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Xiaojun Shan and Jun Zhuang
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Appendix

Appendix A: Proofs

In order to prove theorems in the paper, we first provide and prove Lemma 1.

Lemma 1 *$L(c, \hat{h}(c), h', q)$ in Equation (11) is continuous and strictly convex in c.*

Proof for Lemma 1: Since we assume that $p_i(c_i)$ is continuous and strictly convex in c_i for all i, $\max_{i=1,\cdots,n} \{p_i(c_i)x_i\}$ is continuous and strictly convex in c, since the scaled continuous and strictly convex function and the max of continuous and strictly convex functions are also continuous and strictly convex. Similarly, $(1-q)\sum_{i=1}^{n} h'_i p_i(c_i)x_i$ is continuous and strictly convex in c, since the linear combination of the continuous and strictly convex functions $p_i(c_i)x_i$'s is continuous and strictly convex. Finally, the summation of the two continuous and strictly convex functions is continuous and strictly convex.

Remarks: Note that we assume that the defender and the attacker have the same target valuations x_i's. If that does not hold, the objective function as defined in Equation (11) is not continuous. One can construct an example where the objective function for the defender is discontinuous. For example, targets 1 and 2 value 200 and 100 to the defender and value 199.99 and 200 to the attacker. When the defender increases the defense to target 2 by such a small amount that the attacker will attack target 1 instead. The expected payoff for the attacker changes continuously while the expected payoff for the defender changes discontinuously. By assuming the same target valuations, the game is zero-sum and both the defender's and the attacker's payoffs change continuously in c. If the target valuations differ for the defender and attacker, then the expected loss of the defender becomes $p_k(c_k)x_k$ where $k \in \operatorname{argmax}_i p_i(c_i)y_i$ with y_i's being the attacker's values, which needs not be continuous.

A.1 Proof for Theorem 1

The existence of the equilibrium follows from Lemma 1 and the fact that the set of feasible defender strategies is compact and strictly convex, and the strategic attacker's best response function is assumed, using the existence theorem for a subgame perfect Nash equilibrium (see Kuhn, 1953; Selten, 1965, 1975). Note that in those games the players maximize their utilities, while this paper deals with a game where the defender minimizes expected losses. Results from a maximization game assuming quasiconcave objective function would be equivalently applied to a minimization game assuming quasiconvex objective function.

The uniqueness of the equilibrium follows from Lemma 1 and the fact that the set of feasible defender strategies is compact and strictly convex, since a continuous and strictly convex function obtains a unique minimal point on a compact and convex set. Note that the uniqueness is proven by adapting an optimization approach.

A.2 Proof for Theorem 2

First, since the government's objective function in Equation (11) is strictly convex in government's defensive resource allocation c, any local minimum must also be global minimum. We now show that c^* is the equilibrium (global) solution by showing that any local changes from c^* will not decrease the value of the objective function. In particular, if $R_i^* = A_i^* p_i(c_i^*) x_i = W^* > 0,\ \forall\ i \in J^*$ (or equivalently $c_i^* > 0$ according to Equation (15)), where W^* is a constant, R_i could be total expected loss (if $q \in (0, 1]$, and $J \neq P$, q is the probability that the terrorist is strategic, J is set of defended targets and P is set of targets attracting the non-strategic attacker), expected loss from an attack by a strategic terrorist (if $q \in (0, 1]$, and $J = P$), or reduction in expected loss from an attack by a non-strategic terrorist (if $q = 0$) for target i, $p_i(c_i^*)$ is success probability of an attack, and x_i is valuation of target i, we have that (h^*, c^*) is the equilibrium defined by Definition 1 in Section 2.2 (h or $h(c)$ is the vector representing endogenously-determined probabilities that a strategic terrorist will attack any targets). Second, we have $R_i^* \leq W^*$ for all $i \notin J^*$. Since we have $h^* = \hat{h}(c^*)$, we only need to show Equation (13) is satisfied for all $i \in J^*$; i.e., $c^* = \underset{c}{\operatorname{argmin}}\ L(c, \hat{h}(c), h', q)$ (h' is the vector representing exogenously determined probabilities that a non-strategic terrorist will attack any targets). There are three cases.

(a.1) $q \in (0, 1]$ and $J^* \neq P^*$. For any targets i and j, suppose $c_i^* > 0$, and $c_j^* > 0$ in any particular solution c^*, and we have $R_i^* = A_i^* p_i(c_i^*) x_i = \left[qr \frac{I_i^*}{||P^*||} + (1-q) h_i' \right] p_i(c_i^*) x_i$, $\forall\ i \in P^*$, $R_j^* = A_j^* p_j(c_j^*) x_j = (1-q) h_j' p_j(c_j^*) x_j,\ \forall\ j \in J^* - P^*$. We want to show that if a positive constant $W^* = R_i^* = R_j^*,\ \forall\ i \in P^*, \forall\ j \in J^* - P^*$, Equation (13) is satisfied. We only consider four cases due to symmetry: (a.1.1), c_i $i \in P^*$ is increased while c_k $k \in P^*$ is decreased, (a.1.2), c_i $i \in P^*$ is increased while c_k $k \in J^* - P^*$ is decreased, (a.1.3), c_i $i \in J^* - P^*$ is increased while c_k $k \in J^* - P^*$ is decreased, (a.1.4), c_i $i \in J^* - P^*$ is increased while c_k $k \in P^*$ is decreased. In particular,

(a.1.1) If we increase c_i^* by a small $\epsilon > 0$, for $i \in P^*$, and decrease c_k^* by ϵ, $k \in P^*, k \neq i$. Then h_i^* becomes 0 and h_k^* becomes r and P^* becomes $\{k\}$ and the change of total expected losses is:

$$\begin{aligned}
&\Delta L^*(c^*, \hat{h}(c^*), h', q) \\
=\ & qr(p_k(c_k^* - \epsilon)x_k - p_k(c_k^*)x_k) + (1-q)h_i'(p_i(c_i^* + \epsilon)x_i - p_i(c_i^*)x_i) + (1-q) \\
& h_k'(p_k(c_k^* - \epsilon)x_k - p_k(c_k^*)x_k) \\
=\ & \frac{(1-q)h_i'}{qh_i^* + (1-q)h_i'} W^* \left(\frac{p_i(c_i^* + \epsilon)}{p_i(c_i^*)} - 1 \right) + \frac{qr + (1-q)h_k'}{qh_k^* + (1-q)h_k'} W^* \left(\frac{p_k(c_k^* - \epsilon)}{p_k(c_k^*)} - 1 \right) \\
& \text{(Using Equation 7)} \\
\geq\ & W^* \left[\left(\frac{p_i(c_i^* + \epsilon)}{p_i(c_i^*)} - 1 \right) + \left(\frac{p_k(c_k^* - \epsilon)}{p_k(c_k^*)} - 1 \right) \right] \qquad (17)
\end{aligned}$$

Using Taylor expansion, we have

$$\lim_{\epsilon \to 0} \left(\frac{p_i(c_i^* + \epsilon)}{p_i(c_i^*)} - 1 \right) = \lim_{\epsilon \to 0} \frac{p_i(c_i^*) + \epsilon p_i'(c_i^*) + O(\epsilon^2) - p_i(c_i^*)}{p_i(c_i^*)} = 0,$$

where $\lim_{\epsilon \to 0} \frac{O(\epsilon^2)}{\epsilon} = 0$. Similarly, we have $\lim_{\epsilon \to 0} \left(\frac{p_k(c_k^* - \epsilon)}{p_k(c_k^*)} - 1 \right) = 0$. Therefore, we have $\lim_{\epsilon \to 0} \Delta L^*(c^*, \hat{h}(c^*), h', q) \geq 0$, which implied that such changes will not decrease the expected loss.

(a.1.2) If we increase c_i^* by a small $\epsilon > 0$, for $i \in P^*$, and decrease c_k^* by ϵ, $k \in J^* - P^*$. There are three possibilities. While h_i^* becomes 0, h_k^* remains 0 or becomes $r/||P^*||$ or r (depending on the magnitude of ϵ) and set P^* becomes $P^* - \{i\}$ or $P^* - \{i\} + \{k\}$ or {k}. For the first two possibilities the expected loss caused by the strategic terrorist remains the same and the change of total expected losses is: $\Delta L^*(c^*, \hat{h}(c^*), h', q) = (1-q)h_i'(p_i(c_i^* + \epsilon)x_i - p_i(c_i^*)x_i) + (1-q)h_k'(p_k(c_k^* - \epsilon)x_k - p_k(c_k^*)x_k) = \frac{(1-q)h_i'}{qh_i^* + (1-q)h_i'} W^* \left(\frac{p_i(c_i^* + \epsilon)}{p_i(c_i^*)} - 1 \right) + W^* \left(\frac{p_k(c_k^* - \epsilon)}{p_k(c_k^*)} - 1 \right)$
$\geq W^* \left[\left(\frac{p_i(c_i^* + \epsilon)}{p_i(c_i^*)} - 1 \right) + \left(\frac{p_k(c_k^* - \epsilon)}{p_k(c_k^*)} - 1 \right) \right]$. Similar to case (a.1.1), we have $\lim_{\epsilon \to 0} \Delta L^*(c^*, \hat{h}(c^*), h', q) \geq 0$, which implies that such changes will not decrease the expected loss.

For the third possibility, we know that $p_k(c_k^* - \epsilon)x_k > p_i(c_i^*)x_i \forall i \in P^*$ before the change (since k is the only target attracting the strategic terrorist after the change)

and the change of total expected losses is: $\Delta L^*(c^*, \hat{h}(c^*), h', q) = qr[p_k(c_k^* - \epsilon)x_k - p_i(c_i^*)x_i] + (1-q)h_i'(p_i(c_i^* + \epsilon)x_i - p_i(c_i^*)x_i) + (1-q)h_k'(p_k(c_k^* - \epsilon)x_k - p_k(c_k^*)x_k) > \frac{(1-q)h_i'}{qh_i^* + (1-q)h_i'} W^* \left(\frac{p_i(c_i^*+\epsilon)}{p_i(c_i^*)} - 1\right) + W^* \left(\frac{p_k(c_k^*-\epsilon)}{p_k(c_k^*)} - 1\right)$
$\geq W^* \left[\left(\frac{p_i(c_i^*+\epsilon)}{p_i(c_i^*)} - 1\right) + \left(\frac{p_k(c_k^*-\epsilon)}{p_k(c_k^*)} - 1\right)\right]$.
Similar to case (a.1.1), we have $\lim_{\epsilon\to 0} \Delta L^*(c^*, \hat{h}(c^*), h', q) > 0$, which implies that such changes will only increase the expected loss.

