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Cost of Equity in Homeland Security Resource Allocation in the Face of a Strategic Attacker

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Hundreds of billions of dollars have been spent in homeland security since September 11, 2001. Many mathematical models have been developed to study strategic interactions between governments (defenders) and terrorists (attackers). However, few studies have considered the tradeoff between equity and efficiency in homeland security resource allocation. In this article, we fill this gap by developing a novel model in which a government allocates defensive resources among multiple potential targets, while reserving a portion of defensive resources (represented by the equity coefficient) for equal distribution (according to geographical areas, population, density, etc.). Such a way to model equity is one of many alternatives, but was directly inspired by homeland security resource allocation practice. The government is faced with a strategic terrorist (adaptive adversary) whose attack probabilities are endogenously determined in the model. We study the effect of the equity coefficient on the optimal defensive resource allocations and the corresponding expected loss. We find that the cost of equity (in terms of increased expected loss) increases convexly in the equity coefficient. Furthermore, such cost is lower when: (a) government uses per-valuation equity; (b) the cost-effectiveness coefficient of defense increases; and (c) the total defense budget increases. Our model, results, and insights could be used to assist policy making.

KEY WORDS: Attack-defender games; equity; game theory; homeland security; resource allocation

1. INTRODUCTION

Since September 11, 2001, homeland security in the United States has attracted hundreds of billions of dollars in expenditures. The effectiveness of such large expenditures is obscure and is frequently criticized as reflecting "pork-barrel politics", in which funds are directed toward low-risk targets for political reasons.(1) In the presence of a budget cut for homeland security,(2) it becomes even more important to optimally allocate limited defensive resources, incorporating important factors such as adaptive adversarial behavior and equity (fairness, equality).

Many mathematical models have been developed to study homeland security problems.(3) Specifically, to cope with the adaptiveness of strategic attackers, a number of intelligent risk analysis models(4) and game-theoretic models(5–8) have been used to study defensive resource allocations. Regarding equity, though "pork-barrel politics" is not equivalent to equity, it may result in a more equitable mixture of expenditures (e.g., equitable funding of different geographical areas based on political pressure to get funding for multiple districts, irrespective of need). In practice, the original formula guaranteed that each state received at least 0.75% of State Homeland Security Program (SHSP) and

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Urban Area Security Initiative (UASI), which means almost 40% of the money was allocated without any risk-based optimization; this 0.75% percentage per state was reduced by half to 0.375% in FY 2008.[9] Such homeland security resource allocation practice directly motivates this article, and the pre-2008 and post-2008 percentages correspond to equity coefficients of 0.4 (0.75% $\times$ 52 = 0.4) and 0.2 (0.375% $\times$ 52 = 0.2), respectively, as used in Section 3.1 of this article. Overall, equity is an important topic that has not been extensively studied in homeland security research literature, with only few exceptions. Specifically, Yetman[10] studies how to incorporate equity when screening passengers in an airport. Wang and Zhuang[11] compare discriminatory and nondiscriminatory screening policies facing strategic applicants who attempt to enter an organization but have private information.

Although equity has not been extensively studied in homeland security resource allocation, it has been studied in resource allocation against public risk in general,[12] and is considered as one of the three important performance measures, together with efficiency and effectiveness.[13]

In terms of application areas, equity in public resource allocations has been extensively studied in facility location, global warming, transportation, health care, education, and energy.[14] For example, Wagstaff[15] develops a method to study the tradeoff between equity and efficiency in quality-adjusted life years. Rosegrant and Binswanger[16] study how to improve markets for tradable water rights to achieve efficiency, equity, and sustainability in water resource management. In terms of characterization, equity has been categorized in the following ways: individual versus group versus society equity, vertical versus horizontal equity, and ex-ante versus ex-post equity.[17] Hausken[18] establishes a framework to characterize the relationship between ethics (a concept closely related to equity) and efficiency in organizations. Similarly, Hausken[19] studies self-interested and sympathetic behavior within a game-theoretic framework. The Gini coefficient is probably the most well-known quantitative index for equity, commonly used in measuring inequality of income and wealth.[20] Recently, taking a mathematical programming approach, Bertsimas *et al.*[21] quantify the price of fairness in resource allocation problems by studying proportional and max-min equity. For additional information about how equity is studied in the literature, see Ref. 22 for a comprehensive review of about 20 equity measures, including statistics-related equity measures, minimizing the distance between the best and worst groups, and Hoover's concentration index.

To our knowledge, no previous work has studied the tradeoff between equity and efficiency in homeland security resource allocation, especially considering an adaptive adversary. This is an important gap because considering an adaptive adversary significantly changes the way resources are allocated. Specifically, game-theoretical models generally suggest that resources should be allocated such that the expected losses (or risks) are equal among each defended target (e.g., see Proposition 3 in the article) in order for the terrorists to be indifferent between attacking those targets, which itself is one form of equity.[17,23] The article fills this gap by developing a novel model in which a government allocates defensive resources among multiple potential targets, while reserving a portion of defensive resources (represented by the equity coefficient) for equal distribution (according to geographical areas, population, density, etc.). Such a way to model equity is one of many alternatives (as explained in Section 5.2), but was directly inspired by homeland security resource allocation practice. As a follower to the defender, the attacker observes the defender's resource allocations and endogenously chooses attack probabilities.

We investigate five types of equity in this article. Type-I (per-target) equity is directly inspired by the practice of homeland security resource allocation as introduced above. Type-II (per-valuation), Type-III (per-capita), and Type-IV (per-population density) equity are considered because critical infrastructure, population, and population density are critical factors in homeland security resource allocation.[24] In particular, per capita resource allocation inequalities (e.g., \$5.03 per capita in California vs. \$37.94 per capita in Wyoming in 2004 resulting from the allocation of the general grants[1]) are broadly criticized by researchers[25] and news media.[26] Finally, Type-V (per-weighted capita, based on density-weighted population size) equity is considered because "density-weighted population" (as studied in public transit equity[27]), "is reasonably correlated with the distribution of terrorist threats across the United States."[28] Although our model was directly inspired by homeland

[1]According to Ref. 29, the general grants include those from the Office for Domestic Preparedness (ODP), the Federal Emergency Management Agency (FEMA) and the Transportation Security Administration (TSA), and offices of the Department of Health and Human Services (HHS).

security resource allocation practice at the state level, it is important to note that all the above five types of equity could be applied at the state / district / county / city / territory / tribe / town levels. We also acknowledge that the choice of such specific levels where defensive resources are allocated for equity concerns have a profound impact on how equity is viewed and on how resources are allocated. Finally, we recognize that other forms of equity could be modeled in homeland security, such as defending against different types of threats (e.g., biological terrorist attacks vs. dirty bombs), as explained in details in Section 5.2.

The rest of the article is structured as follows. Section 2 presents notation, assumptions, model formulation, and data sources. Section 3 provides the analytical solution to our equity-constrained optimization model, an algorithm, and some numerical illustrations. Section 4 conducts extensive sensitivity analyses of optimal defensive resource allocations and the corresponding expected losses with regards to three system parameters (type of equity, cost effectiveness of defense, and total budget). Section 5 concludes. The Appendix provides proofs of the propositions in the article and the optimality check for Proposition 3.

2. NOTATION, ASSUMPTIONS, MODEL, AND DATA

2.1. Notation

We use the following notation as listed in Table I throughout the article, including parameters, decision variables, sets, functions, and vectors.

2.2. Assumptions and the Model

The strategic interactions between a government and an attacker are usually modeled as a sequential game.[30] Following this approach, we let the defender move first by distributing a total budget of C among n targets, such that $\sum_{i=1}^{n} c_i = C$. The attacker then observes the defense distribution $c \equiv (c_1, c_2, \ldots, c_n)$, and attacks target i with conditional probability $h_i(c)$, for $i = 1, 2, \cdots, n$, such that $\sum_{i=1}^{n} h_i(c) = 1$, contingent upon the exogenously determined total probability of attack, r. We assume that the attacker chooses a target corresponding to the maximal expected loss $p_i(c_i)v_i$ in his best response function. Best response function refers to strategies that lead to the most preferable outcome for the player, as a function of other players' strategies (see Chapter 9,[31]). We assume that when the maximal expected loss caused by the attacker is the same for two or more targets, those targets are attacked with equal probabilities,

$$\hat{h}_i(c) = \begin{cases} \frac{1}{||S||} & \text{if } i \in S \equiv \{i : h_i(c) > 0\} \\ & = \left\{i : p_i(c_i)v_i = \max\limits_{j=1,\cdots,n} \{p_j(c_j)v_j\}\right\} \\ 0 & \text{otherwise,} \end{cases} \quad (1)$$

where $||S||$ is the cardinality of set S. Note that this assumption is not limiting the results; that is, although this article assumes that the attacker will attack all targets with maximal expected loss, Proposition 1 below implies that all the results for the defender still hold if the attacker chooses any subset of set S to attack.

Proposition 1. *If the attacker chooses any subset $Q \subseteq S$ to attack, all the results for the defender's optimal objective function value and associated decisions remain the same regardless of the value of subset Q.*

We acknowledge that there are other factors affecting terrorists' target choice, such as access to the target, degree of difficulty/success, and funding. However, they are relatively minor; for example, although terrorists in Los Angeles (LA) are more likely to target LA than New York City (NYC) or Washington, DC (DC), the transportation costs from LA to NYC or DC are relatively minor compared to terrorists' funding level (e.g., Osama Bin Laden controlled about \$300 million worth of fortune;[32]) and terrorists' operation costs (e.g., the 9/11/2001 attacks are estimated to cost as much as a half million dollars[33]). The model also takes into account degree of difficulty/success in attacking a target by introducing the success probability function of an attack $p_i(c_i)$, which linearly impacts the expected property loss of target i, $L_i \equiv r h_i(c) p_i(c_i) v_i$. Moveover, the attacker's funding level could be correlated with total probability of attack r.

