



Innovative Applications of O.R.

Hybrid defensive resource allocations in the face of partially strategic attackers in a sequential defender–attacker game

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ABSTRACT

Many models have been developed to study homeland security games between governments (defender) and terrorists (attacker, adversary, enemy), with the limiting assumption of the terrorists being rational or strategic. In this paper, we develop a novel hybrid model in which a centralized government allocates defensive resources among multiple potential targets to minimize total expected loss, in the face of a terrorist being either strategic or non-strategic. The attack probabilities of a strategic terrorist are endogenously determined in the model, while the attack probabilities of a non-strategic terrorist are exogenously provided. We study the robustness of defensive resource allocations by comparing the government's total expected losses when: (a) the government knows the probability that the terrorist is strategic; (b) the government falsely believes that the terrorist is fully strategic, when the terrorist could be non-strategic; and (c) the government falsely believes that the terrorist is fully non-strategic, when the terrorist could be strategic. Besides providing six theorems to highlight the general results, we find that game models are generally preferred to non-game model even when the probability of a non-strategic terrorist is significantly greater than 50%. We conclude that defensive resource allocations based on game-theoretic models would not incur too much additional expected loss and thus more preferred, as compared to non-game-theoretic models.

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1. Introduction

Since September 11, 2001, homeland security in the United States has attracted hundreds of billions of dollars in expenditures (Fig. 1). The effectiveness of such large amounts of expenditure is frequently criticized as reflecting “pork-barrel politics”, in which funds are directed towards low-risk targets for political reasons (e.g., McLaughlin, 2002; O’Beirne, 2003; de Rugy, 2005). Moreover, even though the DHS has implemented a risk-based method in guiding grant allocations since 2006, risk-related measures are still limited (U.S. Government Accountability Office, 2008).

Operations Research (OR) methods are useful in studying counter-terrorism. For example, Harris (2004) discussed the role of probabilistic risk analysis, randomization, and game theory in defending against terrorist attacks. Brown et al. (2005) developed a two-sided optimization model for pre-localization of defense missile platforms considering adaptive adversaries. Kaplan and Kress (2005) modeled and analyzed the operational effectiveness of suicide-bomber-detector schemes in reducing the casualties caused by suicide-bombing. Brown et al. (2005) developed bi-level

and tri-level optimization models to study the defense strategies for critical infrastructures. Lin et al. (2009) built a M/G/1 queue to explore optimal scheduling policies for an antiterrorist surveillance system. Wein (2009) illustrated the close relationship between mathematical modeling and policy recommendations. Baveja and Wein (2009) evaluated and quantified the effectiveness of a two-finger, two-stage biometric strategy for the US-VISIT program using OR methods and Stakelberg game formulations. Jain et al. (2010) developed computer-aided randomized patrol planning systems for airplane transportation security. Recently, Kaplan (2010) employed queueing theory and Markov processes to study how undercover intelligence agents infiltrate and interdict terrorist plots.

One specific area of application of OR methods in homeland security is terrorism risk analysis. Traditional methods of decision and risk analysis do not explicitly take into account the ability of intelligent adversaries to adapt to defenses, and therefore, may overestimate the effectiveness of defensive measures. In contrast, while game theory has been widely applied in counter-terrorism analysis (Azaiez and Bier, 2007; Hausken, 2008; Sandler and Siqueira, 2009; Haphuriwat and Bier, 2011) and other strategic decision-making scenarios (Hausken and Zhuang, 2012), game-theoretic methods have been criticized as attributing excessive levels of knowledge and computational ability to potential terrorists (i.e., assuming players to be fully rational), and

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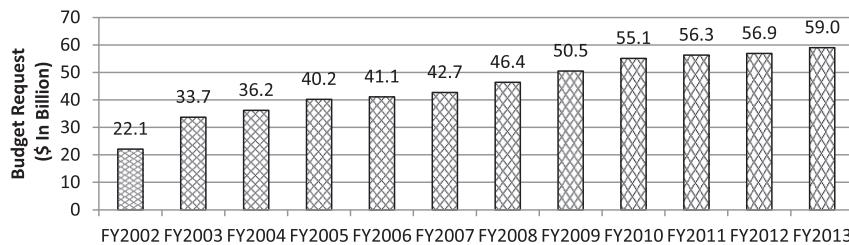


Fig. 1. Annual budget requests for the Department of Homeland Security from Fiscal Year 2002 to 2013. Source: U.S. Department of Homeland Security (2012).

frequently recommending insufficient “hedging” (i.e., protecting against only the few most detrimental attack strategies). Similarly, the justification of defending low-risk targets as in “pork-barrel politics” from a game-theoretic perspective may depend critically on the assumption of game theory about terrorist’s rationality, since irrational terrorists could attack low-risk targets even when that may not be optimal in a game-theoretic sense. Probably as a result, game theory has been less frequently mentioned in risk analysis in recent years (Hall, 2009).

In fact, game theory, and decision and risk analysis complement each other. Decision and risk analysis can model the probable outcomes of a game and evaluate the payoffs of those outcomes. On the other hand, instead of viewing adversary’s decisions as random variables, game-theoretic formulation can help endogenously determine adversary’s decisions (Cox, 2009).

One key difference between terrorism and natural disasters is that terrorists are intelligent and adaptive while natural disasters are not. As a result, a certain government’s optimal strategies in the face of terrorism may significantly differ from the strategies adopted against natural disasters (Powell, 2007; Zhuang and Bier, 2007; Golany et al., 2009; Levitin and Hausken, 2009). Intelligence plays a key role in informing the government of whether, and how much, the terrorist is strategic (Kress and Szechtman, 2009). In particular, Kaplan et al. (2010) found that when the government’s intelligence is poor, it would be easier for strategic insurgents to survive attacks by the government. Overall, decision and risk analysis is useful in devising strategies to deal with natural disasters or non-adaptive threats, while game theory is powerful when coping with terrorism or adaptive threats but usually strongly assumes that the terrorists are fully rational or strategic.

This paper pioneers a novel hybrid approach by integrating the game-theoretic and non-game-theoretic defense allocation models using an adjustable parameter to represent the probability that the terrorist might behave strategically (i.e., will adapt to the observed defense). As a first solid step toward tackling this important problem, we assume that the target government knows the probability of the attacker being strategic and has complete information about the probabilities that each target will be attacked by a non-strategic attacker. Note that the main difference between a strategic and non-strategic attacker lies in their responses to the defender’s allocation decision. On the other hand, the main distinction between game-theoretic and non-game-theoretic models is that game-theoretic models take into account the attacker’s response to the defender’s allocation decision while in non-game-theoretic models, the attacker’s decision is exogenously determined and is not a function of the defender’s allocation decision. As our results will show in Section 4, game-theoretic models are often preferred to non-game-theoretic models, since game-theoretic models often incur lower expected loss for the defender than non-game-theoretic models.

It is instructive to compare non-adaptive threats such as natural disasters and terrorism. Defensive resource allocations against a strategic attacker has been extensively studied in game theory. In particular, Colonel Blotto games were designed to tackle this type of problem (e.g., Shubik and Weber, 1981; Roberson, 2006). One

difference between the game-theoretic portion of our model and traditional Colonel Blotto games is in that our model is sequential while in Colonel Blotto games decisions are made simultaneously. We note that extension of Colonel Blotto games to the realm of sequential, nonzero-sum games has been recently carried out by Powell (2009), which did not consider the scenario that the attacker is partially strategic.

Allowing multiple behavioral types of one player has been pioneered by Kreps et al. (1982). There exist several differences between Kreps et al. (1982) and the current paper. First, Kreps et al. (1982) only allowed the player to take on the alternative type with a very small probability while in the current model the attacker could be non-strategic with any probability between 0 and 1. Second, both behavioral types in the model from Kreps et al. (1982) are fully strategic and play best responses to the other player’s moves while in the current model if the attacker is non-strategic, there is no decision to make and thus his moves are not influenced by the defender’s decision at all.