(a.1.3) If we increase c_i^* by a small ϵ, $i \in J^* - P^*$, and decrease c_k^* by ϵ, $k \in J^* - P^*, k \neq i$. There are three possibilities. While h_i^* remains 0, h_k^* remains 0 or becomes $r/(||P^*|| + 1)$ or r (depending on the magnitude of ϵ) and set P^* becomes $P^* - \{i\}$ or $P^* - \{i\} + \{k\}$ or $\{k\}$. For the first two possibilities the expected loss caused by the strategic terrorist remains the same and the change of total expected losses is: $\Delta L^*(c^*, \hat{h}(c^*), h', q) = (1-q)h_i'(p_i(c_i^* + \epsilon)x_i - p_i(c_i^*)x_i) + (1-q)h_k'(p_k(c_k^* - \epsilon)x_k - p_k(c_k^*)x_k) = W^*(\frac{p_i(c_i^*+\epsilon)}{p_i(c_i^*)} - 1) + W^*(\frac{p_k(c_k^*-\epsilon)}{p_k(c_k^*)} - 1) = W^* \left[\left(\frac{p_i(c_i^*+\epsilon)}{p_i(c_i^*)} - 1\right) + \left(\frac{p_k(c_k^*-\epsilon)}{p_k(c_k^*)} - 1\right)\right]$.
Similar to case (a.1.1), we have $\lim_{\epsilon\to 0} \Delta L^*(c^*, \hat{h}(c^*), h', q) = 0$, which implies that such change will not decrease the expected loss.

For the third possibility, $p_k(c_k^* - \epsilon)x_k > p_j(c_j^*)x_j \forall j \in P^*$ before the change (since k is the only target attracting the strategic terrorist after the change) and the change of total expected losses is: $\Delta L^*(c^*, \hat{h}(c^*), h', q) = qr(p_k(c_k^* - \epsilon)x_k - p_k(c_k^*)x_k) + (1-q)h_i'(p_i(c_i^* + \epsilon)x_i - p_i(c_i^*)x_i) + (1-q)h_k'(p_k(c_k^* - \epsilon)x_k - p_k(c_k^*)x_k) > W^* \left(\frac{p_i(c_i^*+\epsilon)}{p_i(c_i^*)} - 1\right) + W^* \left(\frac{p_k(c_k^*-\epsilon)}{p_k(c_k^*)} - 1\right) \geq W^* \left[\left(\frac{p_i(c_i^*+\epsilon)}{p_i(c_i^*)} - 1\right) + \left(\frac{p_k(c_k^*-\epsilon)}{p_k(c_k^*)} - 1\right)\right]$. Similar to case (a.1.1), we have $\lim_{\epsilon\to 0} \Delta L^*(c^*, \hat{h}(c^*), h', q) > 0$, which implies that such change will only increase the expected loss.

(a.1.4) If we increase c_i^* by a small ϵ, $i \in J^* - P^*$, and decrease c_k^* by ϵ, $k \in P^*$. Then h_k^* becomes r and P^* becomes $\{k\}$ and the change of total expected losses is: $\Delta L^*(c^*, \hat{h}(c^*), h', q) = qr[p_k(c_k^* - \epsilon)x_k - p_k(c_k^*)x_k] + (1-q)h_i'[p_i(c_i^* + \epsilon)x_i - p_i(c_i^*)x_i] + (1-q)h_k'[p_k(c_k^* - \epsilon)x_k - p_k(c_k^*)x_k] = W^*(\frac{p_i(c_i^*+\epsilon)}{p_i(c_i^*)} - 1) + \frac{qr + (1-q)h_k'}{qh_k^* + (1-q)h_k'} W^*(\frac{p_k(c_k^*-\epsilon)}{p_k(c_k^*)} - 1) \geq W^* \left[\left(\frac{p_i(c_i^*+\epsilon)}{p_i(c_i^*)} - 1\right) + \left(\frac{p_k(c_k^*-\epsilon)}{p_k(c_k^*)} - 1\right)\right]$. Similar to case (a.1.1.1), we have $\lim_{\epsilon\to 0} \Delta L^*(c^*, \hat{h}(c^*), h', q) \geq 0$, which implies that such change will not decrease the expected loss.

In summary, all possible deviations from the solution c^* do not decrease the expected losses, and thus we have $c^* = \underset{c}{\operatorname{argmin}}\ L(c, \hat{h}(c), h', q)$. From Equation (9), we have $h^* = \hat{h}(c^*)$. Therefore, according to Definition 1, both (12) and (13) are satisfied and thus (h^*, c^*) is the equilibrium defined by Definition 1 in Section 2.2. Then, we will show that $R_i^* = A_i^* p_i(c_i^*)x_i \leq W^* \ \forall i \notin J^*$ (or equivalently $c_i^* = 0$ according to Equation (15)). For any targets i and j, suppose $c_i^* = 0, c_j^* > 0$ for one particular solution c^*, and we have $R_i^* = A_i^* p_i(c_i^*)x_i = \left[qr\frac{I_i^*}{||P^*||} + (1-q)h_i'\right] p_i(c_i^*)x_i, i \notin J^*$ and $R_j^* = A_j^* p_j(c_j^*)x_j = \left[qr\frac{I_j^*}{||P^*||} + (1-q)h_j'\right] p_j(c_j^*)x_j, j \in J^*$. If $R_i^* \leq R_j^*$, c_i^* cannot be decreased by re-allocating defensive resources from targets i to j, thus $W^* = R_j^* \geq R_i^*$. If $R_i^* > R_j^*$, c_i^* will be increased while c_j^* will be decreased until $R_i^* = R_j^*$ and we no longer have $c_i^* = 0$. Therefore, $R_i^* > R_j^* = W^*$ is not possible in equilibrium.

(a.2) $q \in (0, 1]$ and $J^* = P^*$. For any targets i and j, suppose $c_i^* > 0$, and $c_j^* > 0$ for any particular solution c^*, and we have $R_i^* = A_i^* p_i(c_i^*)x_i = \frac{rI_i^*}{||P^*||} p_i(c_i^*)x_i \ \forall \ i \in P^*$, and $R_j^* = A_j p_j(c_j^*)x_j = \frac{rI_j^*}{||P^*||} p_j(c_j^*)x_j \ \forall \ j \in P^*, \ j \neq i$. We want to show that if a positive constant $W^* = R_i^* = R_j^* \ \forall \ i \in P^*, \ \forall \ j \in P^*, \ j \neq i$, Equation (13) is satisfied.

We increase c_i^* by a small $\epsilon > 0$, $i \in P^*$, and decrease c_k^* by ϵ, $k \in P^*, k \neq i$. Then h_k^* becomes r and P^* becomes $\{k\}$ and the change of total expected losses is: $\Delta L^*(c^*, \hat{h}(c^*), h', q) = qr(p_k(c_k^* - \epsilon)x_k - p_k(c_k^*)x_k) + (1-q)h_i'(p_i(c_i^* + \epsilon)x_i - p_i(c_i^*)x_i) + (1-q)h_k'(p_k(c_k^* + \epsilon)x_k - p_k(c_k^*)x_k) = \frac{qr + (1-q)h_k'}{qh_k^* + (1-q)h_k'} W^* \left[\frac{p_k(c_k^* - \epsilon)}{p_k(c_k^*)} - 1\right] + \frac{(1-q)h_i'}{qh_i^* + (1-q)h_i'} W^* \left[\frac{p_i(c_i^* + \epsilon)}{p_i(c_i^*)} - 1\right]$ $\geq W^* \left[\left(\frac{p_k(c_k^* - \epsilon)}{p_k(c_k^*)} - 1\right) + \left(\frac{p_i(c_i^* + \epsilon)}{p_i(c_i^*)}\right)\right]$.

Similar to case (a.1.1), we have $\lim_{\epsilon \to 0} \Delta L^*(c^*, \hat{h}(c^*), h', q) \geq 0$, which implies that such change will not decrease the expected loss. Note that i and k are symmetric and such changes include both increases and decreases.

Therefore, the above deviation from solution c^* do not decrease the expected losses, and thus we have $c^* = \underset{c}{\text{argmin}} \ L(c, \hat{h}(c), h', q)$. From Equation (9), we have $h^* = \hat{h}(c^*)$. Therefore, according to Definition 1, both (12) and (13) are satisfied and thus (h^*, c^*) is the equilibrium defined by Definition 1 in Section 2.2.

Then, we will show that $R_i^* = A_i^* p_i(c_i^*)x_i \leq W^* \ \forall i \notin J^*$ (or equivalently $c_i^* = 0$ according to Equation (15)). For any targets i and j, suppose $c_i^* = 0, c_j^* > 0$, for one particular solution c^*, and we have $R_i^* = A_i^* p_i(c_i^*)x_i = \left[qr\frac{I_i^*}{||P^*||}\right] p_i(c_i^*)x_i$ and $R_j^* = A_j^* p_j(c_j^*)x_j = \left[qr\frac{I_j^*}{||P^*||}\right] p_j(c_j^*)x_j$. If $R_i^* < R_j^*$, c_i^* cannot be decreased by re-allocating defensive resources from targets i to j, thus $W^* = R_j^* > R_i^*$. Similarly, $R_i^* > R_j^* = W^*$ is not possible in equilibrium.

(a.3) $q = 0$. For any targets i and j, suppose $c_i^* > 0$, and $c_j^* > 0, i, j \in J^*, j \neq i$ for any particular solution c^*, and we have $R_i^* = h_i' \frac{dp_i(c_i^*)}{dc_i^*} x_i$ and $R_j^* = h_j' \frac{dp_j(c_j^*)}{dc_j^*} x_j$. We show that if a positive constant $W^* = R_i^* = R_j^* \ \forall\ i \in J^*,\ \forall\ j \in J^*$ and $j \neq i$, Equation (13) is satisfied.

If we increase c_i^* by $\epsilon > 0$, $i \in J^*$, and decrease c_k^* by ϵ, $k \in J^*$. We have $\Delta L^*(c^*, \hat{h}(c^*), h', q) = h_i' p_i(c_i^* + \epsilon)x_i - h_i' p_i(c_i^*)x_i + h_k' p_k(c_k^* - \epsilon)x_k - h_k' p_k(c_k^*)x_k = \frac{-p_i(c_i^*)}{\frac{dp_i(c_i^*)}{dc_i^*}} W^* \left[\frac{p_i(c_i^*+\epsilon)}{p_i(c_i^*)} - 1\right] + \frac{-p_k(c_k^*)}{\frac{dp_k(c_k^*)}{dc_k^*}} W^* \left[\frac{p_k(c_k^*-\epsilon)}{p_k(c_k^*)} - 1\right]$.

Similar to case (a.1.1), we have $\lim_{\epsilon \to 0} \Delta L^*(c^*, \hat{h}(c^*), h', q) \geq 0$, which implies that such change will not decrease the expected loss.