The objective of the government is to minimize the expected loss $L(c, \hat{h}(c), e)$ by allocating defensive resources C employing two parallel schemes such that a portion $(100 \times e\%)$ of the total defensive resources is reserved for equal allocations, and the rest $(100 \times (1 - e)\%)$ is used for risk-based minimization considering the attacker's best response $\hat{h}_i(c)$.

Table I. Main Notation in this Article

Parameters	
C	Total budget of the defensive resources
$r \in [0, 1]$	Total probability of attack for the attacker
$\lambda \geq 0$	Cost-effectiveness coefficient of defensive investment
n	Number of targets in the system
i	Index for target i, for $i = 1, 2, \ldots, n$
$e \in [0, 1]$	Equity coefficient indicating the reserved portion of the total defense budget
v_i	Valuation of target i
s_i	Population size of target i
d_i	Population density of target i
w_i	Density-weighted population size of target i
Decision Variables	
c	$\equiv (c_1, c_2, \cdots, c_n)$. Vector denoting defensive resource allocations
$\tilde{c}_i \leq c_i$	Government's reserved defensive resource allocation to target i
c'_i	$\equiv c_i - \tilde{c}_i$. Government's nonreserved defensive resource allocation to target i
$\tilde{c}$	$\equiv (\tilde{c}_1, \tilde{c}_2, \cdots, \tilde{c}_n)$. Vector denoting reserved defensive resources
$h_i(c)$	Endogenously determined conditional probability that the attacker will attack target i given an attack, as a function of c. We have $h_i(c) \geq 0$, and $\sum_{i=1}^{n} h_i(c) = 1$
Sets	
D	$\equiv \{i : c'_i > 0, i = 1, 2, \ldots, n\}$. Set of targets with positive nonreserved defenses
S	$\equiv \{i : h_i(c) > 0, i = 1, 2, \ldots, n\}$. Set of targets that is attacked with positive probabilities
$Q \subseteq S$	Any subset of S
Functions and Vectors	
$p_i(c_i)$	Success probability of an attack on target i, as a function of the defensive resource allocation to target i, c_i. We assume that $p_i(c_i)$ is continuous, convex, and decreasing in c_i (i.e., $\frac{\partial p_i(c_i)}{\partial c_i} \leq 0$, $\frac{\partial^2 p_i(c_i)}{\partial c_i^2} \geq 0$)
I_i	$= 1$ if $h_i(c) > 0$; 0 if $h_i(c) = 0$. Indicator function for the event $\{h_i(c) > 0\}$.
$\hat{h}_i(c)$	Attacker's best response function for target i as a function of c
$h(c)$	$\equiv (h_1(c), h_2(c), \ldots, h_n(c))$. Vector denoting attacker's probabilities of attacking
$\hat{h}(c)$	$\equiv (\hat{h}_1(c), \hat{h}_2(c), \ldots, \hat{h}_n(c))$. Vector denoting attacker's best response
$L_i(c_i, h_i(c), e)$	$\equiv r h_i(c) p_i(c_i) v_i$. Expected loss for target i to the government
$L(c, h(c), e)$	$\equiv \sum_{i=1}^{n} L_i(c_i, h_i(c), e) = \sum_{i=1}^{n} r h_i(c) p_i(c_i) v_i$. Total expected loss to the government

That is,

$$\begin{aligned} \min_c \quad & L(c, \hat{h}(c), e) = \sum_{i=1}^{n} r \hat{h}_i(c) p_i(c_i) v_i \\ \text{subject to:} \quad & \sum_{i=1}^{n} c_i = C, c_i \geq \tilde{c}_i, \end{aligned} \tag{2}$$

where $\tilde{c}_i, i = 1, \ldots, n$, is defined to be one of the following five types of equity in Equations (3)–(7), respectively.

$$\text{Type-I (per-target): } \tilde{c}_i = eC\frac{1}{n} \tag{3}$$

$$\text{Type-II (per-valuation): } \tilde{c}_i = eC\frac{v_i}{\sum_{i=1}^{n} v_i} \tag{4}$$

$$\text{Type-III (per-capita): } \tilde{c}_i = eC\frac{s_i}{\sum_{i=1}^{n} s_i} \tag{5}$$

$$\text{Type-IV (per-density): } \tilde{c}_i = eC\frac{d_i}{\sum_{i=1}^{n} d_i} \tag{6}$$

$$\text{Type-V (per-weighted-capita): } \tilde{c}_i = eC\frac{w_i}{\sum_{i=1}^{n} w_i} \tag{7}$$

Note that the three factors $r\hat{h}_i(c)$, $p_i(c_i)$, and v_i in Equation (2) correspond to threat, vulnerability, and consequences, respectively. A similar formula of these three factors is adopted by the U.S. Department of Homeland Security in a standard risk analysis for terrorist attacks.[34] Note that all five types of equity equalize how resources are allocated up front and thus the risk that is actually experienced may not be equal. However, note that when $e < 1$ (less-than-full equity) as to be shown in Proposition 3 in Section 3.1, expected property loss for any target belonging to the set of defended targets (with nonreserved portion of defensive resources) is equal at equilibrium.

Inserting the attacker's best response function $\hat{h}(c)$ defined in Equation (1) to Equation (2), we can rewrite the defender's objective function as,

$$L(c, \hat{h}(c), e) = r \max_{i=1,\cdots,n} \{p_i(c_i)v_i\}. \tag{8}$$

Definition 1. *We call a pair of strategies,* (h^*, c^*), *a Subgame Perfect Nash Equilibrium (or an equilibrium) for the sequential game, if and only if*

$$h^* = \hat{h}(c^*) \tag{9}$$

and

$$c^* = \underset{c}{argmin}\ L(c, \hat{h}(c), e) \tag{10}$$

In other words, $h^* = \hat{h}(c^*)$ is calculated by Equation (1), and c^* is the solution to the optimization problem in Equation (2).

2.3. Data Sources

Although this article is inspired by homeland security resource allocation practice where equity is achieved at the state level, the model is general and could be applied to any level (e.g., state, district, county, city, territory, tribe, and town). Numerical illustration in this article uses the data set at the urban-area level. In particular, Willis *et al.*[28] provide a useful data set of 47 urban areas for homeland security target valuations to illustrate defender-attacker games, where the expected property losses for the 47 U.S. urban areas are the target valuations (v_is, sorted in descending order in column 3 in Table II).[2] The data are only used to illustrate the model, and our model could use other data as inputs to generate new results. Note also that the losses in Table II are potential damages if an attack succeeds with some probability. The attack may not be successful and thus the attack is not always capable of causing the losses in Table II. For the equity calculations, we use populations (s_i's), population densities (d_i's), and density-weighted populations (w_i's) for the 47 urban areas[28] as shown in columns 4–6 in Table II. Column 7 in Table II shows the defense budget allocated to those 47 urban areas from the Office of Grants and Training in FY2004. Because the data on expected property losses in Ref. 28 are from the year of 2004, we use the total FY2004 UASI Grant Allocations ($675M) as the total available defense budget C in our baseline model.

3. SOLUTION

3.1. Analytic Results and Illustrations

We solve the defender-attacker game formulated in Section 2.2 and provide analytical solution in this section. In particular, Proposition 2 discusses the existence and uniqueness of the equilibrium as defined in Definition 1 of Section 2.2, and Proposition 3 presents the equilibrium condition.

Proposition 2. *A Subgame Perfect Nash Equilibrium exists and is unique for the sequential game defined by Equations* (9) *and* (10).

Proposition 3. *First, the objective function in Equation* (2) *is convex in c if the strategic attacker uses his best response function* $\hat{h}(c)$ *defined in Equation* (1). *Second, given* $e < 1$, *consider a pair of strategies,* (h^*, c^*), *and the associated variables,* L_i^*, *and* D^*; *if* $h^* = \hat{h}(c^*)$ *and* L_i^* *equals a positive constant denoted as* W^* *for all* $i \in D^*$; *i.e.,*

$$L_i^* \equiv r h_i^* p_i(c_i^*) v_i = W^*, \quad \forall i \in D^*, \tag{11}$$

such a strategy pair, (h^*, c^*), *qualifies to be a Subgame Perfect Nash Equilibrium defined by Definition 1 in Section* 2.2. *Third, we have* $L_i^* \leq W^*$ *for all* $i \notin D^*$.

Remark. Proposition 3 implies that with nonreserved defensive resources the defender desires to equalize the expected losses for all the defended targets. Moreover, such losses are larger than those for

[2] Expected property losses (e.g., $413 Million for New York City) were calculated in the following way as explained in Ref. 28: "We used the Risk Management Solutions (RMS) Terrorism Risk Model [event-based model, see Ref. 35] to calculate expected annual consequences of terrorist attacks . . . Losses were expressed in terms of . . . total property damage in dollars (buildings, building contents, and business interruption)."