The remainder of the paper is structured as follows. Section 2 formulates the model and discusses data sources. Section 3 provides analytic results, an algorithm, and a numerical example, investigating one particular type of non-strategic attack probabilities: evenly distributed to top N valuable targets. Section 4 introduces two types of false beliefs, defines robustness, and conducts both one-way and two-way sensitivity analyses to investigate the robustness of game-theoretic models. Finally, Section 5 provides conclusion and future research directions. Appendix A contains the proofs of the six theorems for this paper, and Appendix B presents illustrations to Theorem 2 and robustness analysis and sensitivity analyses for three other types of non-strategic terrorist as well as identifying the optimal defensive resource allocations for them.

2. Notation, assumptions, and model formulation

2.1. Notation

We use the following notation throughout the paper:

- $q \in [0, 1]$ and $1 - q$: Probabilities that the terrorist is strategic and non-strategic, respectively.
- n : Number of targets in the system.
- $c_i \geq 0$: Government’s defensive resource allocation to target i , for $i = 1, 2, \dots, n$.
- $c \equiv (c_1, c_2, \dots, c_n)$.
- C : Total budget of the defensive resources. That is,

$$C = \sum_{i=1}^n c_i \quad (1)$$

- J : Set of defended targets. That is, $J \equiv \{i : c_i > 0; i = 1, 2, \dots, n\}$
- r : Total probabilities of attacks for both the strategic and non-strategic attacker.
- $h_i(c)$: Endogenously-determined probability that a strategic terrorist will attack target i , for $i = 1, 2, \dots, n$. We have $h_i(c) \geq 0$, and $\sum_{i=1}^n h_i(c) = r$.

- P : Set of targets that a strategic terrorist attack with strictly positive probabilities. That is,

$$P \equiv \{i : h_i(c) > 0; i = 1, 2, \dots, n\} \quad (2)$$

- I_i : Indicator function for the event $\{h_i(c) > 0\}$, for $i = 1, 2, \dots, n$. That is,

$$I_i = \begin{cases} 1 & \text{if } h_i(c) > 0 \\ 0 & \text{if } h_i(c) = 0 \end{cases} \quad (3)$$

- $I \equiv (I_1, I_2, \dots, I_n)$.

- $\hat{h}_i(c)$: Strategic terrorist's best response function for target i , for $i = 1, 2, \dots, n$, as a function of the observed defensive resource allocations to all targets.

- h'_i : Exogenously-determined probability that a non-strategic terrorist will attack target i , for $i = 1, 2, \dots, n$. We have $h'_i \geq 0$, and $\sum_{i=1}^n h'_i = r$. We assume that the government knows about h'_i , $\forall i = 1, 2, \dots, n$.

- $h(c) \equiv (h_1(c), h_2(c), \dots, h_n(c))$; $\hat{h}(c) \equiv (\hat{h}_1(c), \hat{h}_2(c), \dots, \hat{h}_n(c))$; $h' \equiv (h'_1, h'_2, \dots, h'_n)$.

- $\lambda_i \geq 0$: Cost-effectiveness of defense for target i , for $i = 1, 2, \dots, n$.

- $\lambda = \lambda_i, \forall i = 1, 2, \dots, n$. In Sections 3 and 4, we consider homogeneous λ .

- $p_i(c_i)$: Success probability of an attack on target i , as a function of the defensive resource allocation to target i , c_i , for $i = 1, 2, \dots, n$. We assume that $p_i(c_i)$ is continuous, strictly convex and decreasing in c_i (i.e., $\frac{dp_i(c_i)}{dc_i} < 0, \frac{d^2 p_i(c_i)}{dc_i^2} \geq 0$) and decreasing in λ (i.e., $\frac{dp_i(c_i)}{d\lambda} < 0$). This implies that the more defensive investment for a target, the less likely is that target to suffer from a successful attack, with decreasing marginal effectiveness.

- x_i : Valuation of target i , for $i = 1, 2, \dots, n$. As required by the proof of continuity (see Remark after Lemma 1 in Appendix A.1) and the alternative formulation in Section 2.2, we assume that the government and the terrorist have the same target valuations.

- $L_i(c_i, h_i, h'_i, q)$: Total expected loss of target i , for $i = 1, 2, \dots, n$ to the government. We have:

$$L_i(c_i, h_i, h'_i, q) = q h_i p_i(c_i) x_i + (1 - q) h'_i p_i(c_i) x_i \quad (4)$$

- $L(c, h, h', q)$: Total expected loss to the government. We have

$$\begin{aligned} L(c, h, h', q) &= \sum_{i=1}^n L_i(c_i, h_i, h'_i, q) \\ &= q \underbrace{\sum_{i=1}^n h_i p_i(c_i) x_i}_{[\text{strategic term}]} + (1 - q) \underbrace{\sum_{i=1}^n h'_i p_i(c_i) x_i}_{[\text{non-strategic term}]} \end{aligned} \quad (5)$$

The total expected loss L is a weighted average between expected loss caused by the strategic and non-strategic terrorists. For both terms, the total expected loss equals the summation of expected losses for target i , which equals the multiplication of h_i , $p_i(c_i)$, and x_i . Note that h_i , $p_i(c_i)$, and x_i correspond to threat, vulnerability, and consequences, in a standard risk analysis for terrorism adopted by the DHS (Cox, 2009).

- \bar{c}_i and \hat{c}_i : Government's defensive resource allocation to target i , for $i = 1, 2, \dots, n$, when the government falsely believes that the terrorist is fully strategic (i.e., $q = 1$, false belief I) and non-strategic (i.e., $q = 0$, false belief II), respectively.

- $\bar{c} \equiv (\bar{c}_1, \dots, \bar{c}_n)$; $\hat{c} \equiv (\hat{c}_1, \dots, \hat{c}_n)$.

- $\bar{L}(\bar{c}, h, h', q)$ and $\hat{L}(\hat{c}, h, h', q)$: Total expected loss when the government employs \bar{c} holding false belief I, and employs \hat{c} holding false belief II, respectively.

- T : Preference threshold for game-theoretic models. As Theorem 6d in this paper will show, we have $\bar{L}(\bar{c}, \hat{h}(\bar{c}), h', q) \leq \hat{L}(\hat{c}, \hat{h}(\hat{c}), h', q)$ if and only if $1 - q \leq T$.

- $d \equiv \hat{L}(\hat{c}, h, h', q) - \bar{L}(\bar{c}, h, h', q)$: Robustness of the game-theoretic model as a difference between the costs of two false beliefs.

When $d > 0$, game-theoretic models are preferred to non-game theoretic models and vice versa.

- A_i : Total attack probability (if $q \in (0, 1]$, and $J \neq P$), attack probability of a strategic terrorist (if $q \in (0, 1]$, and $J = P$), or normalized marginal risk reduction from an attack by a non-strategic terrorist (if $q = 0$) for target i , for $i = 1, 2, \dots, n$. That is,

$$A_i = \begin{cases} q h_i + (1 - q) h'_i & \text{if } q \in (0, 1], \text{ and } J \neq P \\ q h_i & \text{if } q \in (0, 1], \text{ and } J = P \\ -\frac{dp_i(c_i)}{dc_i} \frac{h'_i}{p_i(c_i)} & \text{if } q = 0 \end{cases} \quad (6)$$

- $R_i \equiv A_i p_i(c_i) x_i$: Total expected loss (if $q \in (0, 1]$, and $J \neq P$), expected loss from an attack by a strategic terrorist (if $q \in (0, 1]$, and $J = P$), or reduction in expected loss from an attack by a non-strategic terrorist (if $q = 0$) for target i , for $i = 1, 2, \dots, n$. That is,

$$R_i \equiv A_i p_i(c_i) x_i = \begin{cases} [q h_i + (1 - q) h'_i] p_i(c_i) x_i & \text{if } q \in (0, 1], \text{ and } J \neq P \\ q h_i p_i(c_i) x_i & \text{if } q \in (0, 1], \text{ and } J = P \\ -\left[\frac{dp_i(c_i)}{dc_i} \frac{h'_i}{p_i(c_i)} \right] p_i(c_i) x_i & \text{if } q = 0 \end{cases} \quad (7)$$

2.2. Assumptions and the model

Following Hao et al. (2009), we assume that the government is strategic, while the terrorist might be either strategic or non-strategic, with probabilities q and $1 - q$, respectively. Following Berman and Gavious (2007), Powell (2007), and Bier et al. (2008), we model the strategic interaction between the government and the terrorist as a sequential game, where the government moves first by distributing a total budget of C among n targets, such that $\sum_{i=1}^n c_i = C$; a strategic terrorist then observes the defense distribution $c = (c_1, c_2, \dots, c_n)$, and attacks target i with probability $h_i(c)$, for $i = 1, 2, \dots, n$, such that $\sum_{i=1}^n h_i(c) = r$. We assume that there is no secrecy or deception in government's defensive resource allocations in contrast to a study by Zhuang et al. (2010). We also did not consider the possibility of deterring the adversary although it is achievable at least within the context of illegal intrusion (Wang and Zhuang, 2011). We assume that the strategic terrorist chooses a target corresponding to the maximal expected loss $p_i(c_i) x_i$. When the maximal expected loss caused by a strategic terrorist is the same for two or more targets, we assume that the strategic terrorist attacks those targets with equal probabilities.