In summary, the above deviation from c^* does not decrease the total expected losses, and thus we have $c^* = \operatorname{argmin}_c L(c, \hat{h}(c), h', q)$. From Equation (9), we have $h^* = \hat{h}(c^*)$. Therefore, according to Definition 1, both (12) and (13) are satisfied and thus (h^*, c^*) is the equilibrium defined by Definition 1 in Section 2.2. Then, we will show that $R_i^* = A_i^* p_i(c_i^*)x_i \leq W^* \ \forall i \notin J^*$ (or equivalently $c_i^* = 0$ according to Equation (15)). For any targets i and j, suppose $c_i^* = 0, c_j^* > 0$, for one particular solution c^*, and we have $R_i^* = A_i^* p_i(c_i^*)x_i = h_i' p_i'(c_i^*)x_i$ and $R_J^* = A_j^* p_j(c_j^*)x_j = h_j' p_j'(c_j^*)x_j$. If $R_i^* < R_j^*$, c_i^* cannot be decreased by re-allocating defensive resources from target i to j even though investing is target j's defense is more cost-effective, thus $W^* = R_j^* > R_i^* = A_i^* p_i(c_i^*)x_i = h_i' p_i'(c_i^*)x_i$. Similarly, $R_i^* > R_j^*$ is not possible in equilibrium.

A.3 Proof for Theorem 3

First, we show that the total probability of attack r does not affect the defender equilibrium allocation c^*. The objective function (11) can be reformulated as $qr \max_{i=1,\ldots,n} p_i(c_i)x_i + (1 - q)r \sum_{i=1}^{n} h_i'' p_i(c_i)x_i = r[q \max_{i=1,\ldots,n} p_i(c_i)x_i + (1-q) \sum_{i=1}^{n} h_i'' p_i(c_i)x_i]$, where $h_i'' = h_i'/r \ \forall i$. Minimizing $r[q \max_{i=1,\ldots,n} p_i(c_i)x_i + (1-q) \sum_{i=1}^{n} h_i'' p_i(c_i)x_i]$ would be equivalent to minimizing $q \max_{i=1,\ldots,n} p_i(c_i)x_i + (1 - q) \sum_{i=1}^{n} h_i'' p_i(c_i)x_i$ and the value of r does not affect the defender equilibrium allocation c^*.

Second, we show that the equilibrium loss increases linearly with r. We can treat $[q \max_{i=1,\ldots,n} p_i(c_i^*)x_i + (1 - q) \sum_{i=1}^{n} h_i'' p_i(c_i^*)x_i]$ as a constant with regard to r. Therefore, the equilibrium loss increases linearly with r.

A.4 Proof for Theorem 4

In this proof, we first show that the algorithm provided in Section 3.2 will always converge. Then we show that the algorithm will converge to the equilibrium solution c^* defined by Definition 1 in Section 2.2.

In order to show convergence of the entire algorithm, we note that the algorithm contains 3 loops as shown in Figure 3 in Section 3.2: Loop A (formed by S_2, S_3, C_2 and C_3); Loop B (formed by S_6, S_7, C_2 and C_7); and Loop C (formed by S_4, S_5, C_2 and C_5). Within each loop, Steps S_3 (in Loop A), S_5 (in Loop C), and S_7 (in Loop B) delete index j satisfying $x_j A_j \leq W$ (in Condition C_2) from set J (i.e., set of targets to be defended). Now we claim that the algorithm always converges because (a) only deletion and no addition are allowed for modification of set J, (b) the government can defend at least one target (according to Equation (1)), and (c) there are finite number of potential targets in set J ($\leq n$). Therefore, the algorithm will always converge.

Secondly, we show that this converging algorithm always converges to the equilibrium solution c^* defined by Definition 1 in Section 2.2. The equilibrium solution c^* can take 3 different forms corresponding to 3 different conditions in Equation (6). In order to reach the equilibrium solution c^*, the algorithm need to find set J^* (Step S_3 (a-b) in Loop A, Step S_5 (a-b) in Loop B, and Step S_7 (a-b) in Loop C) and W^* (Step S_2 and Condition C_2 in Loop A, Step S_4 and Condition C_2 in Loop B, and Step S_6 and Condition C_2 in Loop C). The algorithm proceeds as follows: in Step S_1, the algorithm initializes sets J, P, and I to include all n targets. There are two cases corresponding to whether $q = 0$ or $q \in (0, 1]$: Case 1. the algorithm goes to Loop A (possibly and C); and Case 2. the algorithm goes to Loop B. Loop C is employed only when Loop A does not find the optimal solution as indicated by the un-satisfaction of both Conditions C_4 and C_6.

Case 1. If $q \in (0, 1]$, the algorithm goes to Loop A, and Step S_2 calculates a suboptimal W with $A = qr\frac{I_i}{||P||} + (1-q)h'_i$ according to Equation (16). Note that W will always be smaller than W^*. This is because (a) in Equation (16), the denominator becomes smaller after each iteration (i.e., the size of set J decreases as j's are deleted from set J), (b) in Equation (16), the numerator becomes larger as a result of $\sum_{i \in J} \ln A = \sum_{i \in J} \ln \left[qr\frac{I_i}{||P||} + (1-q)h'_i \right]$ increasing at a faster rate than the decreasing rate of $\sum_{i \in J} \ln x_i$. Recall that x_i's provided in Table 1 in Section 2.3 are sorted in descending order. After each iteration, $\sum_{i \in J} \ln x_i$ will only decrease slightly, while set P will shrink quickly and thus $A = \left[qr\frac{I_i}{||P||} + (1-q)h'_i \right]$ will increase significantly.

We use Lemma 2 below to justify Step 3 (b) in Loop A in the algorithm provided by Figure 3 and Table 3 in Section 3.2, where most targets to be attacked by a non-strategic terrorist are not included in updating sets P and I.

Lemma 2 *Given $q \in (0,1]$ and $J^* \neq P^*$, for any target $i \in J^*$, if $h'_i > 0$ and $h'_i \neq \min_i \{h'_i\}$, we have $i \notin P^*$.*

Proof for Lemma 2 We first show that if h'_i's are different $\forall i \in P^*$, P^* must contain only the element(s) corresponding to the largest $p_i(c_i^*)x_i$. Since h_i defined by Equation (9) is the same $\forall i \in P^*$ and h'_i's are different $\forall i \in P^*$, $[qrh_i + (1-q)h'_i]$ will not be the same $\forall i \in P^*$. Without loss of generality, we let $h'_i < h'_j$ $\forall i, j \in P^*$, then $[qrh_i + (1-q)h'_i] < [qrh_j + (1-q)h'_j]$ and thus $p_i(c_i^*)x_i > p_j(c_j^*)x_j$ for $R_i^* = R_j^*$ (since we have in equilibrium $R_i^* = [qh_i^* + (1-q)h'_i]p_i(c_i^*)x_i = R_j^* = [qh_j^* + (1-q)h'_j]p_j(c_j^*)x_j$). Then the strategic terrorist will attack target i only and target j no longer belongs to set P, that is, $\forall j \notin P^*$. Therefore, P^* must contain only the element(s) corresponding to $\max_i \{p_i(c_i^*)x_i\}$ (i.e., $\min_i \{h'_i\}$). For all other i's such that $h'_i \neq \min_i \{h'_i\}$, $i \in J^*$ but $i \notin P^*$.

Loop A will be repeated and J, W and c will be updated until Condition C_2 is no longer satisfied. Then J^*, W^*, and c^* are obtained. Since Condition C_2 is not satisfied, all c_i^*'s for $i \in J^*$ are positive as seen from Equation (15) (i.e., $c_i^* = \frac{\ln x_i + \ln A_i^* - \ln W^*}{\lambda}$ $\forall\, i \in J^*$) and also $R_i^* = A_i^* p_i(c_i^*)x_i = W^*$ $\forall i \in J^*$. Note that c^* is unique since J^* and W^* are unique and c^* is calculated based on J^* and W^*. According to Theorem 2, since $R_i^* = A_i^* p_i(c_i^*)x_i = W^*$, (h^*, c^*) is the possible equilibrium defined by Definition 1 in Section 2.2. Therefore, c^* is the possible equilibrium solution to the optimization problem (10). After Loop A, Condition C_4 is checked. If yes, c^* is the equilibrium solution with $J^* \neq P^*$. Otherwise, it is not and the algorithm will check Condition C_6. If yes, c^* is the equilibrium solution with $J^* = P^*$. If not, the algorithm must go to Loop C and restart the iteration process over again.

We obtained Condition C_4 by noting that if $q \in (0,1)$ and $J^* \neq P^*$, at equilibrium we must have $qr\frac{I_i^*}{||P^*||} < (1-q)h'_j$, $\forall\, i \in P^*$ and $\forall j \in J^* - P^*$. This is because $qr\frac{I_i^*}{||P^*||} \geq (1-q)h'_j$, $\forall\, i \in P^*$ and $\forall j \in J^* - P^*$ will lead to $J^* = P^*$ due to the fact that $\frac{I_i^*}{||P^*||}$ is dynamic and based on the relative value of $p_i(c_i^*)x_i$. If $qr\frac{I_i^*}{||P^*||} \geq (1-q)h'_j$, $\forall\, i \in P^*$ and $\forall j \in J^* - P^*$, since $\left[qr\frac{I_i^*}{||P^*||} + (1-q)h'_i\right] p_i(c_i^*)x_i = (1-q)h'_j p_j(c_j^*)x_j$ (equilibrium condition), $p_i(c_i^*)x_i \leq p_j(c_j^*)x_j$, $I_j^* = 1$, and thus $j \in P^*$, which is a contradiction to the assumption that $j \in J^* - P^*$. Therefore, $qr\frac{I_i^*}{||P^*||} < (1-q)h_j'$, $\forall\ i \in P^*$ and $j \in J^* - P^*$.

In Loop C, Step S_4 re-initializes sets J, P and I and calculate a suboptimal W with $A = qr\frac{I_i}{||P||}$ according to Equation (16). Loop C will be repeated and J, W, and c will be updated until Condition C_2 is no longer satisfied. Since only the attack probabilities of the strategic terrorist (i.e., $h_i = r\frac{I_i}{||P||}$) is involved in the updating process, J^* must equal P^*. Then J^*, W^* and c_j^*'s are obtained. Since Condition C_2 is not satisfied, all c_i^*'s for $i \in J^*$ are positive as seen from Equation (15) (i.e., $c_i^* = \frac{\ln x_i + \ln A_i^* - \ln W^*}{\lambda}\ \forall\ i \in J^*$) and also $R_i^* = A_i^* p_i(c_i^*)x_i = W^*\ \forall i \in J^*$. According to Theorem 2, since $R_i^* = A_i^* p_i(c_i^*)x_i = W^*$ and $J^* = P^*$, (h^*, c^*) is the equilibrium defined by Definition 1 in Section 2.2. Therefore, c^* is the equilibrium solution to the optimization problem (10).