Table II. Expected Property Losses, Populations, Population Densities, Density-Weighted Populations, and FY2004 UASI Budget Allocations for 47 Urban Areas in the United States

#	Urban Area	Expected Property Loss ($Million v_i)†	Population (s_i)†	Density (per Square Mile d_i)†	Weighted Population (w_i)†	FY2004 UASI Allocation ($)‡
1	New York City	413.0	9,314,235	8,159	75,991,762,554	47,007,064
2	Chicago	115.0	8,272,768	1,634	13,519,096,414	34,142,222
3	San Francisco	57.0	1,731,183	1,705	2,951,064,038	26,481,275
4	Washington, D.C.	36.0	4,923,153	756	3,723,526,125	29,301,502
5	Los Angeles	34.0	9,519,338	2,344	22,314,867,674	40,404,595
6	Philadelphia, PA-NJ	21.0	5,100,931	1,323	6,749,136,215	23,078,759
7	Boston, MA-NH	18.0	3,406,829	1,685	5,740,709,241	19,131,723
8	Houston	11.0	4,177,646	706	2,948,039,040	19,955,485
9	Newark	7.3	2,032,989	1,289	2,619,713,383	15,054,101
10	Seattle-Bellevue	6.7	2,414,616	546	1,318,032,823	16,516,007
11	Jersey City	4.4	608,975	13,044	7,943,237,618	17,112,311
12	Detroit	4.2	4,441,551	1,140	5,062,484,593	13,754,597
13	Las Vegas	4.1	1,563,282	40	62,076,079	10,531,025
14	Oakland	4.0	2,392,557	1,642	3,927,449,645	7,854,691
15	Orange County	3.7	2,846,289	3,606	10,262,626,470	25,404,219
16	Cleveland	3.0	2,250,871	832	1,871,707,337	10,460,465
17	San Diego	2.8	2,813,833	670	1,885,205,299	10,479,947
18	Miami	2.7	2,253,362	1,158	2,609,185,020	20,108,247
19	Minneapolis-St. Paul	2.7	2,968,806	490	1,453,687,745	19,146,642
20	Denver	2.5	2,109,282	561	1,183,064,989	8,646,361
21	Baltimore	2.4	2,552,994	979	2,498,144,264	15,918,745
22	Atlanta	2.3	4,112,198	672	2,761,386,037	10,744,248
23	Dallas	2.1	3,519,176	569	2,002,093,120	12,198,661
24	St. Louis	2.1	2,603,607	407	1,060,496,877	10,785,053
25	Portland	2.0	1,918,009	381	731,703,925	8,161,143
26	Phoenix	1.9	3,251,876	223	725,649,640	12,200,204
27	San Jose	1.7	1,682,585	1,304	2,193,476,169	9,982,442
28	Charlotte	1.1	1,499,293	444	665,682,378	7,404,955
29	Kansas City	1.1	1,776,062	329	583,476,273	13,295,646
30	Milwaukee	1.1	1,500,741	1,028	1,542,728,464	10,177,999
31	New Haven	1.1	542,149	1,261	683,670,545	9,632,961
32	Buffalo	1.0	1,170,111	747	873,657,856	10,095,856
33	Pittsburgh	1.0	2,358,695	510	1,202,742,683	11,978,479
34	Cincinnati	0.9	1,646,395	493	811,141,960	12,751,270
35	Tampa	0.9	2,395,997	938	2,247,784,596	9,275,359
36	New Orleans	0.8	1,337,726	394	526,405,217	7,152,827
37	Columbus	0.7	1,540,157	490	755,141,752	8,707,544
38	Indianapolis	0.7	1,607,486	456	733,470,541	10,151,880
39	Sacramento	0.7	1,628,197	399	649,623,296	8,024,926
40	Louisville	0.6	1,025,598	495	507,651,616	8,987,662
41	Orlando	0.6	1,644,561	471	774,794,778	8,765,211
42	Memphis	0.5	1,135,614	378	428,953,952	10,067,477
43	Albany	0.4	875,583	272	237,926,588	6,853,481
44	Richmond	0.4	996,512	338	337,254,906	6,543,378
45	San Antonio	0.4	1,592,383	479	762,291,362	6,301,153
46	Baton Rouge	0.2	602,894	380	229,154,762	7,193,806
47	Fresno	0.2	922,516	114	105,084,482	7,076,396
	Total	788.7	122,581,611	58,281	200,768,260,341	**675,000,000**

Sources: †Ref. 28. ‡Ref. 36.

all targets without using any nonreserved defensive resources.

For the rest of the article, following Ref. 30, we consider an exponential form of success probability of an attack,

$$p_i(c_i) = \exp(-\lambda c_i),\ \forall i = 1, 2, \ldots, n, \qquad (12)$$

where λ is the cost-effectiveness of defense. Differentiating Equation (12) with respect to c_i, we have $\frac{dp_i(c_i)}{dc_i} = -\lambda p_i(c_i)$, which means that one extra unit of defensive resources c_i will reduce the probability of a successful attack $p_i(c_i)$ by $100\lambda\%$. We use the data set introduced in Section 2.3 to illustrate Proposition 3. In particular, Table III provides three illustrations when $e = 0, 0.2$, and 0.4, respectively. (As explained in Section 1, the way that some homeland security grants, including SHSP and UASI, allocate funds corresponds to $e = 0.4$ in our model before 2008 and $e = 0.2$ after 2008, respectively.) For illustrating purposes, we let $\lambda = 0.01$ in Table III. We consider more general parameter values of λ in Section 4.2.

For Illustration 1, we have $e = 0$, $D^* = S^* = \{1, 2, 3, 4, 5, 6\}$. Therefore, we have the expected loss for target i: $L_i^* = r(\frac{I_i^*}{||S^*||})p_i(c_i^*)v_i = W^* = 3.47$, for $i = 1, 2, \cdots, 6$, and $L_7^* = L_8^* = \cdots = L_{47}^* = 0.00 < W^*$=3.47, which is consistent with Proposition 3. For Illustration 2, we have $e = 0.2$, $D^* = S^* = \{1, 2, 3\}$. Therefore, we have the expected loss for target i: $L_i^* = r(\frac{I_i^*}{||S^*||})p_i(c_i^*)v_i = W^* = 26.46$, for $i = 1, 2, 3$, and $L_4^* = L_5^* = \cdots = L_{47}^* = 0.00 < W^*$=5.29, which is consistent with Proposition 3. For Illustration 3, we have $e = 0.4$. $D^* = S^* = \{1, 2\}$. Therefore, we have the expected loss for target i: $L_i^* = r(\frac{I_i^*}{||S^*||})p_i(c_i^*)v_i$ =11.37, for $i = 0.4$. Appendix A.4 shows the optimality check of the above three illustrations.

Using the results from Proposition 3, we further study the effects of equity coefficient e and the total probability of attack r on equilibrium solution and payoff. In particular, Proposition 4 below implies that the total expected loss increases in equity coefficient; Proposition 5 discusses three effects of the total probability of attack.

Proposition 4. *$L^*(c^*, \hat{h}(c^*), e)$ weakly increases in e; i.e.,* $\frac{\partial L^*(c^*,\hat{h}(c^*),e)}{\partial e} \geq 0$.

Proposition 5. *First, the total probability of attack r does not affect the equilibrium solution c^* to the defender's optimization problem* (2); *second, r linearly increases the optimal expected loss L^*; third, we have $L^* = 0$ if $r = 0$.*

3.2. Algorithm

In this subsection, we first provide an algorithm based on Proposition 3 to search for the equilibrium defensive resource allocations, and then provide a proposition of convergence. Inserting Equation (12) into Equation (11), we have,

$$\begin{aligned}
& r h_i^* \exp(-\lambda c_i^*) v_i = W^*,\ \forall\, i \in D^* \\
\iff & r h_i^* \exp(-\lambda c_i'^*) v_i' = W^*,\ \forall\, i \in D^* \\
& \text{where } v_i' = v_i \exp(-\lambda \tilde{c}_i) \\
\iff & c_i'^* = \\
& \frac{\ln r + \ln h_i^* + \ln v_i' - \ln W^*}{\lambda}, \quad \forall\, i \in D^*. \qquad (13)
\end{aligned}$$

Using the definition of C as shown in Table I, we have,

$$\begin{aligned}
C &= \sum_{i=1}^{n} c_i^* = \sum_{i=1}^{n} (c_i'^* + \tilde{c}_i) \\
&= \sum_{i=1}^{n} c_i^* + eC \\
\iff (1-e)C &= \sum_{i\in D^*} c_i^* + \sum_{i\notin D^*} c_i^* = \sum_{i\in D^*} c_i^* + 0 \\
&= \sum_{i\in D^*} \frac{\ln r + \ln h_i^* + \ln v_i' - \ln W^*}{\lambda} \\
\iff \lambda(1-e)C &= ||D^*|| \ln r + \sum_{i\in D^*} \ln h_i^* \\
&\quad + \sum_{i\in D^*} \ln v_i' - \sum_{i\in D^*} \ln W^* \\
\iff ||D^*|| \ln W^* &= ||D^*|| \ln r + \sum_{i\in D^*} \ln h_i^* \\
&\quad + \sum_{i\in D^*} \ln v_i' - \lambda(1-e)C \\
\iff W^* &=
\end{aligned}$$

$$\exp\left(\frac{||D^*|| \ln r + \sum_{i\in D^*} \ln h_i^* + \sum_{i\in D^*} \ln v_i' - \lambda(1-e)C}{||D^*||}\right)' \qquad (14)$$

where $||D^*||$ is the cardinality of set D^*.