The objective of the terrorist is to maximize the total expected loss to the government $L(c, \hat{h}(c), h', q)$ by choosing target(s) to attack (we acknowledge that the attacker may have other objectives such as to maximize expected fatalities and business interruption, which could be modeled as a value model; see Keeney, 2007, Keeney and von Winterfeldt, 2011):

$$\max_{h_i(c) \in \mathbb{V}} \left\{ r \sum_{i=1}^n h_i(c) p_i(c_i) x_i \right\} \quad (8)$$

From the terrorist's objective function in (8), we can derive the terrorist's best response function as follows:

$$\hat{h}_i(c) = \begin{cases} \frac{r}{\|P\|} & \text{if } i \in P = \{i : h_i(c) > 0\} = \{i : p_i(c_i) x_i = \max_{j=1, \dots, n} \{p_j(c_j) x_j\}\} \\ 0 & \text{otherwise} \end{cases} \quad (9)$$

where $\|P\|$ is the cardinality of the set P . By contrast, a non-strategic (irrational, bounded-rational) terrorist is assumed to attack target i with exogenously-determined probability $h'_i \geq 0$ for $i = 1, 2, \dots, n$, such that $\sum_{i=1}^n h'_i = r$ and does not have an objective function to maximize. Non-strategic terrorist's behavior could result from erroneous choices (Zhuang, 2010).

The objective of the government is to minimize the total expected loss by allocating defensive resources considering the terrorist's best response $\hat{h}(c)$:

$$\begin{aligned} \min_c L(c, h(c), h', q) &= \sum_{i=1}^n q h_i(c) p_i(c_i) x_i + \sum_{i=1}^n (1-q) h'_i p_i(c_i) x_i \\ \text{s.t. } \sum_{i=1}^n c_i &= C, c_i \geq 0 \end{aligned} \quad (10)$$

If the strategic terrorist has an attack probability distribution over targets with maximum expected loss (his best response function $h = \hat{h}(c)$ defined in Eq. (9) results in one such distribution), we can alternatively formulate the objective function (10) as follows:

$$L(c, \hat{h}(c), h', q) = qr \max_{i=1,\dots,n} \{p_i(c_i)x_i\} + (1-q) \sum_{i=1}^n h'_i p_i(c_i) x_i \quad (11)$$

Definition 1. We call a pair of strategies, (h^*, c^*) , a subgame perfect Nash equilibrium (or equilibrium) if and only if

$$h^* = \hat{h}(c^*) \quad (12)$$

and

$$c^* = \operatorname{argmin}_c L(c, \hat{h}(c), h', q) \quad (13)$$

2.3. Data sources

Following Bier et al. (2008), Hao et al. (2009), and Hu et al. (2011), we use data for expected property loss and expected fatalities for the 47 US urban areas in Willis et al. (2005) as the two possible sets of target valuation, which are shown in columns 3 and 8 in Table 1 (sorted by expected property losses in descending order). Table 1 also shows the defense budget allocated to those 47 urban areas from the Office of Grants and Training in FY 2004 and we use the total FY 2004 Urban Area Security Initiative (UASI)

Grant Allocations (\$675 M) as the total available defense budget C in our model. FY 2005–2010 data are also available as shown in Fig. 2.

3. Solution

3.1. Analytic results and illustrations

Theorem 1. Subgame perfect Nash equilibrium exists and is unique for the sequential game defined by (12) and (13).

Theorem 2. Let (h^*, c^*) be a pair of strategies with corresponding R_i^* , A_i^* and J^* .

- (i) If $h^* = \hat{h}(c^*)$ and $R_i^* = W^* > 0$ for all $i \in J^*$, then (h^*, c^*) qualifies to be a subgame perfect Nash equilibrium.
- (ii) $R_i^* \leq W^*$ for all $i \notin J^*$.

Remark 1. With definitions of (6), (7) of A_i and R_i , Theorem 2 implies that the government desires to equalize: (a) the total expected losses for all the defended targets (set J^*), when $q \in (0, 1]$ and $J^* \neq P^*$; (b) the expected losses caused by the strategic terrorist for all the defended targets, when $q \in (0, 1]$ and $J^* = P^*$; and (c) the marginal reductions in expected loss for all the defended targets, when $q = 0$. Moreover, such losses or loss reductions are larger than those for all undefended targets. We acknowledge that Dresher (1961) provides similar principles for simultaneous-move games (p. 127) without considering the possibility of the attacker being partially rational (strategic) and without allowing for the outcome of conflicts to be stochastic. In contrast, we allow the attacker to be non-strategic. When the attacker is fully non-strategic, no-soft-spot principle might not apply here contingent upon the form of success probability function of an attack. Note that the equilibrium solution is unique guiding the defender to proportionally distribute resources to most valuable targets when facing a fully strategic attacker and distribute resources to targets attracting the non-strategic attacker.

Table 1
Expected property losses and FY 2004 UASI budget allocations for 47 urban areas.

#	Urban areas	x_i^a	FY 2004 UASI Allocations (\$) ^b	#	Urban areas	x_i^a	FY 2004 UASI Allocations (\$) ^b
1	New York City	413.0	47,007,064	24	St. Louis	2.1	10,785,053
2	Chicago	115.0	34,142,222	25	Portland	2.0	8,161,143
3	San Francisco	57.0	26,481,275	26	Phoenix	1.9	12,200,204
4	Washington, D.C.	36.0	29,301,502	27	San Jose	1.7	9,982,442
5	Los Angeles-Long Beach	34.0	40,404,595	28	Kansas City	1.1	7,404,955
6	Philadelphia, PA-NJ	21.0	23,078,759	29	Milwaukee	1.1	13,295,646
7	Boston, MA-NH	18.0	19,131,723	30	New Haven	1.1	10,177,999
8	Houston	11.0	19,955,485	31	Charlotte	1.1	9,632,961
9	Newark	7.3	15,054,101	32	Buffalo	1.0	10,095,856
10	Seattle-Bellevue	6.7	16,516,007	33	Pittsburgh	1.0	11,978,479
11	Jersey City	4.4	17,112,311	34	Cincinnati	0.9	12,751,270
12	Detroit	4.2	13,754,597	35	Tampa	0.9	9,275,359
13	Las Vegas	4.1	10,531,025	36	New Orleans	0.8	7,152,827
14	Oakland	4.0	7,854,691	37	Indianapolis	0.7	8,707,544
15	Orange County (Santa Ana-Anaheim)	3.7	25,404,219	38	Columbus	0.7	10,151,880
16	Cleveland	3.0	10,460,465	39	Sacramento	0.7	8,024,926
17	San Diego	2.8	10,479,947	40	Louisville	0.6	8,987,662
18	Minneapolis-St. Paul	2.7	20,108,247	41	Orlando	0.6	8,765,211
19	Miami	2.7	19,146,642	42	Memphis	0.5	10,067,477
20	Denver	2.5	8,646,361	43	Albany	0.4	6,853,481
21	Baltimore	2.4	15,918,745	44	Richmond	0.4	6,543,378
22	Atlanta	2.3	10,744,248	45	San Antonio	0.4	6,301,153
23	Dallas	2.1	12,198,661	46	Baton Rouge	0.2	7,193,806
				47	Fresno	0.2	7,076,396
					Total	788.7	675,000,000

x_i = Expected property losses (\$ million).

^a Willis et al. (2005).

^b U.S. Department of Homeland Security (2012).