Case 2. If $q = 0$, the algorithm goes to Loop B and Step S_6 calculates a suboptimal W with $A = \lambda h_i'$ according to Equation (16). Loop B will be repeated and J, W, and c will be updated until Condition C_2 is no longer satisfied. Then J^*, W^* and c_j^*'s are obtained. Since Condition C_2 is not satisfied, all c_i^*'s for $i \in J^*$ are positive as seen from Equation (15) (i.e., $c_i^* = \frac{\ln x_i + \ln A_i^* - \ln W^*}{\lambda}\ \forall\ i \in J^*$) and also $R_i^* = A_i^* p_i(c_i^*)x_i = W^*$. According to Theorem 2, since $R_i^* = A_i^* p_i(c_i^*)x_i = W^*$ and $q = 0$, (h^*, c^*) is the equilibrium defined by Definition 1 in Section 2.2. Therefore, c^* is the equilibrium solution to the optimization problem (5).

As shown in Figure 3, depending on whether Conditions C_1, C_4 and C_6 are satisfied, the algorithm employs either 1 (Loop A or B) or 2 loops (Loops A and C) to find the optimal solution c^*. Within Step 3 of Loop A, Condition C_3 needs be checked for up to $n-1$ indices in set J, and each of S_3(a), S_3(b or c, parallel), and S_3(d) requires up to n computations. Thus, Step 3 requires a total of at most $(n-1) + 3n = 4n - 1$ computations. On the other hand, Loop A requires the sum of at most $n-1$ iterations (checking Condition C_2) of looping Step 3, and $2n$ additional computations in S_3(c, e) when C_2 is not satisfied. Therefore, at most $(4n-1)(n-1) + 2n = 4n^2 - 3n + 1$ computations will be needed for Loop A. Loops B and C are independent of Loop A and less complex than Loop A. Therefore, the algorithm requires at most $O(n^2)$ computations in order to find the optimal solution c^*.

A.5 Proof for Theorem 5

If we increase C, the feasible space of optimization model (10) expands and thus the optimal objective function value L^* (weakly) decreases. If we increase λ, $p_i(c_i^*)x_i$ decreases $\forall i$, which implies that both $\max_{i=1,\ldots,n} p_i(c_i^*)x_i$ and $\sum_{i=1}^{n}(1-q)h_i'p_i(c_i^*)x_i$ decrease. Given the alternative formulation (11), the optimal objective function value L^* decreases.

A.6 Proof for Theorem 6

(a) The proof follows immediately from the definition of $\bar{L}(\bar{c}, \hat{h}(\bar{c}), h', q)$ and $\hat{L}(\hat{c}, \hat{h}(\hat{c}), h', q)$. $\bar{L}(\bar{c}, \hat{h}(\bar{c}), h', q)$ equals the objective function value $L^*(c^*, \hat{h}(c^*), h', q)$ when $q = 1$. Therefore, $L^*(c^*, \hat{h}(c^*), h', q) = \bar{L}(\bar{c}, \hat{h}(\bar{c}), h', q)$ when q=1. Similarly, $L^*(c^*, \hat{h}(c^*), h', q) = \hat{L}(\hat{c}, \hat{h}(\hat{c}), h', q)$ when q=0.

(b) First, we prove that $\hat{L}(\hat{c}, \hat{h}(\hat{c}), h', q)$ weakly decreases in $(1-q)$. A strategic terrorist will adjust his attack probabilities h in order to maximize the expected loss. And the strategic terrorist always has the option to choose h' (the non-strategic attack probabilities). Specifically, in equilibrium if the government chooses the optimal defense against h' and the terrorist chooses h' as the strategic attack probability, that is $\sum_{i=1}^{n} \hat{h}_i(\hat{c})p_i(\hat{c}_i)x_i \geq \sum_{i=1}^{n} h_i'p_i(\hat{c}_i)x_i$. The objective value $\hat{L}(\hat{c}, \hat{h}(\hat{c}), h', q)$ is a convex combination of $\sum_{i=1}^{n} \hat{h}_i(\hat{c})p_i(\hat{c}_i)x_i$ and $\sum_{i=1}^{n} h_i'p_i(\hat{c}_i)x_i$. As $1-q$ increases, the weight for $\sum_{i=1}^{n} \hat{h}_i(\hat{c})p_i(\hat{c}_i)x_i$ decreases, while the weight for $\sum_{i=1}^{n} h_i'p_i(\hat{c}_i)x_i$ increases. As a result, $\hat{L}(\hat{c}, \hat{h}(\hat{c}), h', q)$ is (weakly and linearly) decreasing in $1-q$.

Second, $\bar{L}(\bar{c}, \hat{h}(\bar{c}), h', q)$ is a constant not influenced by $1-q$ and thus (weakly and linearly) decreasing in $1-q$. There are two cases.

Case 1. When the strategic terrorist attacks more or equal number of targets than the non-strategic terrorist. The reason is that if the government believes that $1-q=0$, she will allocate the defensive resources to several most valuable targets. When the terrorist is indeed fully strategic and attacks the most vulnerable target in order to yield the maximal expected gain, the terrorist will attack any of the defended targets and the resulting expected losses are the same for all defended targets. On the other hand, when the terrorist is fully non-strategic and attacks most valuable target(s), the resulting expected loss will be the same.

Case 2. When the strategic terrorist will attack less targets than the non-strategic terrorist, $\bar{L}(\bar{c}, \hat{h}(\bar{c}), h', q)$ is strictly decreasing in $1-q$. The reason is that if the terrorist is fully non-strategic and attacks the low-valued targets, which are not worth defending and thus not defended. The resulting expected loss will be lower than if the terrorist is fully strategic and only attacks valuable targets. Therefore, $\sum_{i=1}^{n} \hat{h}_i(\bar{c})p_i(\bar{c}_i)x_i > \sum_{i=1}^{n} h'_i p_i(\bar{c}_i)x_i$. So the objective value $\bar{L}(\bar{c}, \hat{h}(\bar{c}), h', q)$ is a convex combination of $\sum_{i=1}^{n} \hat{h}_i(\bar{c})p_i(\bar{c}_i)x_i$ and $\sum_{i=1}^{n} h'_i p_i(\bar{c}_i)x_i$. As $1-q$ increases, the weight for $\sum_{i=1}^{n} \hat{h}_i(\bar{c})p_i(\bar{c}_i)x_i$ decreases, while the weight for $\sum_{i=1}^{n} h'_i p_i(\bar{c}_i)x_i$ increases. As a result, $\bar{L}(\bar{c}, \hat{h}(\bar{c}), h', q)$ is (weakly and linearly) decreasing in $1-q$ when the strategic terrorist attacks less targets that the non-strategic terrorist.

Third, we prove that $L^*(c^*, \hat{h}(c^*), h', q)$ weakly decreases in $1-q$. Note that c^* might changes to concentrate around defending more valuable targets as $1-q$ decreases depending whether $L^*(c^*, \hat{h}(c^*), h', q) > \hat{L}(\hat{c}, \hat{h}(\hat{c}), h', q)$. If $L^*(c^*, \hat{h}(c^*), h', q) > \hat{L}(\hat{c}, \hat{h}(\hat{c}), h', q)$, c^* will remain the same regardless of the small change in $1-q$. Otherwise, c^* will change as described above. Therefore, when $1-q$ falls into the range where $J^* = P^*$, both c^* and $L^*(c^*, \hat{h}(c^*), h', q)$ remains the same. When $1-q$ changes from the value associated with $J^* = P^*$ to the value associated with $J^* \neq P^*$, $L^*(c^*, \hat{h}(c^*), h', q)$ must decrease since otherwise c^* will remain the same and so will $L^*(c^*, \hat{h}(c^*), h', q)$, contradicting the assumption that $J^* \neq P^*$. When $1-q$ falls into the range where $J^* \neq P^*$, $J^* \neq P^*$ implies that at least one target, say i, is only attracting the non-strategic attacker and not the strategic attacker. As $1-q$ increases (i.e., the attacker is more likely to be non-strategic), $L_i^*(c^*, \hat{h}(c^*), h', q)$ will decrease for target i. Since Theorem 2 tells us R^* equals $L_i^*(c^*, \hat{h}(c^*), h', q)$, $L^*(c^*, \hat{h}(c^*), h', q) = ||J^*|| L_i^*(c^*, \hat{h}(c^*), h', q)$, and $||J^*||$ weakly decreases in $1-q$, $L^*(c^*, \hat{h}(c^*), h', q)$ weakly decreases in $1-q$ when $1-q$ falls into the range where $J^* \neq P^*$. Therefore, we claim that $L^*(c^*, \hat{h}(c^*), h', q)$ weakly decreases in $1-q$.

(c) By definition, c^* is the optimal solution and $L^*(c^*, \hat{h}(c^*), h', q)$ is the corresponding objective value as seen from Equations (13) and (5). So, $L^*(c^*, \hat{h}(c^*), h', q)$ is the minimal among all L's. $\bar{L}(\bar{c}, \hat{h}(\bar{c}), h', q), \hat{L}(\hat{c}, \hat{h}(\hat{c}), h', q)$ are two feasible objective values and belong to the set of all L's. And thus $L^*(c^*, \hat{h}(c^*), h', q) \leq \bar{L}(\bar{c}, \hat{h}(\bar{c}), h', q)$, $L^*(c^*, \hat{h}(c^*), h', q) \leq \hat{L}(\hat{c}, \hat{h}(\hat{c}), h', q)$.

(d) Note that $\bar{L}(\bar{c}, \hat{h}(\bar{c}), h', q)$ is a constant regardless of the value of q when the strategic terrorist attacks more or equal number of targets than the non-strategic terrorist (see proof for Theorem 6(b)). When $q = 1$, $\bar{L}(\bar{c}, \hat{h}(\bar{c}), h', q) = L^*(c^*, \hat{h}(c^*), h', q) \leq \hat{L}(\hat{c}, \hat{h}(\hat{c}), h', q)$ by Theorem 6(c). Similarly, when $q = 0$, $\hat{L}(\hat{c}, \hat{h}(\hat{c}), h', q) = L^*(c^*, \hat{h}(c^*), h', q) \leq$

$\bar{L}(\bar{c}, \hat{h}(\bar{c}), h', q)$ by Theorem 6(c). Since $\bar{L}(\bar{c}, \hat{h}(\bar{c}), h', q)$ and $\hat{L}(\hat{c}, \hat{h}(\hat{c}), h', q)$ are (weakly) decreasing in (1-q), their values cross over at a certain value of $1-q$, which is denoted as T. Therefore, $\bar{L}(\bar{c}, \hat{h}(\bar{c}), h', q) \leq \hat{L}(\hat{c}, \hat{h}(\hat{c}), h', q)$ if $1-q < T$, where T is a constant and varies with h'.