Based on Equation (14), we develop an algorithm, for which Fig. 1 shows an illustrative diagram, Table IV presents a detailed description of the steps

Table III. Three Illustrations for Proposition 3

		Illustration 1: $e = 0$				Illustration 2: $e = 0.2$				Illustration 3: $e = 0.4$			
#	v_i	c_i^*	$\tilde{c}_i$	$p_i v_i$	L_i^*	c_i^*	$\tilde{c}_i$	$p_i v_i$	L_i^*	c_i^*	$\tilde{c}_i$	$p_i v_i$	L_i^*
1	413.00	298.75	0.00	20.82	3.47	274.79	2.87	26.46	5.29	249.37	5.74	34.11	11.37
2	115.00	170.90	0.00	20.82	3.47	146.93	2.87	26.46	5.29	121.52	5.74	34.11	11.37
3	57.00	100.71	0.00	20.82	3.47	76.75	2.87	26.46	5.29	51.34	5.74	34.11	11.37
4	36.00	54.75	0.00	20.82	3.47	30.80	2.87	26.46	5.29	5.74	5.74	33.99	0.00
5	34.00	49.04	0.00	20.82	3.47	25.08	2.87	26.46	5.29	5.74	5.74	32.10	0.00
6	21.00	0.86	0.00	20.82	3.47	2.87	2.87	20.40	0.00	5.74	5.74	19.83	0.00
7	18.00	0.00	0.00	18.00	0.00	2.87	2.87	17.49	0.00	5.74	5.74	17.00	0.00
8	11.00	0.00	0.00	11.00	0.00	2.87	2.87	10.69	0.00	5.74	5.74	10.39	0.00
⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮
47	0.20	0.00	0.00	0.20	0.00	2.87	2.87	7.09	0.00	5.74	5.74	0.19	0.00
Total	782.00	675.00	0.00	237.63	20.82	675.00	135.00	255.69	26.46	675.00	270.00	288.34	34.11
D^*		{1, 2, 3, 4, 5, 6}				{1, 2, 3, 4, 5}				{1, 2, 3}			
S^*		{1, 2, 3, 4, 5, 6}				{1, 2, 3, 4, 5}				{1, 2, 3}			
W^*		3.47				5.29				11.37			

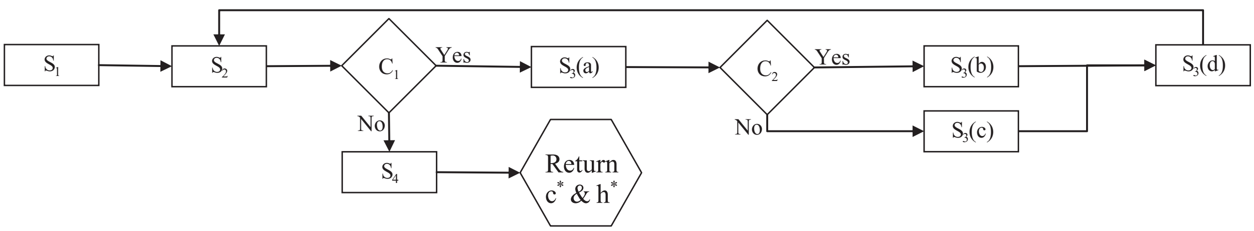

Fig. 1. Illustrative diagram for the algorithm when equity is considered.

Table IV. Description of Steps and Conditions in the Algorithm Shown in Fig. 1

S_1	Let $D = \{1, 2, \ldots, n\}$, $S = \{1, 2 \ldots, n\}$ and $I = \{1, 1, \ldots, 1\}$; replace v_i with $v'_i = v_i \exp(-\lambda \tilde{c}_i)$.
S_2	Solve for W using Equation (14).
S_3	(a) Identify such index j('s) satisfying C_1 with smallest v'_j. Let $c'_j = 0$. Delete j('s) from set D.
	(b) Let $c'_j = 0$ and delete j('s) from set D.
	(c) Update $c'_j = \frac{1}{\lambda}\left[\ln\left(r \frac{I_j}{\lVert S \rVert}\right) + \ln v'_j - \ln W\right]$, $\forall j \in D$.
	(d) Update sets S and I.
S_4	Calculate $c_i^* = c_i'^* + \tilde{c}_i, \forall i \in D$.
C_1	There exists some index $j \in D$ such that $v'_j h_j \leq W$.
C_2	$I_j = 0, \forall j \in D$.

and conditions, and Proposition 6 provides results on convergence and computational complexity.

Proposition 6. *The algorithm provided by Fig.* 1 *and Table* IV *in Section* 3.2 *always converges to an equilibrium defined by Definition* 1 *in Section* 2.2. *The algorithm requires* $O(n^2)$ *computations, where n is the number of targets in the system.*

4. SENSITIVITY ANALYSES

For the sensitivity analyses, we adopt the following baseline parameter values: Type-I (per-target) equity, $\lambda = 0.01$ (as used in Table III), and C=\$675M (UASI total budget for FY 2004 as shown in Table II). We study the sensitivity analysis for each of the three parameters in Sections 4.1, 4.2, and 4.3, respectively.

4.1. Sensitivity Analysis of Five Types of Equity

Using the data set introduced in Section 2.3 and the algorithm provided in Section 3.2, we solve for the optimal defensive resource allocations at equilibrium with different values of e. We study five types of equity as defined by Equations (3)–(7) in Section 2.2: (I) per-target equity; (II) per-valuation equity (where target valuations equal expected property losses); (III) per-capita equity; (IV) per-density equity; and (V) per-weighted-capita equity.

As shown in Fig. 2(I_1), where the defender employs Type-I (per-target) equity: if $e = 0$ (i.e., no consideration of equity), the defender allocates resources to the top six valuable targets. As e increases (i.e., more equal distribution is implemented), more targets will be defended, and eventually all 47 targets are equally defended when $e = 1$. If $e = 0$, the expected property loss is \$20.82M and increases convexly in e and eventually becomes \$357.75M as shown in Fig. 2(I_2). Similar patterns in resource allocations and the corresponding expected property loss are observed in Figs. 2(II_1-II_2, III_1-III_2, IV_1-IV_2, V_1-V_2), where Type-II to Type-V equity types are employed, respectively.

Comparing across five equity types shown in Figs. 2(I)–(V), we find that: (1) employing different types of equity results in different optimal defensive resource allocations, and the cost of equity increases convexly in equity coefficient e for all five types; (2) Type-II (per-valuation) equity yields the lowest expected property loss for any given equity coefficient; and (3) Type-I (per-target) equity results in the highest expected property loss.

4.2. Sensitivity Analysis of Cost-Effectiveness of Defense

In this subsection, we conduct sensitivity analysis with regard to the cost effectiveness of defense λ. As the cost effectiveness of defense increases from 0.001 in Fig. 3(a) to 0.01 in Fig. 3(b) and to 0.05 in Fig. 3(c), more valuable urban areas are defended (e.g., when $e = 0$ from 1 to 6 and to 25, respectively) and the expected property loss decreases (e.g., when $e = 0$ from \$210.28M to \$20.82M and to \$1.92M, respectively; and when $e = 1$ from \$401.31M to \$309.89M and to \$98.23M, respectively). This comparison implies that higher defense effectiveness leads to lower cost of equity, and more targets to be defended given any level of equity.

4.3. Sensitivity Analysis of Total Budget

In this subsection, we conduct sensitivity analysis with regard to total defense budget. As defense budget C increases from \$100M in Fig. 4(a) to \$675M in Fig. 4(b) and to \$3000M in Fig. 4(c), more top valuable urban areas are defended (e.g., when $e = 0$ from 1 to 6 and to 45, respectively) and the corresponding expected property loss decreases (e.g., when $e = 0$ from \$151.93M to \$20.82M and to \$0.30M, respectively; and when $e = 1$ from \$395.79M to \$309.88M and to \$5.86M, respectively). This comparison implies that higher defense budget leads to lower cost of equity, and more targets to be defended given any level of equity.

5. CONCLUSION AND DISCUSSION

5.1. Conclusion and Policy Implications

Equity constitutes a major practical and political concern in allocation of public resources, including defensive resources. To the best of our knowledge, no prior study has investigated equity issues in homeland security resource allocations when facing a strategic attacker. In this article, we develop a novel model where a certain portion (represented by an equity coefficient) of the total defense budget is reserved for equity distribution. We investigate the manner that optimal defensive resource allocations change as a function of such equity coefficient. We find that the cost of equity (increased expected loss) increases convexly in the equity coefficient for all five possible equity types, and the per-valuation and per-target equity results in the lowest and highest expected losses, respectively.

Our results show that high defense cost-effectiveness or large defense budget may compensate for a small portion of resources reserved for equity allocations. That is, if the defender has a higher cost-effective defensive system or a larger budget, the defender can afford higher level of equity by reserving more resources for equity allocations. On the other hand, the defender needs to be cautious in reserving defensive resources for equity allocation when the budget is low or the defensive investments are not effective.