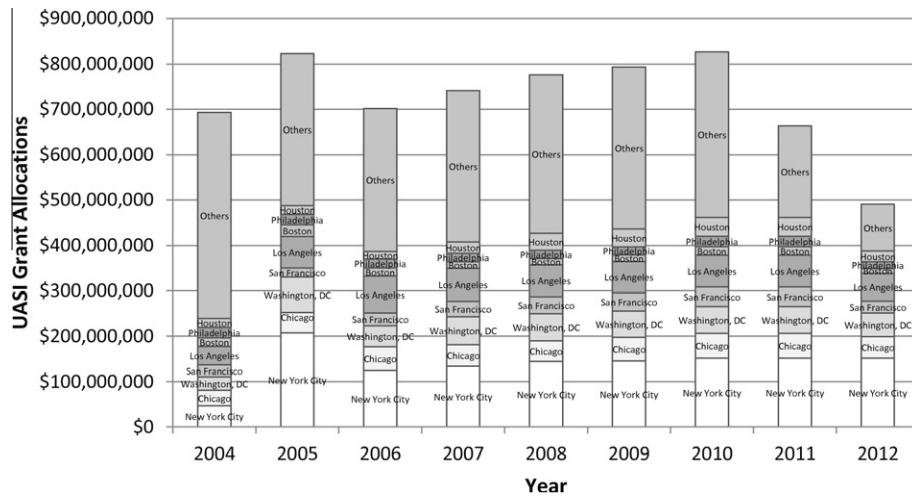


Fig. 2. UASI grant allocations to U.S. urban areas from 2004 to 2012. Sources: U.S. Department of Homeland Security (2012).

For the rest of the paper, following Bier et al. (2008), Hao et al. (2009) and Golalikhani and Zhuang (2011), we consider an exponential form of success probability:

$$p_i(c_i) = e^{-\lambda_i c_i}, \quad \forall i = 1, 2, \dots, n \quad (14)$$

where λ_i is the cost-effectiveness of defense for target i . For simplicity, we only consider homogeneous $\lambda_i = \lambda \forall i = 1, 2, \dots, n$.

A number of other differences exist between Dresher (1961) and the current paper. In Dresher (1961), no two targets have the same valuation while the scenario that multiple targets sharing same target valuation is allowed in our paper. As a consequence, if a target is not defended, it would not be attacked in Dresher (1961) while it can still be attacked in our case. Dresher (1961) assumes that if more units of defense are allocated to a particular target than units of attack, the attack would not succeed for sure, which might not be a realistic assumption. On the other hand, we assume that the success probability of an attack is a function of defense to that target and it is impossible to achieve perfect protection with finite resources in practice. Dresher (1961) assumes that the attacker also has a budget constraint and the outcome of the battle depends on the comparison of the amount of defense forces and attack forces. In the current paper, the total attack probability r corresponds to the attacker's total budget while the attack probability $h_i(c)$ corresponds to the expected value of the ratio of the budget allocated to target i to the total budget but the actual ratio will be either 0 or 1.

We use the data set introduced in Section 2.3 to illustrate Theorem 2. In particular, Table 2 provides three illustrations corresponding to the three conditions in Eqs. (6) and (7). We provide Appendix A.3 to accompany Table 2, where three illustrations for three conditions in Theorem 2 are shown to be at optimality. For illustrative purposes, we let $\lambda = 0.01$, $r = 1$ and $h' = \{0.5r, 0.5r, 0, \dots, 0\}$ in Table 2. (We consider more general parameter values of λ and h' in Sections 3.3, 4.1 and 4.2.)

For Illustration 1 in Table 2, we have $q = 0.5 \in (0, 1]$, and $P^* = \{3, 4, 5\} \neq \{1, 2, 3, 4, 5\} = J^*$. Therefore, according to the first condition in Eq. (7) we have the total expected loss for target $i: R_i^* = [qr \frac{l_i}{\|P^*\|} + (1-q)h'_i] p_i(c_i^*) x_i = 4.09 = W^*$, for $i = 1, 2, \dots, 5$, and $R_6^* = R_7^* = \dots = R_{47}^* = 0.00 < 4.09 = W^*$, which is consistent with Theorem 2. For Illustration 2, we have $q = 0.8 \in (0, 1]$, $P^* = \{1, 2, 3, 4, 5, 6\} = J^*$. Therefore, according to the second condition in Eq. (7) we have the expected loss for target i caused by the strategic terrorist: $R_i^* = (qr \frac{l_i}{\|P^*\|}) p_i(c_i^*) x_i = 2.79 = W^*$, for $i = 1, 2, \dots, 6$, and $R_7^* = R_8^* = \dots = R_{47}^* = 0.00 < 2.79 = W^*$, which is consistent with Theorem 2. For Illustration 3, we have $q = 0$. Therefore, according to the third condition in Eq. (7) we have the marginal reduction in expected loss caused by the non-strategic terrorist for target $i: R_i^* = h'_i \frac{dp_i(c_i)}{dc_i} x_i = 0.04 = W^*$ for $i = 1, 2$, and $R_3^* = R_4^* = \dots = R_{47}^* = 0.00 < 0.04 = W^*$.

Table 2
Three illustrations for Theorem 2.

#	x_i	Illustration 1: $q = 0.5$				Illustration 2: $q = 0.8$				Illustration 3: $q = 0$			
		c_i^*	$p_i(c_i^*)x_i$	R_i^*	L_i^*	c_i^*	$p_i(c_i^*)x_i$	R_i^*	L_i^*	c_i^*	$p_i(c_i^*)x_i$	R_i^*	L_i^*
1	413.0	322.85	16.36	4.09	4.09	298.41	20.89	2.79	4.87	400.46	7.53	0.04	3.77
2	115.0	194.99	16.36	4.09	4.09	170.56	20.89	2.79	4.87	272.57	7.53	0.04	3.77
3	57.0	84.26	24.54	4.09	4.09	100.37	20.89	2.79	2.79	0.00	57.00	0.00	0.00
4	36.0	38.31	24.54	4.09	4.09	54.42	20.89	2.79	2.79	0.00	36.00	0.00	0.00
5	34.0	32.59	24.54	4.09	4.09	48.71	20.89	2.79	2.79	0.00	34.00	0.00	0.00
6	21.0	0.00	21.00	0.00	0.00	0.52	20.89	2.79	2.79	0.00	21.00	0.00	0.00
7	18.0	0.00	18.00	0.00	0.00	0.00	18.00	0.00	0.00	0.00	18.00	0.00	0.00
8	11.0	0.00	11.00	0.00	0.00	0.00	11.00	0.00	0.00	0.00	11.00	0.00	0.00
:	:	:	:	:	:	:	:	:	:	:	:	:	:
47	0.2	0.00	0.20	0.00	0.00	0.00	0.20	0.00	0.00	0.00	0.20	0.00	0.00
Total	788.7	675.00	240.06	20.45	20.45	675.00	238.04	16.71	20.89	675.00	275.76	0.08	7.53
J^*	{1,2,3,4,5}				{1,2,3,4,5,6}				{1,2}				
P^*	{3,4,5}				{1,2,3,4,5,6}				\emptyset				
W^*	4.09				2.79				0.04				

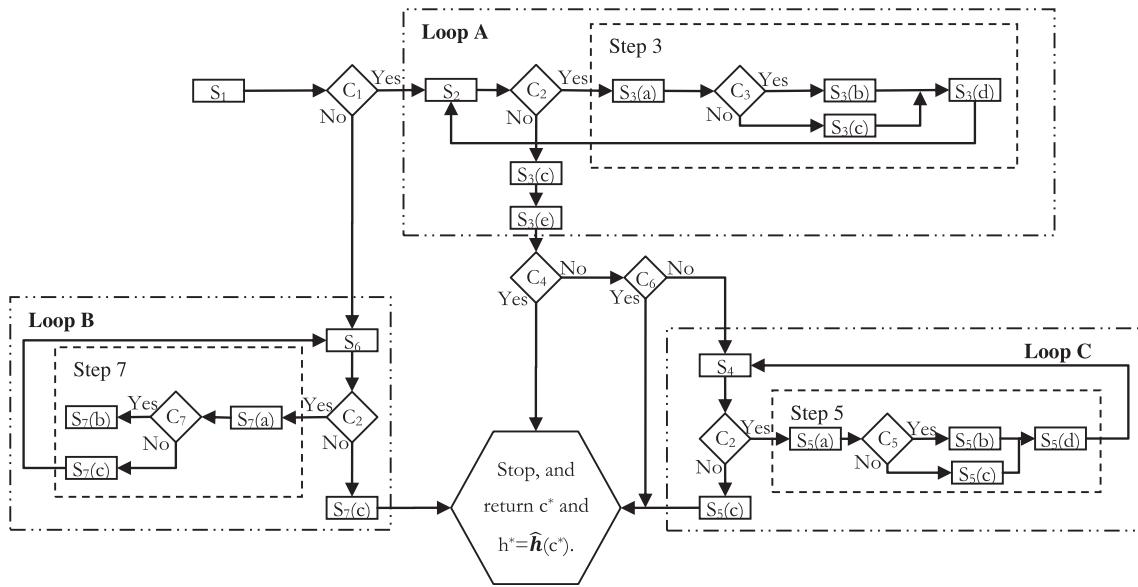


Fig. 3. Illustrative diagram for the algorithm.