Appendix B: Examples and illustrations

B.1 Illustrations for Theorem 2

For all three illustrations, we consider two scenarios of re-allocation, and observe that both re-allocations increase the total expected loss. Note that in this optimality check, we examine a higher precision level than that was used in Table 2 in Section 3.1, in order to show more precise changes after the re-allocations.

Illustration 1 ($q = 0.5$): Suppose that the government re-allocates one unit of defensive resources from targets 1 to 2 (both in set J^*). This will decrease c_1^* from 322.8465 to 321.8465, and increase c_2^* from 194.9949 to 195.9949. Thus, $p_1(c_1^*)x_1$ increases from 16.3624 to 16.5268, while $p_2(c_2^*)x_2$ decreases from 16.3624 to 16.1995. Therefore, we have higher expected loss for target 1 ($L_1^* = 4.1317$, increased from 4.0906), lower expected loss for target 2 ($L_2^* = 4.0499$, decreased from 4.0906), and higher total expected loss ($L^* = 20.4534$, increased from 20.4530), which means that this re-allocation is not optimal.

Illustration 2 ($q = 0.8$): Suppose the government re-allocates one unit of defensive resources from targets 1 to 2, (both in set J^*). This will decrease c_1^* from 298.4142 to 297.4142, and increase c_2^* from 170.5627 to 171.5627. Thus, $p_1(c_1^*)x_1$ increases from 20.8907 to 21.1007, while $p_2(c_2^*)x_2$ decreases from 20.8907 to 20.6829. Therefore, we have higher expected loss for target 1 ($L_1^* = 4.9235$, increased from 4.8745), lower expected loss for target 2 ($L_2^* = 4.8260$, decreased from 4.8745), and higher total expected loss ($L^* = 20.8911$, increased from 20.8906), which means that this re-allocation is not optimal.

Illustration 3 ($q = 0$): Suppose the government re-allocates one unit of defensive resources from targets 1 to 2 (both in set J^*). This will decrease c_1^* from 400.4258 to 399.4258, and increase c_2^* from 272.5742 to 273.5742. Thus, $p_1(c_1^*)x_1$ increases from 7.5322 to 7.6079, while $p_2(c_2^*)x_2$ decreases from 7.5322 to 7.4572. Therefore, we have higher expected loss for target 1 ($L_1^* = 3.8040$, increased from 3.7661), lower expected loss for target 2 ($L_2^* = 3.7286$, decreased from 3.7661), and higher total expected loss ($L^* = 7.5326$, increased from 7.5322), which means that this re-allocation is not optimal.

B.2 Three Other Types of Non-strategic Terrorist and Robustness Analysis

We study three other types of non-strategic terrorist, whose attack probabilities have the following characteristics: Type-II non-strategic attack probabilities are proportional to the target valuations; Type-III non-strategic attack probabilities are evenly distributed among top N least valuable targets; and Type-IV non-strategic attack probabilities are inversely proportional to the target valuations.

B.2.1 Type-II Non-strategic Terrorist: attack probabilities are proportional to the target valuations

In this subsection, we investigate a Type-II non-strategic terrorist, whose attack probabilities are proportional to the target valuations. In other words,

$$\text{(Type-II Non-Strategic Terrorist)} \qquad h'_i \propto x_i \Longrightarrow h'_i = \frac{rx_i}{\sum_{i=1}^{n} x_i}, \ \forall i = 1, 2, \cdots, n. \tag{18}$$

Figure 8 shows the optimal defensive budget allocations as a function of the probability that the terrorist is Type-II non-strategic $(1-q)$ when $\lambda = 0.01, 0.05$, and 1, respectively. The results are similar to that for Type-I non-strategic terrorist when $N = 1, 2, 5$ as shown in Figures 4 (a1-a3, b1-b3, c1-c3), and different from the case when $N = 47$, where optimal defensive resource allocations are not sensitive to the values of $1-q$ as shown in Figures 4 (a4, b4 ,c4).

Figure 9 shows the three total expected loss after the government has applied each of the three defensive resource allocation schemes discussed in Subsection 4.1 as a function of the probability that the terrorist is Type-II non-strategic $(1-q)$ when $\lambda = 0.01, 0.05$, and 1, respectively. The results are similar to that for Type-I non-strategic terrorist when $N = 1, 2, 5$ as shown in Figures 5 (a1-a3, b1-b3, c1-c3), and different from the case when $N = 47$, where there is no difference between L^*, $\bar{L}$, and $\hat{L}$ as shown in Figures 5 (a4, b4 ,c4). We observe that $T \geq 0.69$ for all cases.

Figure 10 shows that preference threshold (T) and robustness measure (d) for game-theoretic models as a function of budget (C) when $\lambda = 0.01$, 0.05, and 1, respectively, and a Type-II non-strategic terrorist is concerned. Figures 10 (a1, b1, c1) show that $T > 0.5$ for all cases. Figures 10 (a2, b2, c2) show that $d > 0$ when $1-q$ is smaller (than 0.8 when λ=0.01, 0.05, and than 0.6 when λ=1), game theoretic models perform better than non-game theoretic models.

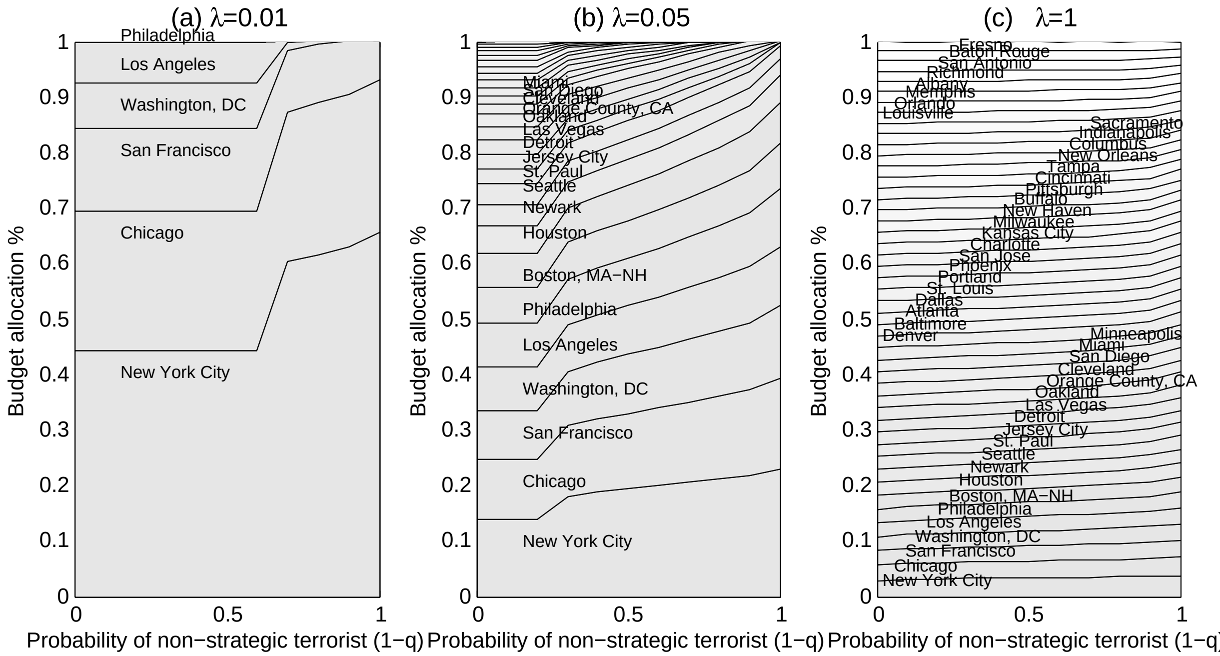

Figure 8: Optimal defensive budget allocations as a function of the probability that the terrorist is Type-II non-strategic $(1-q)$ when $\lambda = 0.01, 0.05$, and 1, respectively.

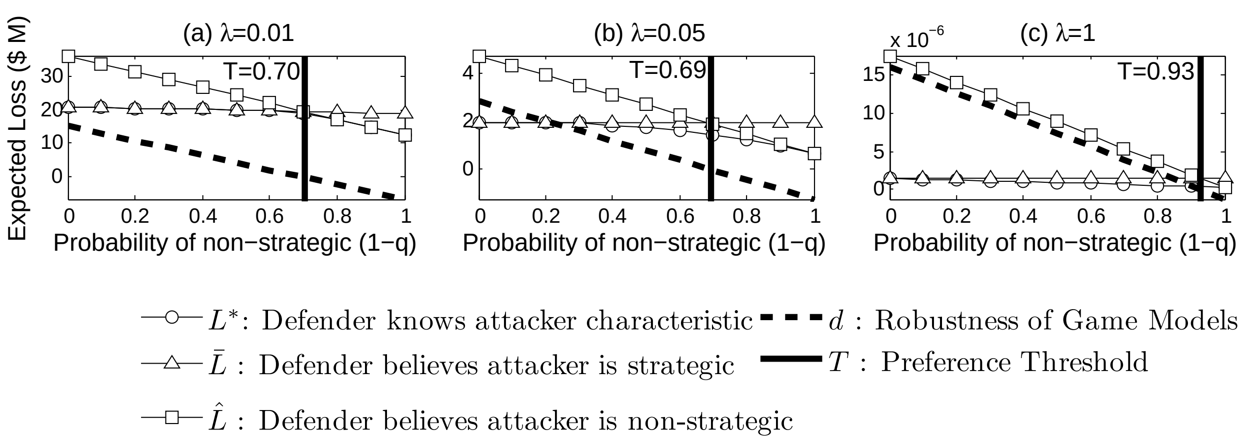

Figure 9: Government's three types of total expected losses and preference threshold (T) and robustness measure (d) as a function of probability that the terrorist is Type-II non-strategic $(1-q)$ when $\lambda = 0.01, 0.05$, and 1, respectively.