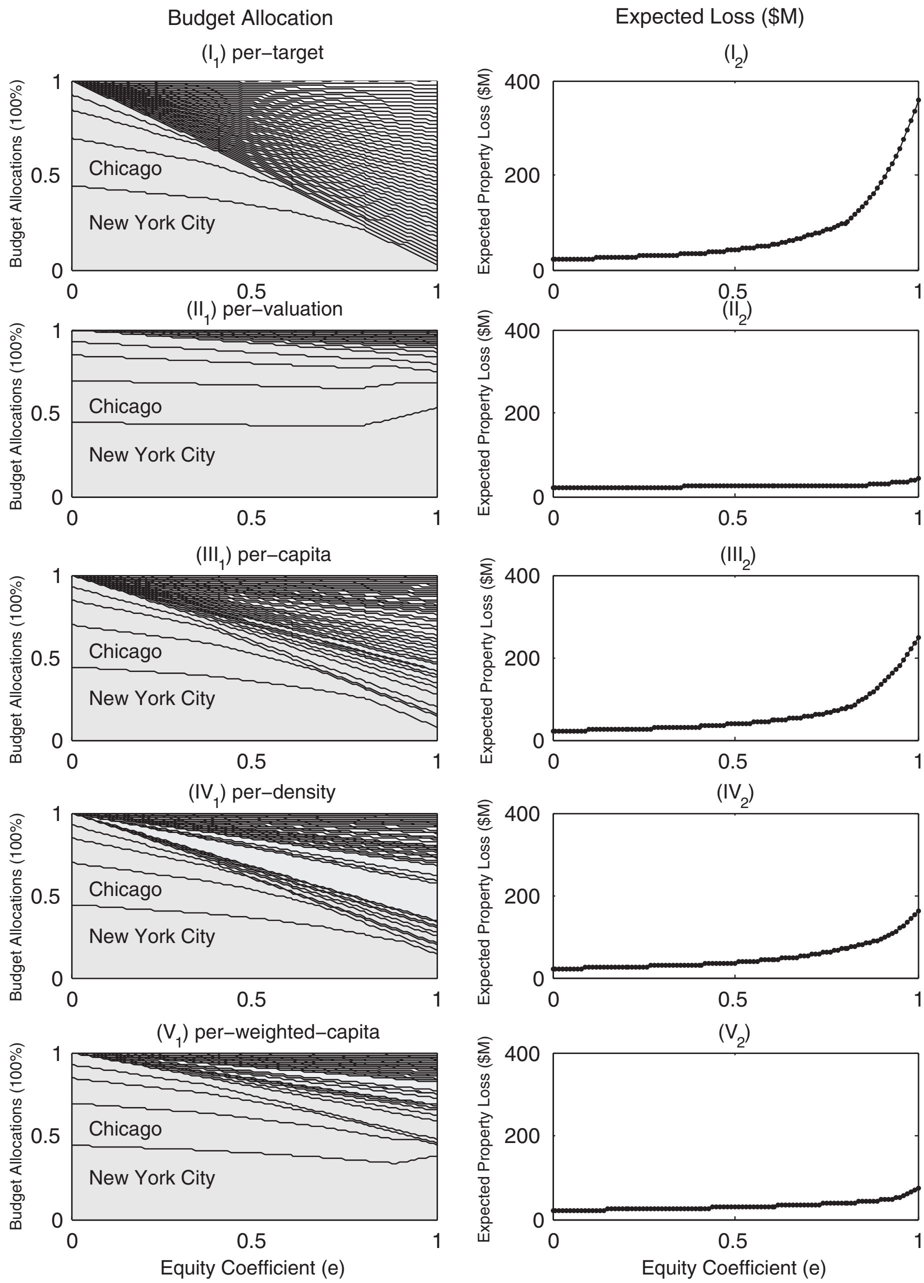

Fig. 2. Optimal defensive resource allocations (c) and the consequent expected property loss ($L(c, \hat{h}(c), e)$) as a function of equity coefficient (e) increasing from 0 to 1 with five types of equity: Type-I (per-target), Type-II (per-valuation), Type-III (per-capita), Type-IV (per-density) and Type-V (per-weighted-capita).

There are potential policy implications. In particular, this article provides a tool where equity coefficient is a tunable parameter, and illustrates a complete Pareto frontier of the equity-efficiency tradeoff in the face of a strategic attacker. The decisionmaker could choose an appropriate (equity, efficiency) pair, based on her own individual preferences. Because our results show that the cost of equity increases convexly in equity coefficient, equity consideration in the middle of the coefficient range would likely lead to a reasonable balance between cost and equity in practice. The exact "optimal level" of equity would depend on factors such as budget, cost-effectiveness of defense, and tolerance of expected loss.

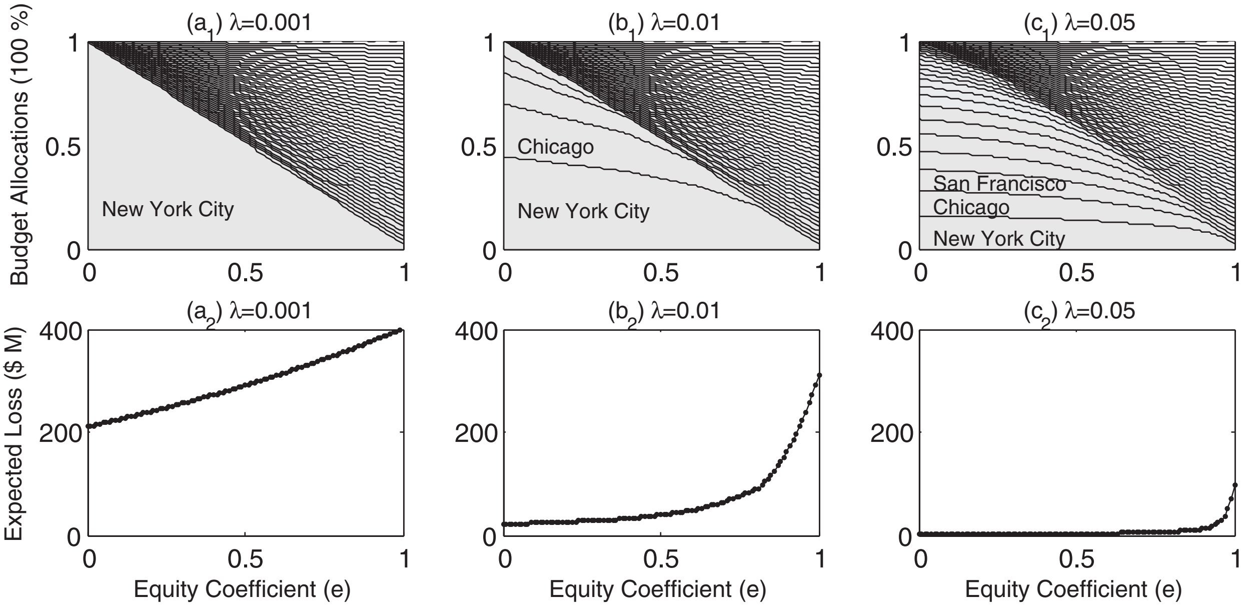

Fig. 3. Optimal defensive resource allocations (c) and the consequent expected property loss ($L(c, \hat{h}(c), e)$) as a function of equity coefficient (e) increasing from 0 to 1 with λ=0.001, 0.01, and 0.05, respectively, when the defender employs Type-I (per-target) equity and C =$675M.

5.2. Future Research Directions

Although the article studies equity in spending and reserving resources defending over targets (e.g., urban areas), there are various alternative methods to model equity in defensive resource allocations. For example, equity could be modeled in balancing defenses against biological attacks versus dirty bomb attacks, or between terrorism and nonterrorism prevention activities. For example, the 9/11 Commission Act of 2007 requires that at least 25% of total available grant (including SHSG, UASI, and Citizens Corps Program) is used for terrorism prevention activities,[37] though such reservation level (currently 25%) could be dynamically optimized considering factors such as (estimated) cost-effectiveness of defense, defense budget, and the "costs" of such reservation.

There are several other interesting future research directions. In particular, most game-theoretic models developed for analyzing resource allocations in homeland security only concern single-period games with increasing interests in multi-period games.[38–41] As a first step toward tackling the equity issue in defensive resource allocations, we focus on a one-period game. However, when the centralized defender allocates resources over time, the history of resource allocations should be taken into account to achieve equity over time (e.g., more allocation to targets A over B in previous periods would lead to more allocation to targets B over A in the next period under a potential consideration of over-time equity). Moreover, in terms of policy making, the implications of switching between equity-based policy and nonequity-based policy would be interesting to pursue to understand effects of equity in cumulative defensive resource allocations. Therefore, more research on equity issues in multi-period games is of interest, where each period need not be a year although in practice budgets are managed annually.

For simplicity, this article assumed complete information, though in practice, the defender still decides upon defense investments with incomplete information of effectiveness. For example, it may take time for the defense measures to be effective and the defender may even never know how effective the security investments are. Further research could investigate the defender's optimal allocation decisions with equity consideration in a game of incomplete information, including the information about cost-effectiveness of defense.