From the numerical illustrations, we observe that the set of targets to be attacked by the strategic terrorist (set P^*) is always a subset of defended targets (set J^*) (i.e., $P^* = \{3, 4, 5\} \subseteq \{1, 2, 3, 4, 5\} = J^*$, $P^* = \{1, 2, 3, 4, 5, 6\} = J^*$, and $P^* = \emptyset \subseteq \{1, 2\} = J^*$ in Illustration 1, 2, and 3, respectively), although we have not proved this result generally. In other words, at equilibrium of those numerical illustrations, the government defends all targets that attract the strategic terrorist.

Theorem 3. The defender's equilibrium allocation c^* does not depend on r and the equilibrium loss is a linear function of r .

3.2. Algorithm

In this subsection, we first provide an algorithm based on Theorem 2 to search for the equilibrium defense resource allocations, and then provide a theorem of convergency. Combining Eqs. (7) and (14), we have: $A_i e^{-\lambda_i c_i^*} x_i = W^*, \forall i \in J^*$, which is equivalent to

$$\begin{aligned} c_i^* &= \frac{\ln x_i + \ln A_i^* - \ln W^*}{\lambda_i}, \quad \forall i \in J^* \\ &= \{i : c_i^* > 0, \quad i = 1, \dots, n\} \end{aligned} \quad (15)$$

Inserting Eq. (15) into Eq. (1) and noting that we have $c_i^* = 0 \quad \forall i \notin J^*$, we have:

$$\begin{aligned} C &= \sum_{i=1}^n c_i^* = \sum_{i \in J^*} c_i^* + \sum_{i \notin J^*} c_i^* = \sum_{i \in J^*} \frac{\ln x_i + \ln A_i^* - \ln W^*}{\lambda_i} \leftrightarrow W^* \\ &= \exp \left\{ \frac{\sum_{i \in J^*} \frac{\ln x_i}{\lambda_i} + \sum_{i \in J^*} \frac{\ln A_i^*}{\lambda_i} - C}{\sum_{i \in J^*} \frac{1}{\lambda_i}} \right\} \end{aligned} \quad (16)$$

Based on Eq. (16), we develop an algorithm, for which Fig. 3 shows an illustrative diagram and Table 3 presents a detailed description of the steps and conditions in the algorithm.

Theorem 4. The algorithm provided by Fig. 3 and Table 3 in Section 3.2 always converges to an equilibrium defined by Definition 1 in Section 2.2. The algorithm requires $O(n^2)$ computation, where n is the number of targets in the system.

Table 3
Description of steps and conditions in the algorithm shown in Fig. 3.

Steps
S_1 Let $J = \{1, 2, \dots, n\}$, $P = \{1, 2, \dots, n\}$ and $I = \{1, 1, \dots, 1\}$
S_2 Solve for W using Eq. (16) with $A_i = qr \frac{h_j}{\ P\ } + (1-q)h'_j$
S_3 (a) Identify such index j 's satisfying C_2 with smallest x_j . Let $c_j = 0$. Delete j 's from set J (b) Let $c_j = 0$ and delete j 's from set J (c) Save old c_j and update $c_j = \frac{1}{\lambda_j} \left[\ln x_j + \ln \left(qr \frac{h_j}{\ P\ } + (1-q)h'_j \right) - \ln W \right], \forall j \in J$ (d) Update sets P and I using Eqs. (2), (3) excluding most valuable attracting NS-terrorist (e) Replace $c_j \forall j \in J$ with old c_j 's and adjust c_j so that $p_j(c_j)x_j = \max\{x_i \forall j \in J - P, i \notin J\}$
S_4 (a) Let $J = \{1, 2, \dots, n\}$, $P = \{1, 2, \dots, n\}$ and $I = \{1, 1, \dots, 1\}$ (b) Solve for W using Eq. (16) with $A_i = qr \frac{h'_i}{\ P\ }$
S_5 (a) Identify such index j 's satisfying C_2 with smallest x_j . Let $c_j = 0$. Delete j 's from set J (b) Let $c_j = 0$ and delete j 's from set J (c) Update $c_j = \frac{1}{\lambda_j} \left[\ln x_j + \ln \left(qr \frac{h'_j}{\ P\ } - \ln W \right) \right], \forall j \in J$ (d) Update sets P and I using Eqs. (2), (3)
S_6 Solve for W using Eq. (16) with $A_i = \lambda h'_i$
S_7 (a) Identify such index j 's satisfying C_2 with smallest x_j . Let $c_j = 0$. Delete j 's from set J (b) Let $c_j = 0$ and delete j 's from set J (c) Update $c_j = \frac{1}{\lambda_j} (\ln x_j + \ln \lambda h'_j - \ln W), \forall j \in J$
Conditions
C_1 $q \in (0, 1]$
C_2 There exists some index $j \in J$ such that $x_j A_j \leq W$
C_3 $qr \frac{h_j}{\ P\ } + (1-q)h'_j = 0 \forall j \in J$
C_4 $J \neq P$ and $qr \frac{h_j}{\ P\ } < (1-q)h'_j \forall i \in P$ and $j \in J - P$
C_5 $I_j = 0, \forall j \in J$
C_6 $J = P$ and $L(c, \hat{h}(c), h', q) \leq L^*(c^*, \hat{h}(c^*), h', q = 1)$
C_7 $h'_j = 0 \forall j \in J$
C_8 $x_i > p_j(c_j)x_j \forall i \notin J, j \in J$

3.3. Numerical illustrations

Using the data set introduced in Section 2.3 and the algorithm provided in Section 3.2, we solve for the optimal defense resource allocations at equilibrium with different values of $1 - q$ when $\lambda = 0.01, 0.05$ and 1 , respectively. We study Type-I non-strategic

terrorist's behavior: attack probabilities are equal for the top N most valuable urban areas, and zero for the other $n - N$ urban areas. Other three types of non-strategic terrorist's behavior are studied in Appendix A.8. (Recall that we have $n = 47$ as shown in Table 1, and targets are sorted by their valuations in descending order.) In other words, we have:

$$(\text{Type-I Non-Strategic Terrorist}) \quad h'_i = \begin{cases} \frac{r}{N} & \text{for } i = 1, 2, \dots, N \\ 0 & \text{for } i = N+1, N+2, \dots, n \end{cases}$$

Fig. 4 shows the government's optimal defense budget allocations at equilibrium (in terms of the percentage allocations of the total budget) as a function of the probability that the terrorist is non-strategic ($1 - q$), when $r = 1$, $N = 1, 2, 5, 47$ and $\lambda = 0.01, 0.05$ and 1 , respectively. Three columns in **Fig. 4** show that if the terrorist is fully strategic ($1 - q = 0$), the government will allocate the defense budget to the top 6, 25 and 47 urban areas regardless of the value of N , for $\lambda = 0.01, 0.05$ and 1 , respectively. (Note that the defensive resource allocation for Philadelphia in the first column and for some other urban areas may be too small to read.) When the cost-effectiveness of defense is high ($\lambda = 1$), more urban areas will be defended. Moreover, all panels in **Fig. 4** show that as the probability that the terrorist is non-strategic ($1 - q$) increases, the defense budget allocations for top N urban areas (weakly) increase, while the allocations for other urban areas (weakly) decrease; when the terrorist is fully non-strategic ($1 - q = 1$), urban areas, other than the top N , will not be defended (because they will not be attacked by the non-strategic terrorist). When $N = 47$, the government will allocate defense resources to the top 6, 25 and 47 urban areas regardless of the value of $1 - q$ when $\lambda = 0.01, 0.05$ and 1 , respectively.