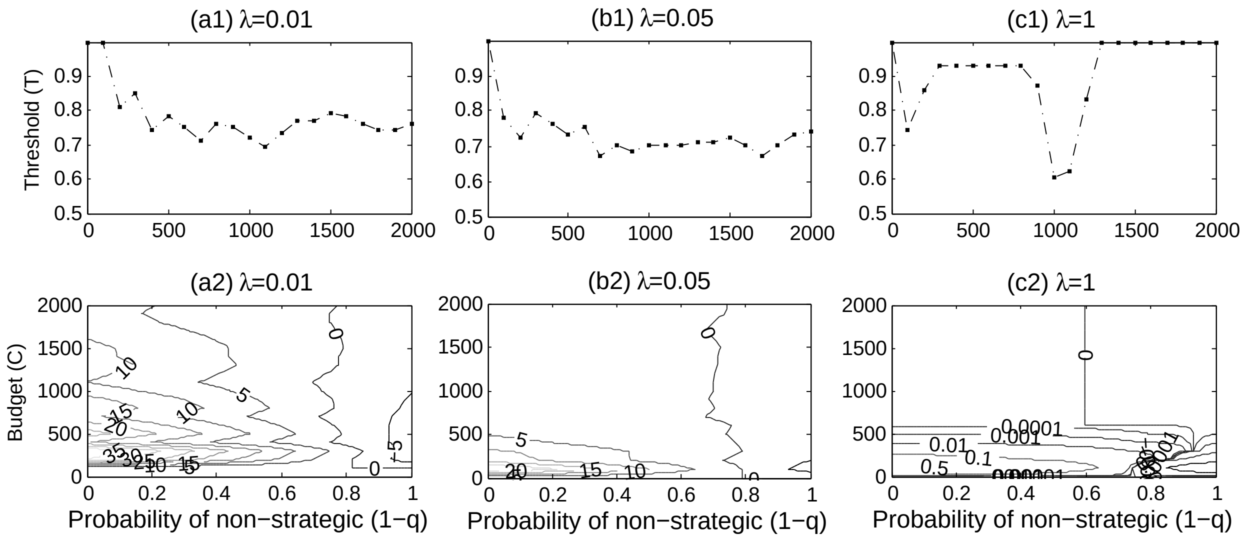

Figure 10: Preference threshold (T) and robustness measure (d) as a function of budget (C) when $\lambda = 0.01$, 0.05, and 1, respectively, and a Type-II nonstrategic terrorist is concerned.

B.2.2 Type-III Non-strategic Terrorist: attack probabilities are even distributed among top N least valuable targets

In this subsection, we investigate a Type-III non-strategic terrorist: attack probabilities are equal for top N least valuable urban areas, and zero for other $n - N$ urban areas. In other words, we have:

$$\text{(Type-III Non-Strategic Terrorist)} \qquad h'_i = \begin{cases} \dfrac{r}{N} & \text{for } i = n-N+1, n-N+2, \cdots, n \\ 0 & \text{for } i = 1, 2, \cdots, n-N \end{cases}$$

Figure 11 shows the government's optimal defensive budget allocations as a function of the probability that the terrorist is Type-III non-strategic $(1-q)$, when $r = 1$, $N = 1$, 2, 5, and 47, and $\lambda = 0.01$, 0.05 and 1, respectively. The results are different from that for Type-I non-strategic terrorist, which is shown in Figures 4, in that optimal defensive resource allocations are not sensitive to the values of $1-q$ unless $1-q = 1$.

Figure 12 shows the three total expected losses after the government has applied each of the three defensive resource allocation schemes discussed in Section 4.1 as a function of the probability that the terrorist is Type-III non-strategic (1-q) when $r = 1$, $N = 1$, 2, 5, and 47, and $\lambda = 0.01$, 0.05, and 1, respectively. We find that $T = 1$ for all ranges. The results are different from that for Type-I non-strategic terrorist when $N = 1$, 2, and 5, as shown in Figures 5 (a1-a3, b1-b3, c1-c3).

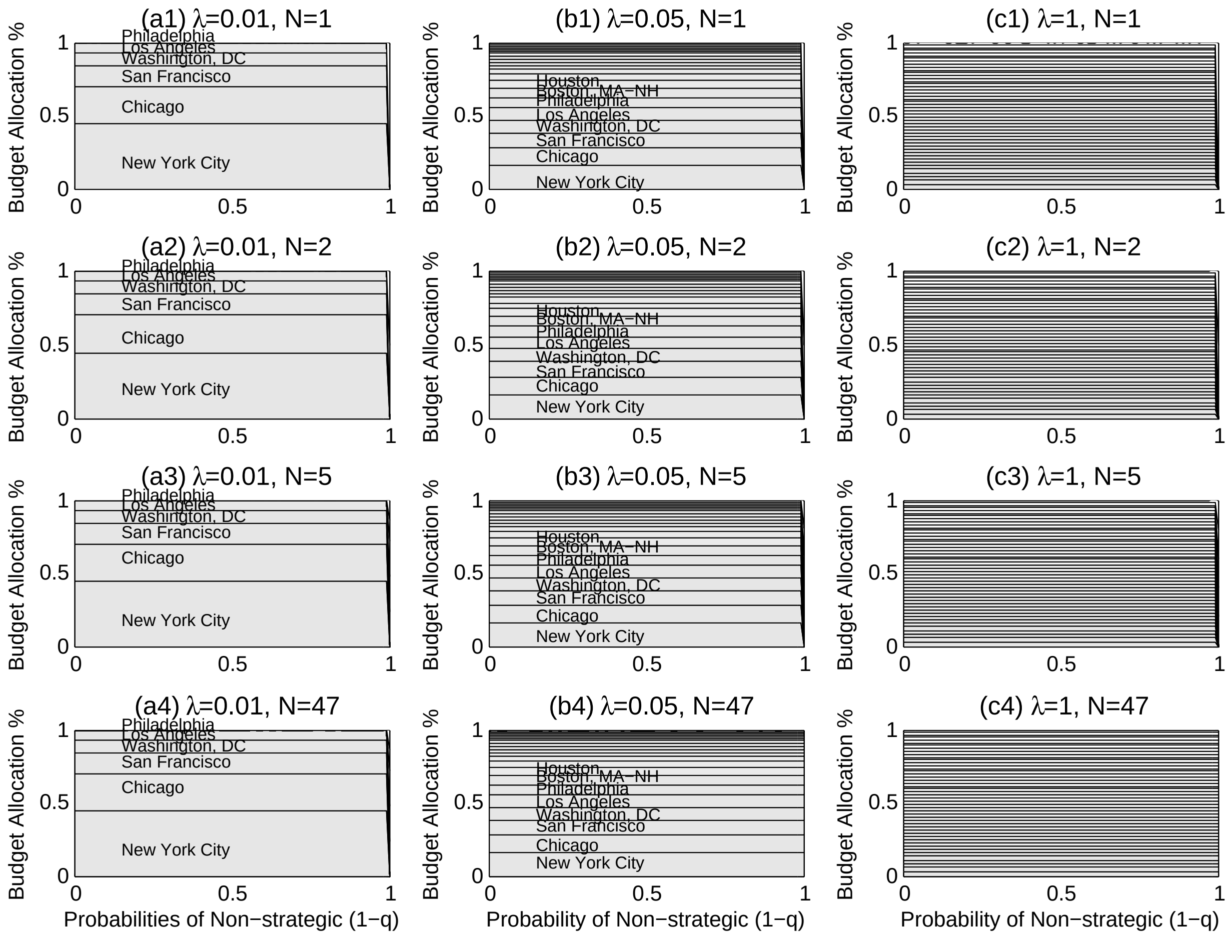

Figure 11: Optimal defensive budget allocations as a function of probability that the terrorist is Type-III non-strategic (1-q) when $r = 1$, $N = 1$, 2, 5, and 47, and $\lambda = 0.01$, 0.05 and 1, respectively.

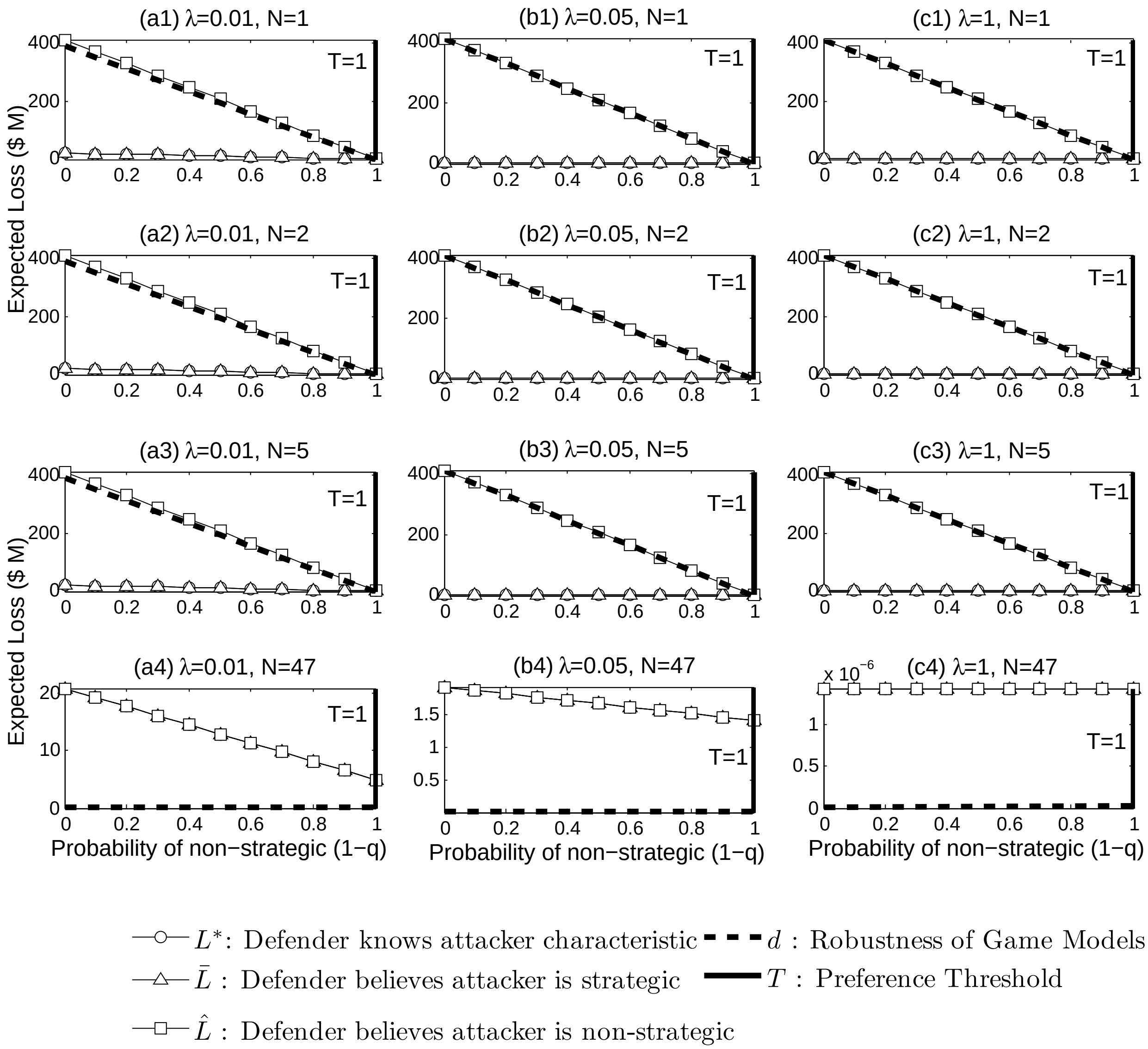

Figure 12: Government's three types of expected loss as a function of the probability that the terrorist is Type-III non-strategic (1-q) when $r = 1$, $N = 1$, 2, 5, and 47, and $\lambda = 0.01$, 0.05 and 1, respectively.

