightarrow f(x) = H_{11}x_1^2 + 2H_{12}x_1x_2 + H_{22}x_2^2 + c_1x_1 + c_2x_2 + d$

Positivity of Quadratic Functions

- Consider quadratic functions with only second degree terms:

$$x \in \mathbb{R}^n, \quad f(x) = x^T H x$$

$H \in \mathbb{R}^{n \times n}$, H symmetric

- We can know what values $f(x)$ will take (range) by simply analyzing H:

    1. $f(x) \geq 0$ if and only if $H \succeq 0$ (nonnegative eigenvalues) positive semidefinite $\lambda_i \geq 0$ $f(x) \in [0, \infty)$
    2. $f(x) > 0$ if and only if $H \succ 0$ (positive eigenvalues) pos. definite $\lambda_i > 0$ $f(x) > 0$ for $x \neq 0$
    3. $f(x) \leq 0$ if and only if $H \preceq 0$ (nonpositive eigenvalues) $\lambda_i \leq 0$ $f(x) \in (-\infty, 0]$
    4. $f(x) < 0$ if and only if $H \prec 0$ (negative eigenvalues)
    5. What if H contains both positive and negative eigenvalues? $f(x) \in (-\infty, \infty)$

Positivity of Quadratic Functions – Example 1

- Consider the following quadratic function:

  * $f(x) = x_1^2 + 2x_1x_2 + 1x_2^2$

$f(x): \mathbb{R}^2 \longrightarrow \mathbb{R}$

- What values does this function take?

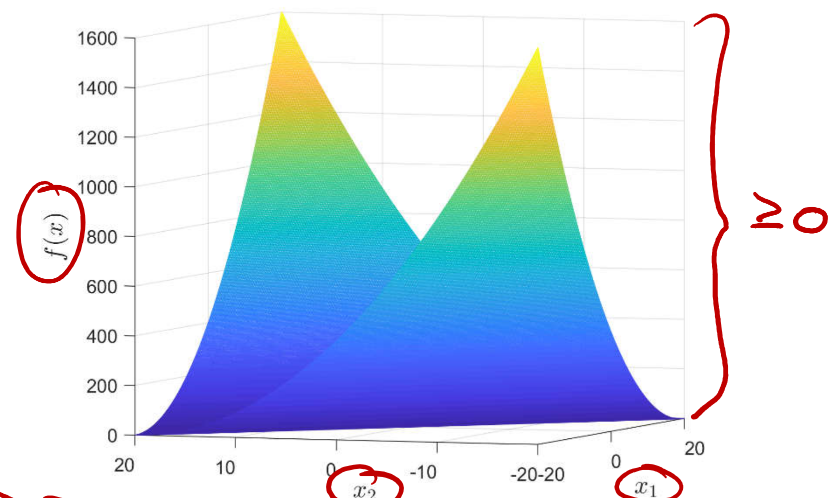
$f(x) = x^T H x$ where $x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$

$f(x) = \underbrace{(x_1 \;\; x_2) \begin{pmatrix} H_{11} & H_{12} \\ H_{12} & H_{22} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}}_{?}$

$f(x) = (x_1 \;\; x_2) \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$

$H = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$ eigenvalues $\boxed{\begin{matrix} \lambda_1 = 0 \\ \lambda_2 = 2 \end{matrix}}$

$\det(H - \lambda I) = 0$ $\lambda_{1,2} \geq 0 \Rightarrow f(x) \geq 0$



Positivity of Quadratic Functions – Example 2

- Consider the following quadratic function:

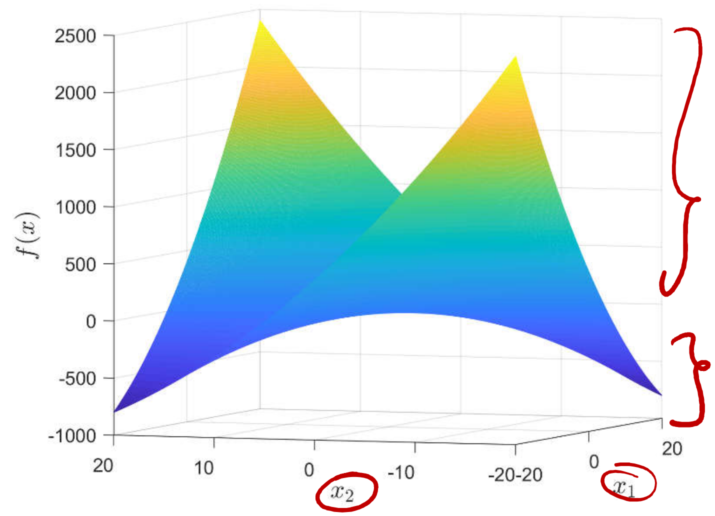
$$\boxed{f(x) = x_1^2 + 4x_1x_2 + 1x_2^2}$$

- What values does this function take? (Range)

$$f(x) = x^T \underbrace{\begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}}_{H} x \qquad x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

$$H = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix} \quad \text{eigenvalues} \quad \begin{matrix} \lambda_1 = -1 \\ \lambda_2 = 3 \end{matrix}$$

$$f(x) \in (-\infty, \infty)$$



Gradient Definition

- For a function of one variable, the derivative signifies the rate of change at a certain point

$$f(x): \mathbb{R}' \to \mathbb{R}' \qquad \frac{df(x)}{dx} \text{ rate of change of } f \text{ w.r.t } x$$

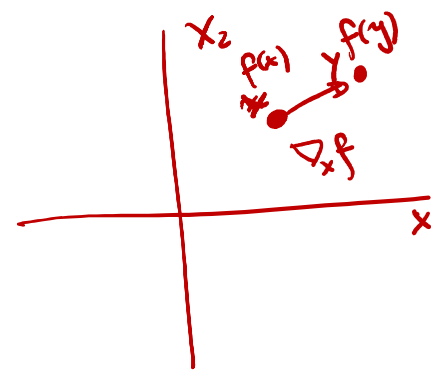
$f(x) = 3x^2 \quad \frac{df}{dx} = 6x$

- For a function of several variables, this is analogous to the **gradient** (vector)

$$f(x): \mathbb{R}^n \to \mathbb{R}$$

$x \in \mathbb{R}^n \Rightarrow x = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \quad f(x) = f(x_1, \ldots, x_n) \quad f(y) \geq f(x)$

$$\nabla f_x(x) = \begin{pmatrix} \frac{\partial f}{\partial x_1} \\ \frac{\partial f}{\partial x_2} \\ \vdots \\ \frac{\partial f}{\partial x_n} \end{pmatrix} = \begin{pmatrix} \text{r.o.c. of } f \text{ w.r.t. } x_1 \\ \text{'' '' '' '' } x_2 \\ \vdots \\ \text{'' '' '' '' } x_n \end{pmatrix}$$



- Notice that the gradient is a vector!

points where f increases the most

Gradient Example

- Consider again a quadratic function:

$$f(x) = 3x_1^2 + \underline{10x_1x_2} + 5x_1 + 10 \qquad x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \qquad f(x): \mathbb{R}^2 \longrightarrow \mathbb{R}$$

$$\nabla_x f = \begin{pmatrix} \frac{\partial f}{\partial x_1} \\ \frac{\partial f}{\partial x_2} \end{pmatrix} = \begin{pmatrix} 6x_1 + 10x_2 + 5 \\ 10x_1 \end{pmatrix}$$

Next

- Introduction to optimization with power systems applications