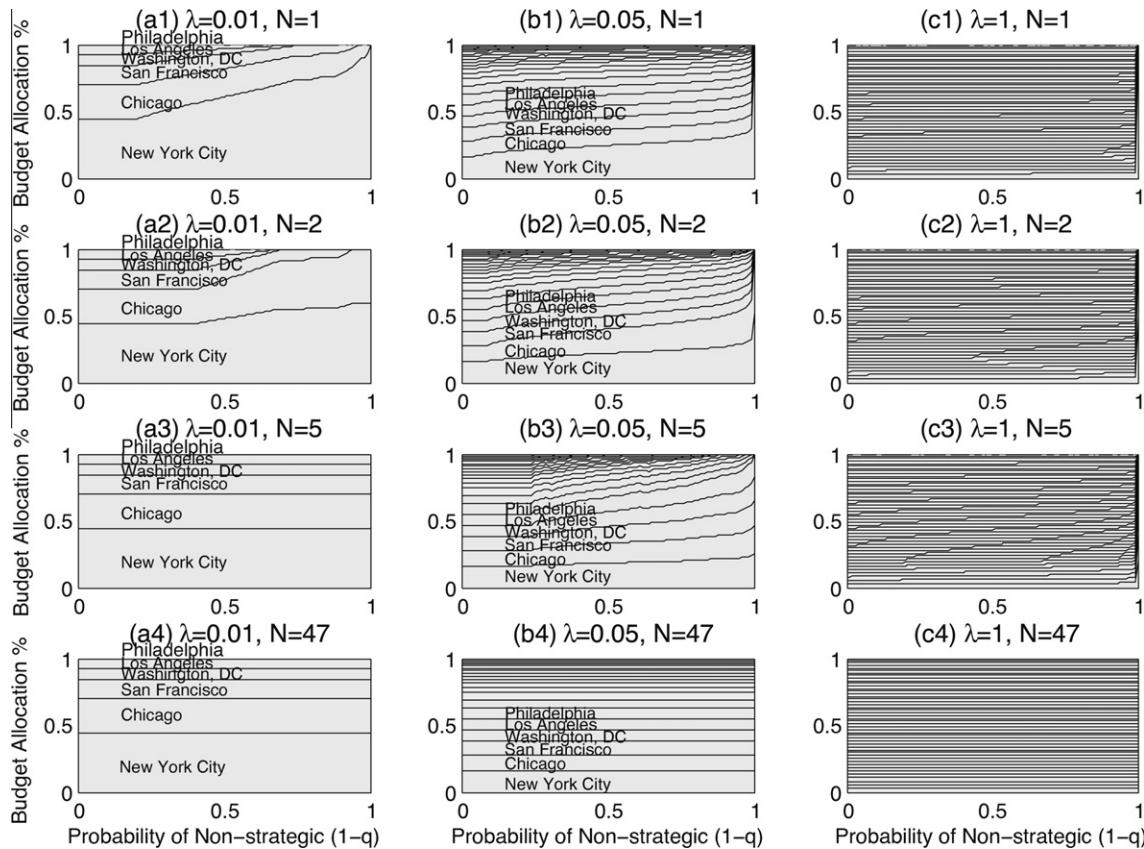


Fig. 4. Optimal defensive budget allocations as a function of probability that the terrorist is non-strategic ($1 - q$) when $r = 1$, $N = 1, 2, 5, 47$ and $\lambda = 0.01, 0.05$ and 1 , respectively.

In general, **Fig. 4** suggests that the optimal defense budget allocations are not too sensitive to the probability that the terrorist is non-strategic ($1 - q$), especially when the values of $1 - q$ are small, the cost-effectiveness of defense (λ) is large, or the number of top valuable urban areas that are attacked by the non-strategic terrorist (N) is large.

Theorem 5. *The optimal loss (L^*) decreases when budget (C) increases, or cost-effectiveness of defense (λ) increases.*

4. Robustness analyses

4.1. Two false beliefs and definitions of robustness

In this section, we compare optimal defensive resource allocations c^* and the corresponding optimal total expected loss, $L^*(c^*, \hat{h}(c^*), h', q)$, with two other alternative allocation schemes. The motivation is that, in practice, the government may (falsely) believe that the terrorist is fully strategic (or non-strategic) and use game-theoretic (or non-game theoretic) models to guide defensive resource allocations. We are interested in evaluating the costs of these two false beliefs: $\bar{L}(\bar{c}, \hat{h}(\bar{c}), h', q)$ and $\hat{L}(\hat{c}, \hat{h}(\hat{c}), h', q)$, which could be obtained by solving the optimization problem (10) with forcing $q = 1$, and $q = 0$, respectively, with corresponding resource allocations \bar{c} and \hat{c} .

Theorem 6. *For a subgame perfect Nash equilibrium (h^*, c^*):*

- (a) $L^*(c^*, \hat{h}(c^*), h', q) = \bar{L}(\bar{c}, \hat{h}(\bar{c}), h', q)$ when $q = 1$; and
- $L^*(c^*, \hat{h}(c^*), h', q) = \hat{L}(\hat{c}, \hat{h}(\hat{c}), h', q)$ when $q = 0$.

- (b) $L^*(c^*, \hat{h}(c^*), h', q), \bar{L}(\bar{c}, \hat{h}(\bar{c}), h', q)$, and $\hat{L}(\hat{c}, \hat{h}(\hat{c}), h', q)$ all (weakly) decrease in $1-q$.
- (c) $L^*(c^*, \hat{h}(c^*), h', q) \leq \bar{L}(\bar{c}, \hat{h}(\bar{c}), h', q)$ and $L^*(c^*, \hat{h}(c^*), h', q) \leq \hat{L}(\hat{c}, \hat{h}(\hat{c}), h', q)$ for the values of all h', q .
- (d) For all h', q , there exists a constant $T \geq 0$ such that $\bar{L}(\bar{c}, \hat{h}(\bar{c}), h', q) \leq \hat{L}(\hat{c}, \hat{h}(\hat{c}), h', q)$ if and only if $1-q \leq T$.

Remark 2. Theorem 6b implies that the defender can achieve lower expected loss when she faces a strategic attacker with higher probability regardless of the validity of the belief held by the defender since the attacker becomes more likely to be non-strategic and thus predictable by the defender.