Figure 13 shows preference threshold (T) for game-theoretic models as a function of budget (C) when $r = 1$, $N = 1$, 2, 5, and 47, and $\lambda = 0.01$, 0.05, and 1, respectively, and a Type-III non-strategic terrorist is concerned. We find that T will always be approximately 1.

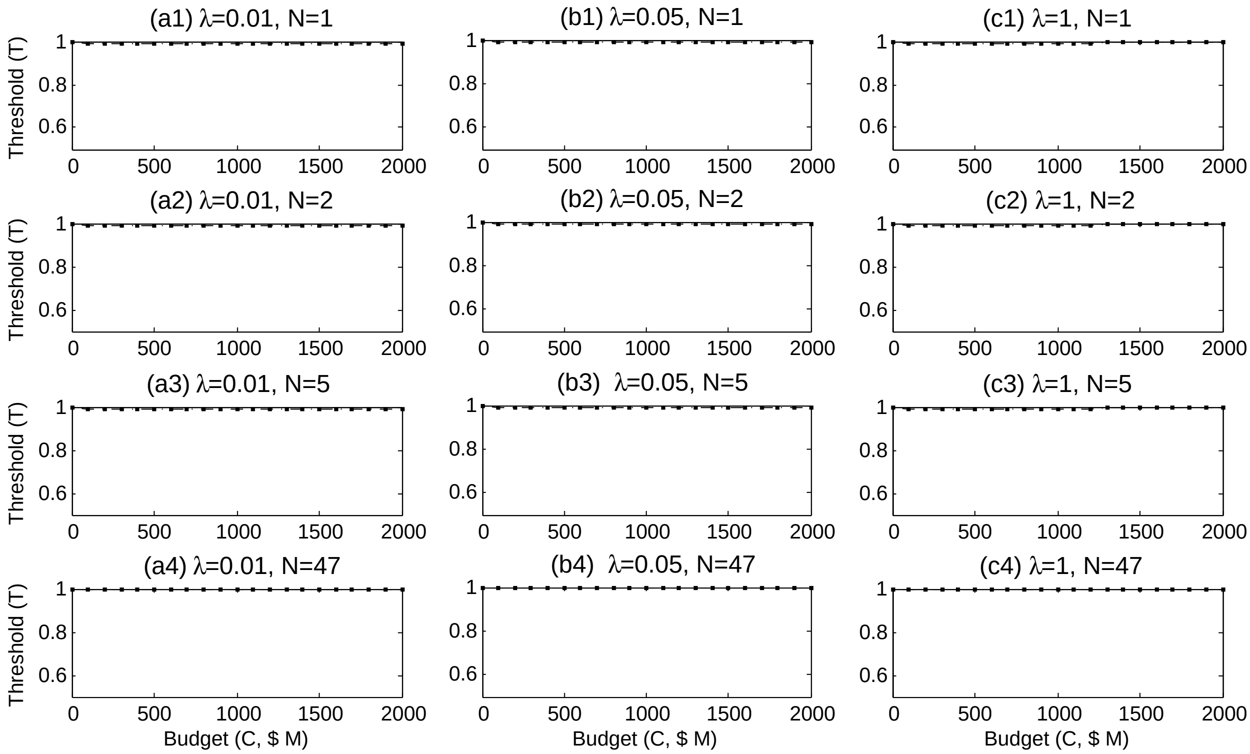

Figure 13: Preference threshold (T) for game-theoretic models as a function of budget (C) when $r = 1$, $N = 1$, 2, 5, and 47, and $\lambda = 0.01$, 0.05, and 1, respectively, and a Type-III non-strategic terrorist is concerned.

Figure 14 shows robustness measure (d) for game-theoretic models as a function of budget (C) and the probability that the terrorist is Type-III non-strategic when $r = 1$, $N = 1$, 2, 5, and 47, and $\lambda = 0.01$, 0.05, and 1, respectively. We find that d is always non-negative. Those findings suggest that game-theoretic models are highly preferred regardless of the value of λ and N.

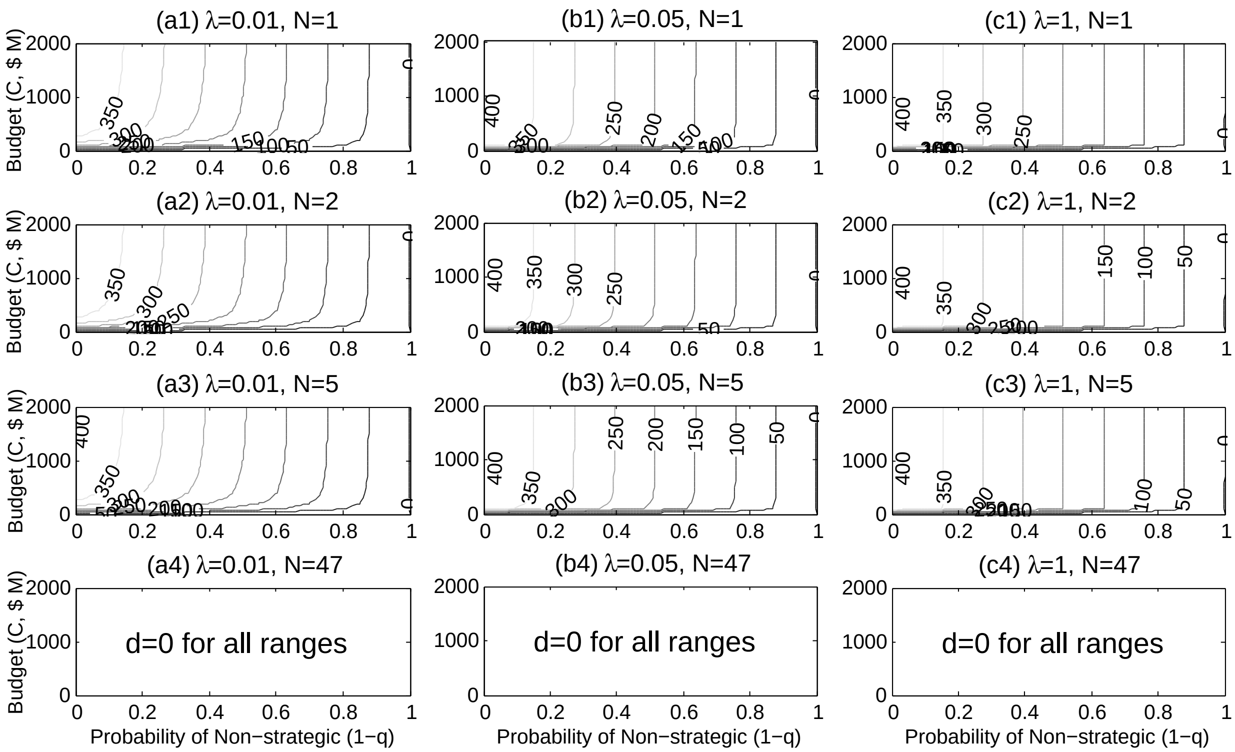

Figure 14: Robustness (d) for game-theoretic models as a function of budget (C) and the probability that the terrorist is Type-III non-strategic when $r = 1$, $N = 1$, 2, 5, and 47, and $\lambda = 0.01$, 0.05, and 1, respectively.

B.2.3 Type-IV Non-strategic Terrorist: attack probabilities are inversely proportional to the target valuations

In this subsection, we investigate a Type-IV non-strategic terrorist: attack probabilities are inversely proportional to target valuations. In other words,

$$\text{(Type-IV Non-Strategic Terrorist)} \qquad h'_i \propto \frac{1}{x_i} \Longrightarrow h'_i = \frac{r}{x_i \sum_{i=1}^{n} \frac{1}{x_i}}, \ \forall i = 1, 2, \cdots, 47. \quad (19)$$

Figure 15 shows optimal defensive budget allocations as a function of the probability that the terrorist is Type-IV non-strategic $(1-q)$ when $r=1$, $\lambda = 0.01, 0.05$, and 1, respectively. The results are different from that for Type-I non-strategic terrorist, which is shown in Figures 4, in that optimal defensive resource allocations are not sensitive to the values of $1-q$ unless $1-q$ approaches 1.

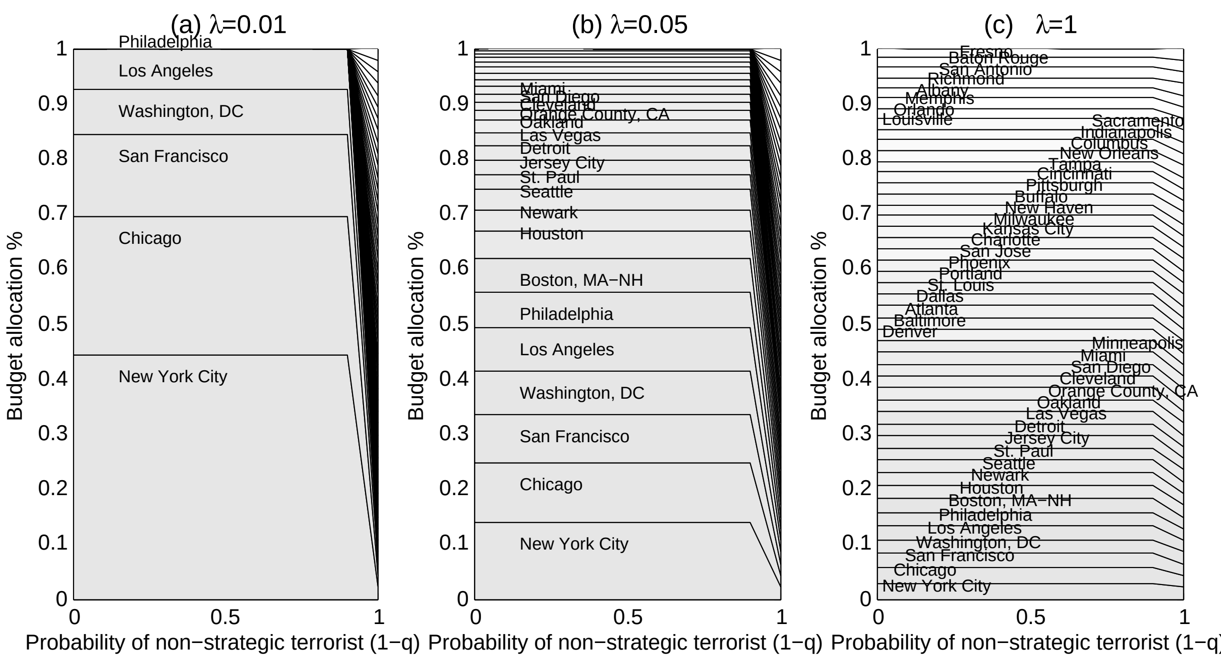

Figure 15: Optimal defensive budget allocations as a function of probability that the terrorist is Type-IV non-strategic (1-q) when $r = 1$, $\lambda = 0.01, 0.05$ and 1, respectively.