Although the defender and the attacker could value targets similarly, a relaxation of the assumption of common target valuation would be an interesting direction to explore. In particular, an equity model could be studied with multi-attribute and

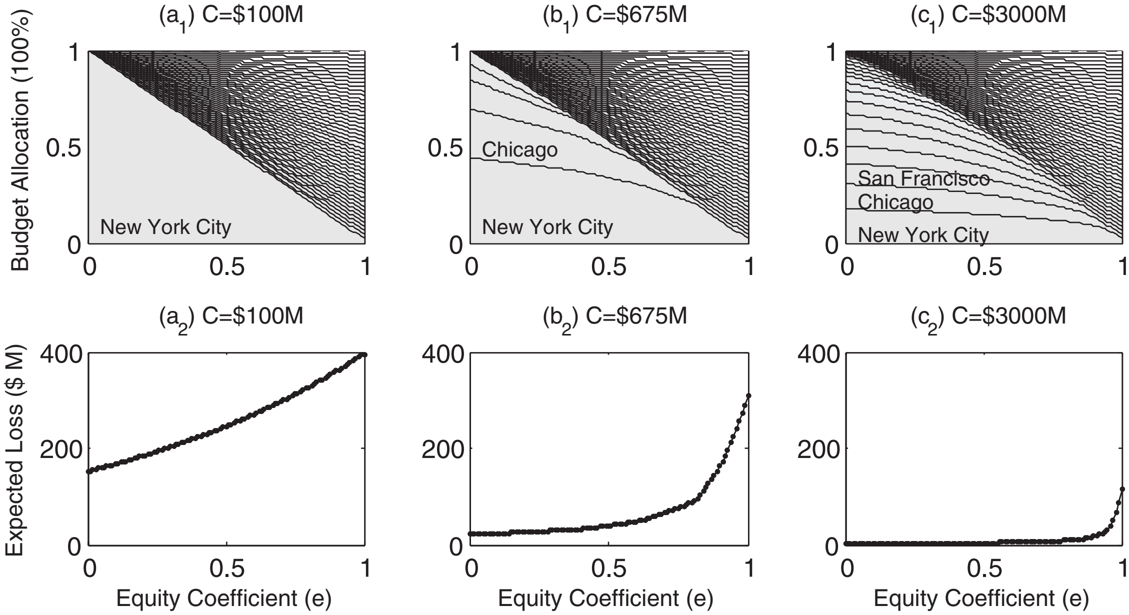

Fig. 4. Optimal defensive budget allocations (c) and the consequent expected property loss ($L(c, \hat{h}(c), e)$) as a function of equity coefficient (e) increasing from 0 to 1 when C=\$100M, \$675M, and \$3000M, respectively, when the defender employs Type-I (per-target) equity and $\lambda = 0.01$.

multi-objective utility functions for both attacker[42,43] and defender.[44] Other forms of success probability function of attack could also be studied.[45] Moreover, a more complicated game with equity constraints considering the strategy of deception and secrecy,[39,46] could be explored, especially when the defender could have private information such as target valuations and cost-effectiveness of defense. Finally, the case where defensive investments decrease both the consequences and the probability of a successful attack could be explored.

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APPENDIX

A.1. PROOF FOR PROPOSITION 1

If the attacker chooses any subset Q of S to attack, the best response function for the attacker becomes the following:

$$\hat{h}_i(c) = \begin{cases} \dfrac{1}{||Q||} & \text{if } i \in Q \subseteq S \equiv \{i : h_i(c) > 0\} \\ & = \{i : p_i(c_i)v_i = \max\limits_{j=1,\cdots,n} \{p_j(c_j)v_j\}\} \\ 0 & \text{otherwise,} \end{cases} \tag{15}$$

where $||Q||$ is the cardinality of set Q. Inserting Equation (15) into objective function Equation (2), we have:

$$\begin{aligned} \min_c L(c, \hat{h}(c), e) &= r \sum_{i=1}^{n} \hat{h}_i(c) p_i(c_i) v_i \\ &= r \sum_{i \in Q} \hat{h}_i(c) p_i(c_i) v_i \\ &= r \sum_{i \in Q} \frac{1}{||Q||} \max_{i=1,\ldots,n} \{p_i(c_i) v_i\} \\ &= r \max_{i=1,\cdots,n} \{p_i(c_i) v_i\}. \end{aligned} \tag{16}$$

Note that Equation (16) is identical to Equation (8), which is equivalent to objective function Equation (2). Therefore, the defender's optimization problem remains the same by allowing the attacker to choose any subset Q of S to attack, and all the results for the defender's utility and allocation remain the same regardless of subset Q. If the attacker chooses any subset $Q \subseteq S$ to attack, all the results for the defender's optimal objective function value and associated decisions remain the same regardless of the value of subset Q.

A.2. PROOF FOR PROPOSITION 2

To prove Proposition 2 in the article, we first provide and prove Lemma 1.

Lemma 1. *$L(c, \hat{h}(c), e)$ given in Equation (8) is continuous and convex in c.*

Proof for Lemma 1. As we assume that $p_i(c_i)$ is continuous and convex in c_i for all i, $\max_{i=1,\cdots,n}\{p_i(c_i)v_i\}$ is continuous and convex in c, because the scaled continuous and convex function and the max of continuous and convex functions are also continuous and convex. Because the linear combination of the continuous and convex functions $p_i(c_i)v_i$ is continuous and convex, $L(c, \hat{h}(c), e)$ is continuous and convex.

Remarks. Note that we assume that the defender and the attacker have the same target valuations v_is. If that does not hold, the objective function as defined in Equation (8) is not continuous. One can construct an example where the objective function for the defender is discontinuous. For example, targets 1 and 2 are of value 300 and 200 to the defender and value 299.99 and 300 to the attacker. When the defender increases the defense to target 2 by such a small amount, the attacker will attack target 1 instead. The expected payoff for the attacker changes continuously whereas the expected payoff for the defender changes discontinuously. By assuming the same target valuations, the game is zero-sum and both the defender's and the attacker's payoffs change continuously in c.

The existence of the equilibrium follows from Lemma 1 and the fact that the set of feasible defender strategies is compact and convex, and the attacker's best response function is assumed, using the existence theorem for a pure-strategy Nash equilibrium.[47] Note that Theorem 1 in Dasgupta-Maskin (1986) deals with a game where the players maximize their utilities, while a minimization game for the defender is coped with in this article. The equivalence between quasi-concavity of the objective function for maximization problem and quasi-convexity of the objective function for minimization problem complete the proof (from Lemma 1, the objective function as defined in Equation (2) is convex).

The uniqueness of the equilibrium follows from Lemma 1 and the fact that the set of feasible defender strategies is compact and convex because a continuous and convex function obtains a unique minimal point on a compact and convex set.

A.3. PROOF FOR PROPOSITION 3

First, given that we assume that $p_i(c_i)$ is convex in c_i for all i, $\max_{i=1,\cdots,n}\{p_i(c_i)v_i\}$ must also be convex in c because the scaled convex function is convex and the max of convex functions is also convex.

Second, because the defender's objective function in Equation (2) is convex in c, any local minimum must also be global minimum. We now show that c^* is the equilibrium (global) solution by showing that any local changes from c^* will not decrease the value of the objective function. In particular, if $L_i^* \equiv A_i^* p_i(c_i^*)v_i = W^* > 0,\ \forall\, i \in D^*$, where W^* is a constant, we have that (h^*, c^*) is the equilibrium defined by Definition 1 in Section 2.2. Because we have $h^* = \hat{h}(c^*)$, we only need to show Equation (10) is satisfied; i.e., $c^* = \underset{c}{argmin}\ L(c, \hat{h}(c), e)$. Note that $c_i^* = c_i'^* + \tilde{c}_i,\ \forall i$, and $\tilde{c}_i$ can be treated as a constant for any given e and type of equity employed. Therefore, identifying c^* is equivalent to identifying c'^*.

If $e = 1$, the solution c^* would most likely not be the equilibrium solution because c^* is obtained without considering the objective function Equation (2). Therefore, $e < 1$ is a required condition for Proposition 3 to hold. For any given $e < 1$, note that set $D^* = S^*$.

For any targets i and j, suppose $c_i^* > \tilde{c}_i$, and $c_j^* > \tilde{c}_j$ in any particular solution c^*, and we have $L_i^* \equiv h_i^* p_i(c_i^*)v_i = (\frac{rI_i^*}{||S^*||})p_i(c_i^*)v_i,\ \ \forall i \in D^*$, $L_j^* \equiv h_j^* p_j(c_j^*)v_j,\ \ \forall\, j \in D^*, j \neq i$. We want to show that if a positive constant $W^* = L_i^* = L_j^*,\ \ \forall\, i \in D^*$, $\forall\, j \in D^*, j \neq i$, Equation (10) is satisfied. There are two subcases:

(1) If we increase c_i^* by a small $\epsilon > 0$, $i \in D^*$, and decrease c_j^* by ϵ, $j \in D^*, j \neq i$. Then h_i^* becomes 0 and h_j^* becomes 1 and S^* becomes $\{j\}$ and the change of the total expected loss is: $\Delta L^*(c^*, \hat{h}(c^*), e) = p_j$

$(c_j^* - \epsilon)v_j - p_j(c_j^*)v_j = W^*(\frac{p_j(c_j^*-\epsilon)}{p_j(c_j^*)} - 1) > 0$, using Equation (11) and the assumption $\frac{\partial p_i(c_i)}{\partial c_i} < 0$.

(2) If we decrease c_i^* by a small $\epsilon > 0$, for $i \in S^*$, and increase c_j^* by ϵ, $j \in S^*, j \neq i$. Then h_i^* becomes 1 and h_j^* becomes 0 and S^* becomes $\{i\}$ and the change of the total expected loss is: $\Delta L^*(c^*, \hat{h}(c^*), e) = p_i(c_i^* - \epsilon)v_i - p_i(c_i^*)v_i = W^*(\frac{p_i(c_i^*-\epsilon)}{p_i(c_i^*)} - 1) > 0$, using Equation (11) and the assumption $\frac{\partial p_i(c_i)}{\partial c_i} < 0$.

In summary, all possible deviations from the solution c^* increase the expected loss, and thus we have $c^* = \underset{c}{argmin}\ L(c, \hat{h}(c), e)$. From Equation (1), we have $h^* = \hat{h}(c^*)$. Therefore, according to Definition 1, both Equations (9) and (10) are satisfied and thus (h^*, c^*) is the equilibrium defined by Definition 1 in Section 2.2.

Second, we show the second part of Proposition 3; that is, we have if an equilibrium (h^*, c^*) defined by Definition 1 in Section 2.2 is reached, $L_i^* \equiv h_i^* p_i(c_i^*)v_i \leq W^*\ \forall i \notin D^*$.