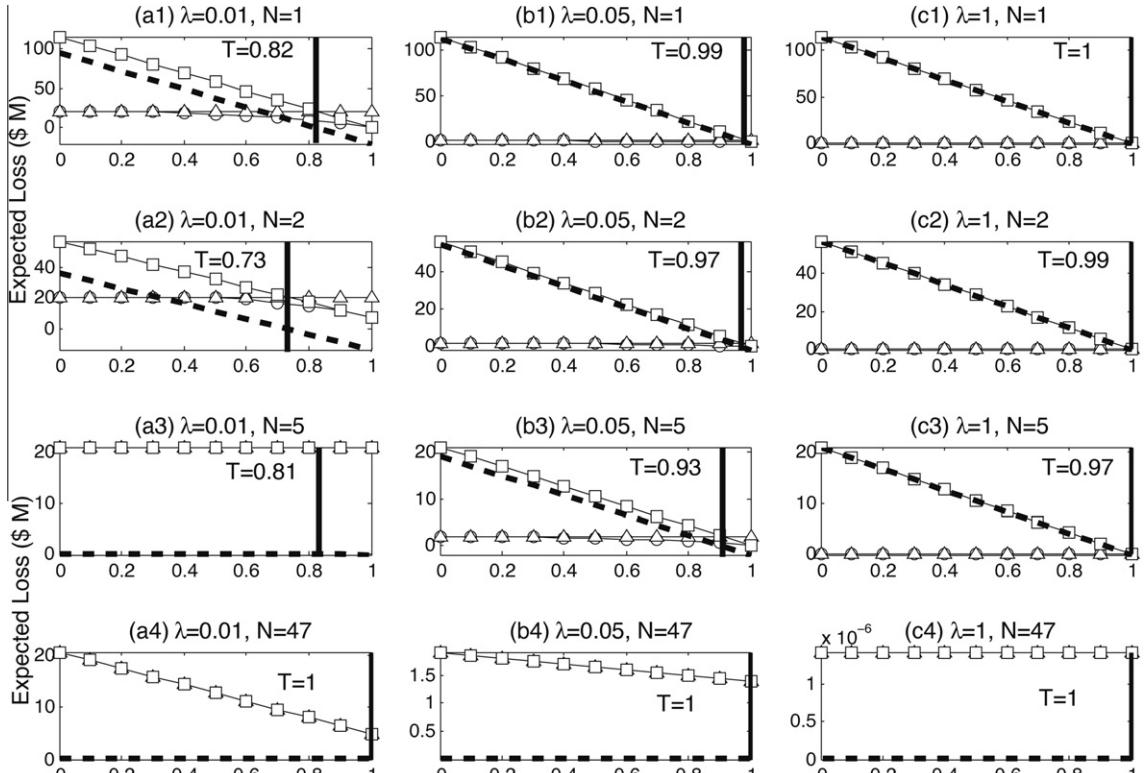
Fig. 5 shows the expected loss for the government applying each of the three defense resource allocation schemes discussed in Section 4.1, as a function of the probability that the terrorist is non-strategic ($1-q$) when $r=1, N=1, 2, 5$, and 47 and $\lambda=0.01, 0.05$, and 1 , respectively. Fig. 5 also shows $d = \hat{L}(\hat{c}, \hat{h}(\hat{c}), h', q) - \bar{L}(\bar{c}, \hat{h}(\bar{c}), h', q)$ to compare the costs of these two false beliefs. The case $d=0$ happens when $1-q=T$ according to Theorem 6d. This means that the game-theoretic model and the non-game-theoretic model perform equally well in terms of the consequent total expected loss. The cases $d>0$ (and $d<0$) means that the game-theoretic model is superior (and inferior) to the non-game-theoretic model. We have the following observations:

First, in all panels in Fig. 5, as predicted by Theorem 6a–c, we observe that: (a) $L^*=\bar{L}$ when $q=1$; and $L^*=\hat{L}$ when $q=0$; (b) L^*, \bar{L}, \hat{L} all (weakly) decrease as $1-q$ increases; and (c) $L^* \leq \bar{L}$ and $L^* \leq \hat{L}$ for all h', q .

Second, when $1-q \leq T, \bar{L}(\bar{c}, \hat{h}(\bar{c}), h', q) \leq \hat{L}(\hat{c}, \hat{h}(\hat{c}), h', q)$ for $T=0.82, 0.73, 0.81, 1, 0.99, 0.97, 0.93, 1, 1, 0.99, 0.97$, and 1 in Fig. 5a1–a4, b1–b4 and c1–c4, respectively, as predicted by Theorem 6d. It is important to observe that we have large values of T ($T \geq 0.73$) for all cases, which means that as long as the probability that the terrorist is non-strategic is less than 0.73, game-theoretic models are always preferred to non-game theoretic models. Note these numbers are significantly greater than 0.5. This observation also implies that if the government has non-informative beliefs (i.e., equally likely to strategic or non-strategic), then the government should strictly prefer to using game-theoretic models to guide defensive resource allocations.

Third, as shown in Fig. 5a4, b4 and c4, when $N=47, L^*, \bar{L}$ and \hat{L} are the same regardless of the value of $1-q$. The reason is that whether the government believes that the terrorist is strategic or not, the government will defend the top 6 (when $\lambda=0.01$), 25 (when $\lambda=0.05$) and 47 (when $\lambda=1$) urban areas, respectively, which results in the same total expected loss. This is consistent with the results in Fig. 4a4, b4, and c4.

Finally, Fig. 5 suggests that given the attack behavior is unknown, it is safer to allocate defensive resources based upon



—○— L^* : Defender knows attacker characteristic —△— \bar{L} : Defender believes attacker is strategic —□— \hat{L} : Defender believes attacker is non-strategic
 ——— d : Robustness of Game Models ——— T : Preference Threshold

Fig. 5. Government's total expected loss after three types of defensive resource allocations and corresponding preference threshold (T) and robustness measure (d) as a function that the probability that the terrorist is non-strategic ($1-q$) when $r=1, N=1, 2, 5, 47$ and $\lambda=0.01, 0.05$ and 1 , respectively.

the belief that the terrorist is fully strategic (i.e., $1 - q = 0$) than based on the belief that the terrorist is fully non-strategic (i.e., $1 - q = 1$).

This is because $\bar{L}(\bar{c}, \bar{h}(\bar{c}), h', q)$ could only be slightly higher than $L^*(c^*, \bar{h}(c^*), h', q)$ for large values of $1 - q$, while $\hat{L}(\bar{c}, \bar{h}(\bar{c}), h', q)$ could be significant larger than $L^*(c^*, \bar{h}(c^*), h', q)$. Such a difference slightly increases in cost-effectiveness of defense (λ), decreases in the number of targeting attracting the non-strategic terrorist (N) and the probability that the terrorist is non-strategic ($1 - q$).

4.2. Sensitivity analyses

We also conduct the following sensitivity analyses for preference threshold (T) and difference measure of robustness (d) as used in the numerical illustration. In order to understand the effects of varying the budget, we conduct sensitivity analysis with regard to budget. Instead of setting budget C be to a fixed number (i.e., \$675 M as introduced in Section 2.3). We consider a range between \$0 M and \$2000 M. Fig. 6 shows preference threshold (T) for game-theoretic models as a function of budget (C) when

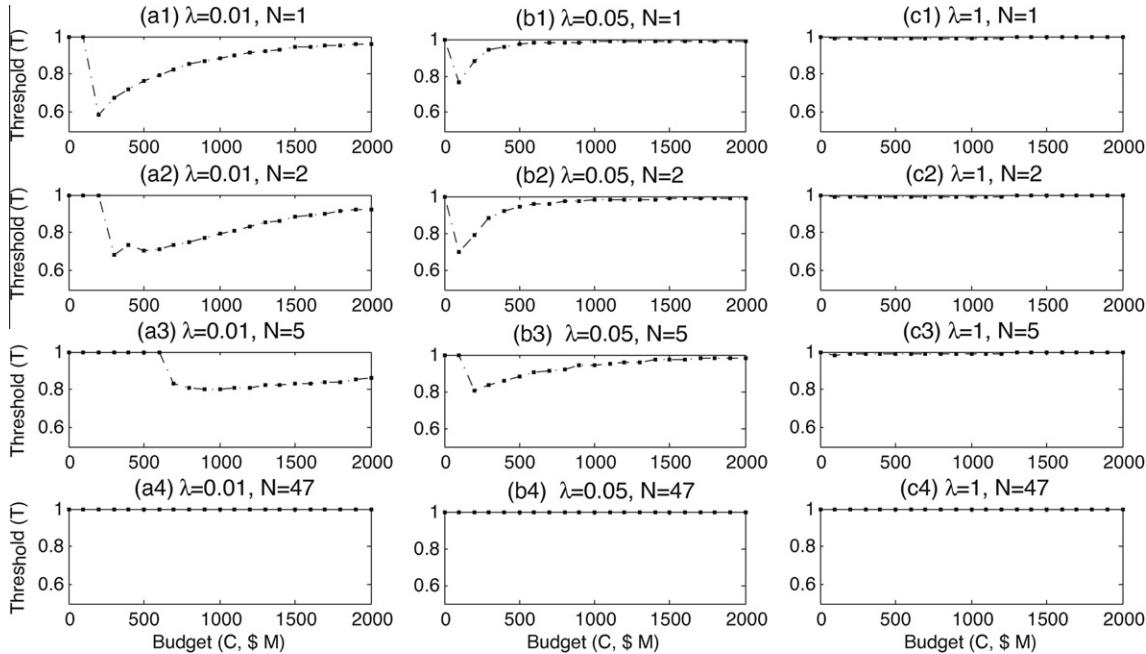


Fig. 6. Preference threshold (T) as a function of budget (C) when $r = 1$, $\lambda = 0.01, 0.05$, and 1 , and $N = 1, 2, 5$, and 47 , respectively, and Type-I non-strategic terrorist is concerned.