Figure 16 shows the expected losses for the government after the government has applied each of the three defensive resource allocation schemes discussed in Section 4.1 as a function of the probability that the terrorist is Type-IV non-strategic (1-q) when $r = 1$, $\lambda = 0.01, 0.05$, and 1, respectively. We find that $T = 1$ for all cases. The results are different from that for

Type-I non-strategic terrorist when $N = 1$, 2, and 5, as shown in Figures 5 (a1-a3, b1-b3, c1-c3).

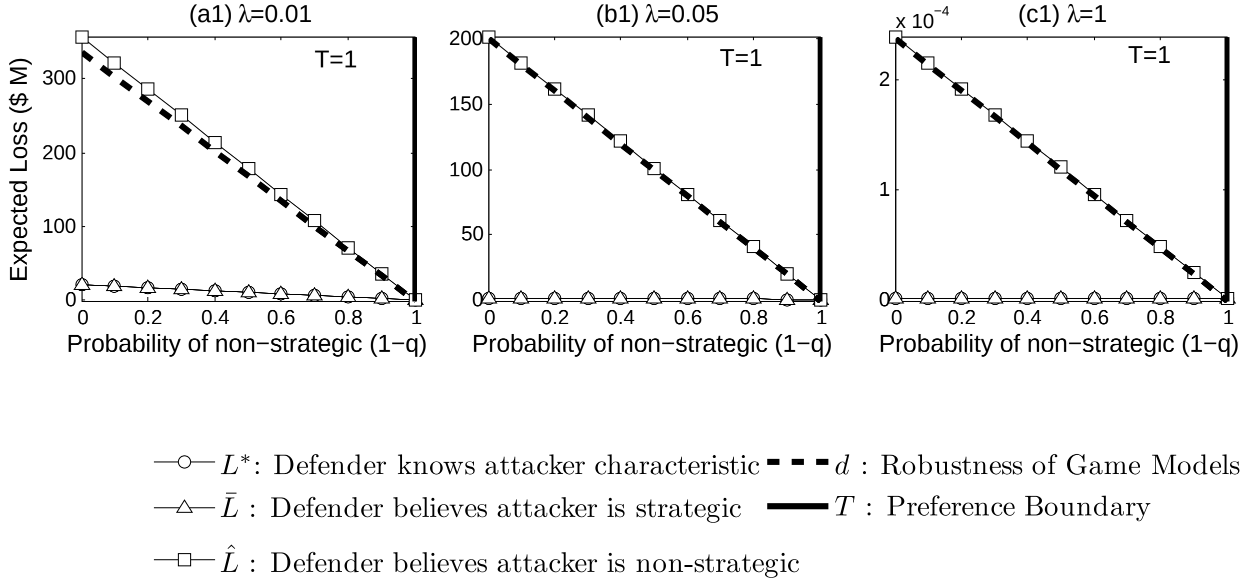

Figure 16: Government's three types of expected losses and corresponding preference threshold (T) and robustness measure (d) as a function of probability that the terrorist is Type-IV non-strategic $(1 - q)$ when $r = 1$, $\lambda = 0.01, 0.05$, and 1, respectively.

Figures 17 shows preference threshold (T) and robustness (d) for game-theoretic models as a function of budget (C) and the probability that the terrorist is Type-IV non-strategic when $\lambda = 0.01$, 0.05, and 1, respectively. Figures 17 (a1, b1, c1) show that T will always be approximately 1. Figures 17 (a2, b2, c2) show that d will always be non-negative. Those findings suggest that game-theoretic models are highly preferred regardless of the value of λ and N.

B.3 Example where T could be less than or greater than or equal to 0.5

Figure 18 shows that T could be less than or greater than or equal to 0.5 where we assume that all the target valuations are equal to \$16.78 M ($x_i = C/n =$ \$788.8/47 M $\forall i$, the actual numerical value does not matter). In particular, Figures 18 (a1-a3) show that $T < 0.5$ when λ is low and N is small; Figures 18 (b1-b3) show that $T = 0.5$ when λ is medium and N is small; and Figures 18 (a4, b4, c1-c4) show that $T = 1$ when λ is high or N is large.

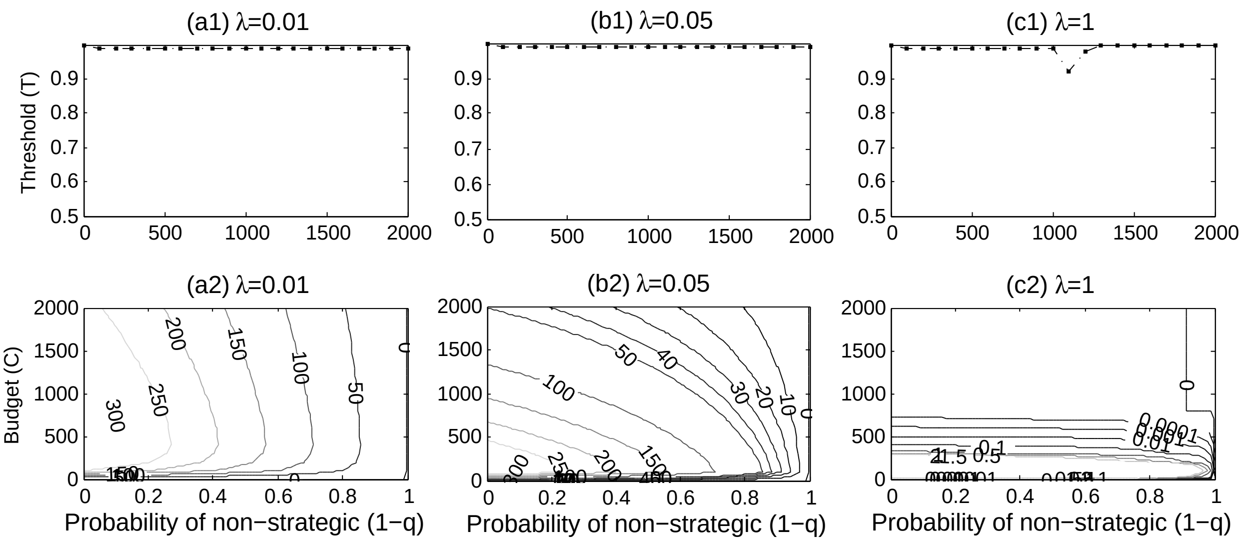

Figure 17: Preference threshold (T) and robustness (d) for game-theoretic models as a function of budget (C) and the probability that the terrorist is Type-IV non-strategic when $r = 1$, $\lambda = 0.01, 0.05$, and 1, respectively.

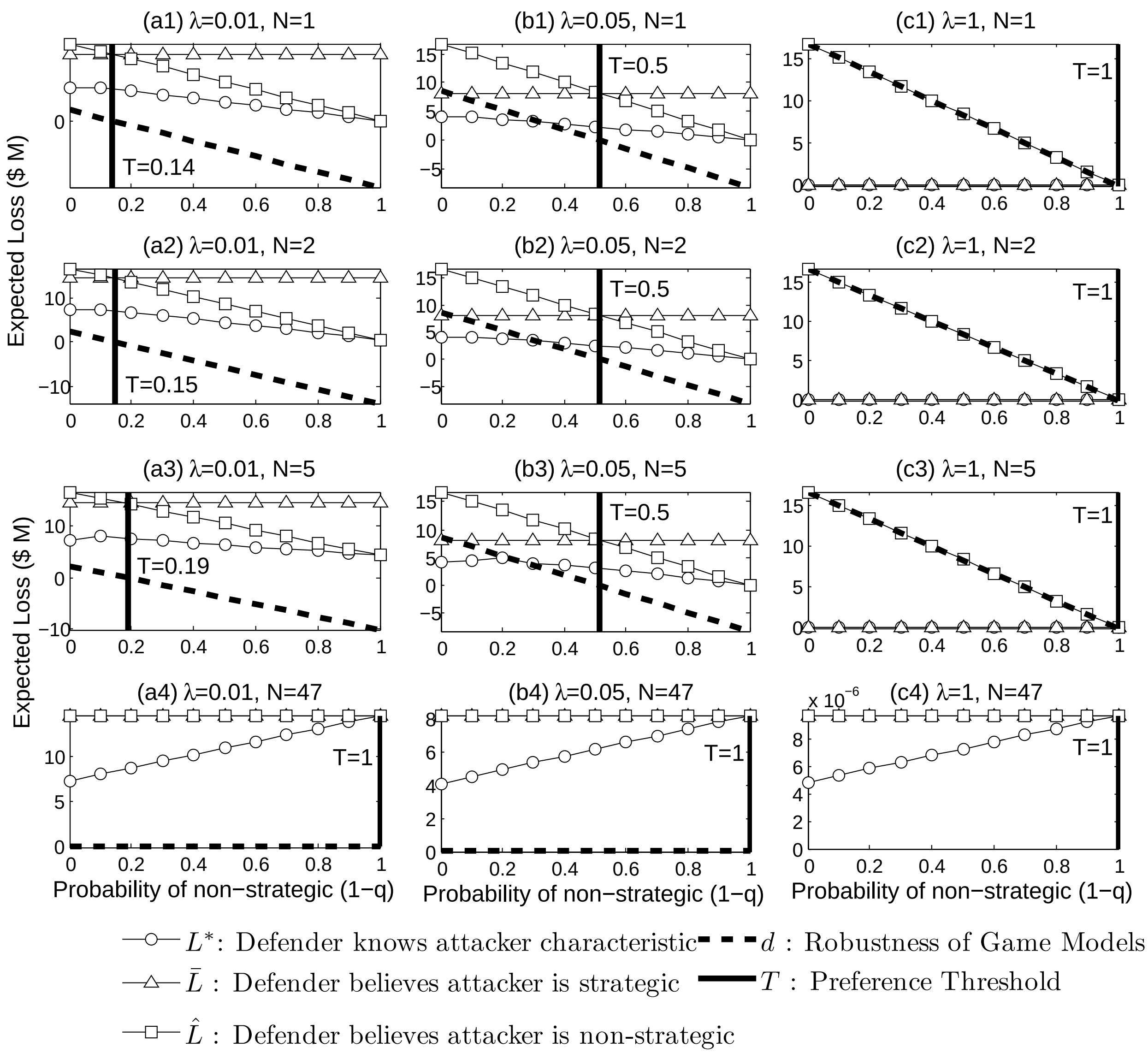

Figure 18: Example where T could be less than or greater than or equal to 0.5, when assuming that $x_i = C/N = \$16.78$ M.