For any targets i and j, suppose $c_i^* = \tilde{c}_i, c_j^* > \tilde{c}_j$, for one particular solution c^*, and we have $L_i^* \equiv h_i^* p_i(c_i^*)v_i = (\frac{rI_i^*}{||S^*||})p_i(c_i^*)v_i,\ \forall i \notin D^*$ and $L_j^* \equiv h_j^* p_j(c_j^*)v_j = (\frac{rI_j^*}{||S^*||})p_j(c_j^*)v_j, \forall\ j \in D^*$. If $L_i^* \leq L_j^*$, c_i^* cannot be decreased by reallocating defensive resources from targets i to j, thus $W^* \equiv L_j^* \geq L_i^*$. In contrast, if $L_i^* > L_j^* = W^*$, h_i becomes 1 while h_j becomes 0, which is a contradiction to the assumption that $i \notin D^*$. Therefore, $L_i^* > L_j^* = W^*\ \forall i \notin D^*, j \in D^*$ is not possible in equilibrium and thus $L_i^* \leq L_j^* = W^*\ \forall i \notin D^*, j \in D^*$.

A.4. OPTIMALITY CHECK OF THREE ILLUSTRATIONS FOR PROPOSITION 3

For all three illustrations, we consider two scenarios of reallocation, and observe that both reallocations increase the expected loss.

Illustration 1 ($e = 0$):

(i) Suppose that the defender reallocates one unit of defensive resources from targets 1 to 2 (both in set D^*). This will decrease c_1^* from 298.75 to 297.75, and increase c_2^* from 170.90 to 171.90. Thus, $p_1(c_1^*)v_1$ increases from 20.82 to 21.03, while $p_2(c_2^*)v_2$ decreases from 20.82 to 20.61. Therefore, target 1 becomes the only target attracting the attacker and we have higher total expected loss ($L^* = 21.03$, increased from 20.82), which means that this reallocation is not optimal.

(ii) Suppose that the defender reallocates one unit of defensive resources from targets 1 (in set D^*) to 7 (outside set D^*). This will decrease c_1^* from 298.75 to 297.75, and increase c_7^* from 0.00 to 1.00. Thus, $p_1(c_1^*)v_1$ increases from 20.82 to 21.03, while $p_7(c_7^*)v_7$ decreases from 18.00 to 17.82 without suffering from any attacks because after the increase in c_7^*, target 7 becomes even more unattractive to the attacker. Therefore, target 1 becomes the only target attracting the attacker and we have higher total expected loss ($L^* = 21.03$, increased from 20.82), which means that this reallocation is not optimal.

Illustration 2 ($e = 0.2$):

(i) Suppose the defender reallocates one unit of defensive resources from targets 1 to 2 (both in set D^*). This will decrease c_1^* from 274.79 to 273.79, and increase c_2^* from 146.93 to 147.93. Thus, $p_1(c_1^*)v_1$ increases from 26.46 to 26.72, while $p_2(c_2^*)v_2$ decreases from 26.46 to 26.20. Therefore, target 1 becomes the only target attracting the attacker and we have higher total expected loss ($L^* = 26.72$, increased from 26.46), which means that this reallocation is not optimal.

(ii) Suppose the defender reallocates one unit of defensive resources from targets 1 (in set D^*) to 6 (outside D^*). This will decrease c_1^* from 274.79 to 273.79, and increase c_6^* from 2.87 to 3.87. Thus, $p_1(c_1^*)v_1$ increases from 26.46 to 26.72, while $p_6(c_6^*)v_6$ decreases from 20.41 to 20.20 without suffering from any attacks because after the increase in c_4^*, target 6 becomes even less attractive to the attacker. Therefore, target 1 becomes the only target attracting the attacker and we have higher total expected loss ($L^* = 26.72$, increased from 26.46), which means that this reallocation is not optimal.

Illustration 3 ($e = 0.4$):

(i) Suppose the defender reallocates one unit of defensive resources from targets 1 to 2 (both in set D^*). This will decrease c_1^* from 249.37 to 248.37, and increase c_2^* from 121.52 to 122.52.

Thus, $p_1(c_1^*)v_1$ increases from 34.11 to 34.46, while $p_2(c_2^*)v_2$ decreases from 34.11 to 33.77. Therefore, target 1 becomes the only target attracting the attacker and we have higher total expected loss ($L^* = 34.46$, increased from 34.11), which means that this reallocation is not optimal.

(ii) Suppose the defender reallocates one unit of defensive resources from targets 1 (in set D^*) to 4 (outside D^*). This will decrease c_1^* from 249.37 to 248.37, and increase c_4^* from 5.74 to 6.74. Thus, $p_1(c_1^*)v_1$ increases from 34.11 to 34.46, while $p_3(c_3^*)v_3$ decreases from 33.99 to 33.65 without suffering from any attacks because after the increase in c_4^*, target 4 becomes even less attractive to the attacker. Therefore, target 1 becomes the only target attracting the attacker and we have higher total expected loss ($L^* = 34.46$, increased from 34.11), which means that this reallocation is not optimal.

A.5. PROOF FOR PROPOSITION 4

In the proof, $L(c, \hat{h}(c), e)$ could be simplified to $L(e)$ as we assume that other system parameters are fixed. Recall that the following five types of equity are considered in the article.

$$\text{Type-I (per-target): } \tilde{c}_i = eC\frac{1}{n}$$
$$\text{Type-II (per-valuation): } \tilde{c}_i = eC\frac{v_i}{\sum_{i=1}^n v_i}$$
$$\text{Type-III (per-capita): } \tilde{c}_i = eC\frac{s_i}{\sum_{i=1}^n s_i}$$
$$\text{Type-IV (per-density): } \tilde{c}_i = eC\frac{d_i}{\sum_{i=1}^n d_i}$$
$$\text{Type-V (per-weighted-capita): } \tilde{c}_i = eC\frac{w_i}{\sum_{i=1}^n w_i}$$

For any given $L^*(e)$, the feasible set $\mathscr{C}(e) = \{(c_1, c_2, \cdots, c_n) : \sum_{i=1}^n c_i = C, c_i \geq \tilde{c}_i \forall i\}$. If $e_1 > e_2 \geq 0$, we must have $\mathscr{C}(e_1) \subseteq \mathscr{C}(e_2)$ because $\tilde{c}_1 > \tilde{c}_2$ for all five types of equity. Because the feasible region for e_1 is smaller than that for e_2, we must have $L^*(e_1) \geq L^*(e_2)$ for the minimization problem. Therefore, we proved that $L^*(e)$ weakly increases in e. That is, $\frac{\partial L^*(e)}{\partial e} \geq 0$.

A.6. PROOF FOR PROPOSITION 5

First, note the total probability of attack r does not affect the feasible region. Given the alternative formulation in Equation (8), r does not affect $\max_{i=1,\ldots,n} p_i(c_i)v_i$. Minimizing over $r \max_{i=1,\ldots,n} p_i(c_i)v_i$ is equivalent to $\max_{i=1,\ldots,n} p_i(c_i)v_i$ and thus r does not affect the optimal solution c^*. Second, we show that the optimal objective function value increases linearly with r. We can treat $\max_{i=1,\ldots,n} p_i(c_i^*)v_i$ as a constant with regard to r. Therefore, the optimal objective function value increases linearly with r. Third, directly from Equation (8), we have that $L^* = 0$ if $r = 0$.

A.7. PROOF FOR PROPOSITION 6

In this proof, we first show that the algorithm provided in Section 3.2 will always converge. Then we show that the algorithm will converge to the equilibrium solution c^* defined by Definition 1 in Section 2.2.

To show convergence of the entire algorithm, we note that the algorithm contains one loop as shown in Fig. 1 in Section 3.2, which is formed by S_2, S_3, C_1, and C_2. Step S_3 deletes index j satisfying $v_j A_j \leq W$ (in Condition C_1) or $I_j = 0$ (in Condition C_2) from set D (i.e., set of targets to be defended). Now we claim that the algorithm always converges because (a) only deletions and no additions are allowed for modification of set D, (b) the defender will defend at least one target, and (c) there are finite number of potential targets in set D ($\leq n$). Therefore, the algorithm always converges.

Second, we show that this converging algorithm always converges to the equilibrium solution c^* defined by Definition 1 in Section 2.2. To reach the equilibrium solution c^*, the algorithm needs to find set D^* (Step S_3 (a–b)) and W^* (Step S_2). The algorithm proceeds as follows: in Step S_1, the algorithm initializes sets D, S, and I to include all n targets and replace v_i with $v_i' \exp(-\lambda\tilde{c}_i)$. Then Step S_2 calculates a suboptimal W with $h_i = \frac{rI_i}{||S||}$ according to Equation (14). Note that W is always smaller than W^*. This is because (a) in Equation (14), the denominator becomes smaller after each iteration (i.e., the size of set D decreases as js are deleted from set D), (b) in Equation (13), the numerator becomes larger as a result of $\sum_{i \in D} \ln h_i = \sum_{i \in D} \ln(\frac{rI_i}{||S||})$ increasing at a faster rate than the decreasing rate of $\sum_{i \in D} \ln v_i'$. After each iteration, $\sum_{i \in D} \ln v_i'$ only decreases slightly because only indices js with smallest v_j' are deleted, while set S shrinks quickly and thus $h_i = (\frac{rI_i}{||S||})$ increases significantly.

The loop will be repeated and D, W, and c' will be updated until Condition C_1 is no longer satisfied. Then D^*, W^*, and c'^* are obtained. As Condition C_1 is not satisfied, all $c_i'^*$'s for $i \in D^*$ are positive as seen from Equation (13) (i.e., $c_i'^* = \frac{\ln h_i^* + \ln v_i' - \ln W^*}{\lambda}$, $\forall\ i \in D^*$) and also $L_i^* = h_i^* p_i(c_i'^*)x_i' = W^*$, $\forall i \in D^*$. Step S_4 calculates $c^* = c^* + \tilde{c}_i$. According to Proposition 3, as $L_i^* = h_i^* p_i(c_i^*)v_i = h_i^* p_i(c_i'^*)v_i' = W^*$, (h^*, c^*) is a possible equilibrium defined by Definition 1 in Section 2.2. Therefore, the algorithm arrives at c^*, which is the equilibrium solution to the optimization problem (2).

As shown in Fig. 1, the algorithm employs 1 loop (Steps S_2 and S_3 and Condition C_1 and C_2). Within Step S_3, Condition C_2 needs be checked for up to $n-1$ indices in set D, and each of S_3(a), S_3(b or c, parallel), and S_3(d) requires up to n computations. Thus, Step 3 requires a total of at most $(n-1) + 3n = 4n - 1$ computations. On the other hand, this loop requires the sum of at most $n-1$ iterations (checking Condition C_1) of looping Step 3, and n additional computations in S_4 when C_1 is not satisfied. Therefore, at most $(4n-1)(n-1) + n = 4n^2 - 4n + 1$ computations will be needed. Therefore, the algorithm requires $O(n^2)$ computations to find the optimal solution c^*.

REFERENCES

1. de Rugy V. Homeland-security pork. Washington Times, 2005 [cited 2012 Sep 20]. Available at: http://www.aei.org/article/22924. Accessed October 10, 2012.
2. Zuckerman J. The 2013 homeland security budget: Misplaced priorities. In Backgrounder: The Heritage Foundation, March 14, 2012 [cited 2012 Sep 20]. Available at: http://thf_media.s3.amazonaws.com/2012/pdf/bg2664.pdf. Accessed October 10, 2012.
3. Atkinson MP, Cao Z., Wein LM. Optimal stopping analysis of a radiation detection system to protect cities from a nuclear terrorist attack. Risk Analysis, 2008; 28(2):353–371.
4. Parnell GS, Smith CM., Moxley FI. Intelligent adversary risk analysis: A bioterrorism risk management model. Risk Analysis, 2010; 30(1):32–48.
5. Hausken K. Probabilistic risk analysis and game theory. Risk Analysis, 2002; 22(1):17–27.
6. Zhuang J, Bier VM. Balancing terrorism and natural disasters—Defensive strategy with endogenous attacker effort. Operations Research, 2007; 55(5):976–991.
7. Golalikhani M, Zhuang J. Modeling arbitrary layers of continuous-level defenses in facing with strategic attackers. Risk Analysis, 2010; 31(4):533–547.
8. Nikoofal ME, Zhuang J. Robust allocation of a defensive budget considering an attacker's private information. Risk Analysis, 2012; 32(5):930–943.
9. U.S. Government Accountability Office. DHS risk-based grant methodology is reasonable, but current version's measure of vulnerability is limited. 2008 [cited 2012 Sep 20]. Available at: http://www.gao.gov/new.items/d08852.pdf. Accessed October 10, 2012.
10. Yetman J. Suicidal terrorism and discriminatory screening: An efficiency-equity trade off. Defence and Peace Economics, 2004; 15(3):221–230.
11. Wang X, Zhuang J. Balancing congestion and security in the presence of strategic applicants with private information. European Journal of Operational Research, 2011; 212(1):100–111.
12. Keeney RL. Equity and public risk. Operations Research, 1980; 28(3):527–534.
13. Savas ES. On equity in providing public services. Management Science, 1978; 24(3):800–808.
14. Leclerc PD, McLay LA, Mayorga ME. Community-based operations research: Decision modeling for local impact and diverse populations. Pp. 97–118 in Johnson MP (ed). Modeling Equity for Allocation in Public Resources. New York: Springer, 2011.
15. Wagstaff A. QALYs and the equity-efficiency trade off. Journal of Health Economics, 1991: 10(1):21–41.
16. Rosegrant MW, Binswanger HP. Markets in tradable water rights: Potential for efficiency gains in developing country water resource allocation. World Development, 1994; 22(11):1613–1625.
17. Fishburn PC. Equity axioms for public risks. Operations Research, 1984; 32(5):901–908.
18. Hausken K. Ethics and efficiency in organizations. International Journal of Social Economics, 1996; 23(9):15–40.
19. Hausken K. Self-interest and sympathy in economic behaviour. International Journal of Social Economics, 1996; 23(7):4–24.
20. Young P. Equity: In Theory and Practice. Princeton, NJ: Princeton University Press, 1994.
21. Bertsimas D, Farias VF., Trichakis N. The price of fairness. Operations Research, 2011; 59(1):17–31.
22. Marsh MT, Schilling DA. Equity measurement in facility location analysis: A review and framework. European Journal of Operational Research, 1994; 74(1):1–17.
23. Fishburn PC, Straffin PD. Equity considerations in public risks evaluation. Operations Research, 1989; 37(2):229–239.
24. Masse T, O'Neil S, Rollins J. The Department of Homeland Security's risk assessment methodology: Evolution, issues, and options for Congress. 2007 [cited 2012 Sep 20]. Available at: http://fpc.state.gov/documents/organization/80208.pdf. Accessed October 10, 2012.
25. Carafano JJ. Homeland security dollars and sense #1: Current spending formulas waste aid to states. 2004. Available at: http://www.heritage.org/research/reports/2004/05/homeland-security-dollars-and-sense-1-current-spending-formulas-waste-aid-to-states. Accessed on October 10, 2012.
26. Lucas F. Homeland security funding "pork" under fire. CNN News, 2008 [cited 2012 Sep 20]. Available at: http://cnsnews.com/news/article/homeland-security-funding-pork-under-fire. Accessed October 10, 2012.
27. Garrett M, Taylor B. Reconsidering social equity in public-transit. Berkeley Planning Journal, 1999; 13(1):6–27.
28. Willis HH, Morral AR, Kelly TK, Medby JJ. Estimatingterrorism risk. 2005 [cited 2012 Sep 20]. Available from: http://www.rand.org/pubs/monographs/2005/RAND MG388.pdf. Accessed October 10, 2012.
29. Ransdell T. Federal formula grants and California: Homeland security. California. 2004 [cited 2012 Sep 20]. Available at: https://www.ppic.org/content/pubs/ffg/FF 104TRFF.pdf. Accessed October 10, 2012.

30. Bier VM, Haphuriwat N., Menoyo J, Zimmerman R, Culpen AM. Optimal resource allocation for defense of targets based on differing measures of attractiveness. Risk Analysis, 2008; 28(3):763–770.
31. Mas-Colell A, Whinston MD, Green JR. Microeconomic Theory. New York: Oxford University Press, 1995.
32. Ackman D. The cost of being Osama bin Laden. 2001 [cited 2012 Sep 20]. Available at: http://www.forbes.com/2001/09/14/0914ladenmoney. Accessed October 10, 2012.
33. del Cid Gómez JM, Miguel J. A financial profile of the terrorism of Al Qaeda and its affiliates. Perspectives on Terror, 2010; 4(4):3–27.
34. Cox LA. Game theory and risk analysis. Risk Analysis, 2009; 29(8):1062–1068.
35. Risk Management Solutions. Managing terrorism risk. Newark, CA: Risk Management Solutions, 2003 [cited 2012 Sep 20]. Available at: http://www.rms.com/publications/terrorism_risk_modeling.pdf. Accessed October 10, 2012.
36. U.S. Department of Homeland Security. Fiscal Year 2004 Urban Areas Security Initiative Grant Program. 2004 [cited 2012 Sep 20]. Available at: http://web.archive.org/web/*/http://www.ojp.usdoj.gov/odp/docs/fy04uasi.pdf. Accessed October 10, 2012.
37. Public Law 110-53. Public law 110-53 — Implementing recommendations of the 9/11 Commission Act of 2007. [cited 2012 Sep 20]. Available at: http://www.gpo.gov/fdsys/pkg/PLAW-110publ53/content-detail.html. Accessed October 10, 2012.
38. Wang C, Bier VM. Impact of intelligence on target-hardening decisions based on multi-attribute terrorist utility. HST'09. IEEE Conference on Technologies for Homeland Security, 2010; 373–380.
39. Zhuang J, Bier VM., Alagoz O. Modeling secrecy and deception in a multiple-period attacker-defender signaling game. European Journal of Operational Research, 2010; 203(2):409–418.
40. Hausken K, Zhuang J. Defending against a terrorist who accumulates resources. Military Operations Research, 2011; 16(1):21–39.
41. Hausken K, Zhuang J. Governments' and terrorists' defense and attack in a T-period game. Decision Analysis, 2011; 8(1):46–70.
42. Keeney RL. Modeling values for anti-terrorism analysis. Risk Analysis, 2007; 27(3):585–596.
43. Wang C, Bier VM. Target-hardening decisions based on uncertain multiattribute terrorist utility. Decision Analysis, 2011; 8(4):286–302.
44. Keeney RL, von Winterfeldt D. A value model forevaluating homeland security decisions. Risk Analysis, 2011; 31(9):1470–1487.
45. Skaperdas S. Contest success functions. Economic Theory, 1996; 7(2):283–290.
46. Zhuang J, Bier VM. Reasons for secrecy and deception in homeland-security resource allocation. Risk Analysis, 2010; 30(12):1737–1743.
47. Dasgupta P, Maskin E. The existence of equilibrium in discontinuous economic games, I: Theory. Review of Economic Theory, 1986; 53(1):1–26.