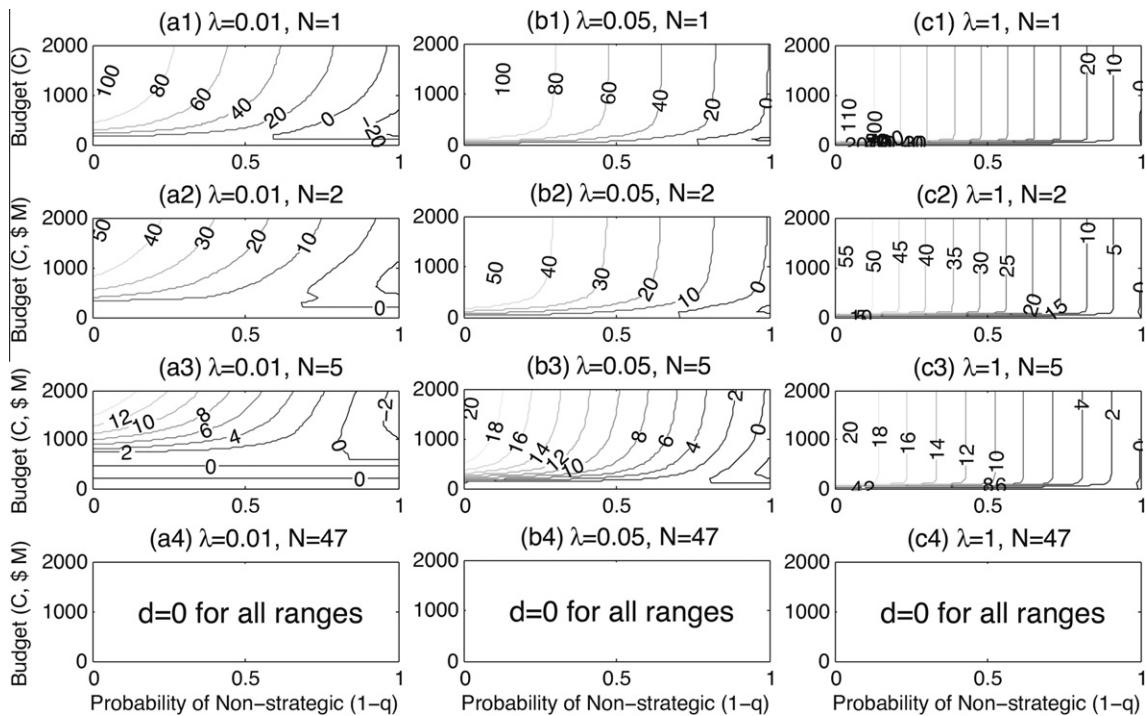


Fig. 7. Robustness (d) for game-theoretic models as a function of budget (C) and the probability that the terrorist is non-strategic when $r = 1$, $\lambda = 0.01, 0.05$, and 1 , and $N = 1, 2, 5$, and 47 , respectively.

$r = 1$, $\lambda = 0.01, 0.05$, and 1 , and $N = 1, 2, 5$, and 47 . One important observation is that T is always greater than 0.58 for all combinations of C , λ and N , similar to our observation from Fig. 5. We acknowledge that T could be less than 0.5 for some extreme cases. (see Appendix A.7 for an example.) Fig. 6a4, b4 and c4 shows that when $N = 47$, T is always 1 regardless of the value of C and λ , which is consistent with Fig. 5a4, b4, and c4. Another observation is that T is generally high when λ or N is high, or C is either low or high. Moreover, we conduct sensitivity analysis with regard to number of attacks.

Next we conduct two-way sensitivity analysis to study the robustness measure d against $1 - q$, C , λ and N . (Note that unlike in studying threshold T , where $1 - q$ is irrelevant, we do need to study the values of $1 - q$ in studying d .) In particular, Fig. 7 shows contours of difference measure of robustness of game-theoretic models (d) with one dimension being budget (C), and the second dimension being the probability that the terrorist is non-strategic ($1 - q$).

Note that when $N = 47$ (Fig. 7a4, b4, and c4), d is always 0 regardless of the values of C and λ . This is consistent with the results in Fig. 4a4, b4, and c4 and Fig. 5a4, b4, and c4.

Overall, Fig. 7 shows that d is generally positive (game-theoretic models are superior to non-game theoretic models in the sense of incurring lower total expected loss) for most of the regions, especially when $1 - q$ is not extremely high, C is either low or high, or λ is high.

5. Conclusion and future research directions

In this paper, we develop a novel hybrid model, where a centralized government allocates defensive resources among multiple potential targets to minimize expected loss caused by an unknown adversary, who could be either strategic or non-strategic. In general, we find that the optimal defensive strategies facing a partially strategic attacker differs from those facing a fully strategic or a fully non-strategic attacker. The defender can achieve a lower expected loss when the probability of facing a non-strategic attacker is high regardless of the defender's belief about the behavioral type of the attacker. From the numerical illustrations, we find that the optimal defense budget allocations are not too sensitive to the probability that the terrorist is non-strategic, especially when the probability is small, the cost-effectiveness of defense is large, or the number of top valuable urban areas that are attacked by the non-strategic terrorist is large.

Note that both fully endogenous and fully exogenous models are special cases of our hybrid model, and we use our hybrid model to compare the robustness of these two models with our extensive numerical illustrations and sensitivity analyses. We find that game-theoretic models are preferred even when the probability that the terrorist is non-strategic is significantly greater than 50% . In particular, defensive resource allocations based on game-theoretic models are generally superior to those based on non-game theoretic models in the sense of incurring lower total expected loss, especially when the budget is either low or high, the cost-effectiveness of defense is high, or the terrorist is more likely to be strategic.

In light of the ongoing controversy of applicability of game theory to terrorism analysis, it is significant that we found optimal defensive resource allocations based on game-theoretic models to be "robust" to the possibility of the terrorist being non-strategic. In other words, assuming the terrorist to be strategic when he is not is "conservative" in the sense of incurring less expected loss than assuming the terrorist is non-strategic when he is not. (We acknowledge that game-theoretic models such as min-max games have been known to be robust against the worst-case scenarios,

but our paper further shows that game-theoretic models are also robust in terms of expected losses, in addition to worst-possible losses, when facing a partially strategic terrorist.).

Our work helps shed light on this important issue. In particular, our results support further development and application of game-theoretic methods, by demonstrating that they can be useful even when the terrorist's behavior may not be fully strategic or game-theoretic. Our results also tend to lend further support to critiques of past defensive resource allocations, which claim the allocations have little emphasis on the defense of the most valuable (and presumably, therefore, most attractive) targets. While it seems intuitively plausible that underestimating terrorists' intelligence and adaptability is likely to be a more costly mistake than overestimating terrorist's capabilities, this logic has not yet had widespread impact on the counter-terrorism measures used in practice; in fact, many methods of risk analysis and defensive resource allocations currently in use by the DHS adopt a risk-analytic or decision-analytic perspective, rather than a game-theoretic perspective. The results of our work have demonstrated that even relatively simple game-theoretic models, which do not pretend to capture the full range of possible terrorist's behavior, can significantly outperform defensive resource allocations based purely on non-game-theoretic models.

Finally, although our results suggest that game-theoretic models are more robust than non-game-theoretic models, we acknowledge that in practice game-theoretic models may be difficult to implement, and may not always scale well to problems of realistic size and complexity.

Interesting future research directions include: (a) A more efficient and general algorithm should be developed for similar models of realistic size and complexity, taking into account more general forms of the success probability function; (b) a more sophisticated objective function could be used to incorporate the terrorist's target valuation and risk preferences (Keeney, 2007). For example, a multi-attribute terrorist utility function with uncertainty can be employed to capture terrorist's preferences (Wang and Bier, 2011); (c) multiple-period attacks allow for updating of the attacker's rationality and examine the manner that robustness was impacted by beliefs that are off-equilibrium path; (d) Allowing more than one networked targets to be impacted by one attack could be considered; and (e) lattice programming could be used to characterize the equilibrium (Veinott, 1992).

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Appendix. Supplementary material

Supplementary material associated with this article can be found, in the online version, at <http://dx.doi.org/10.1016/j.ejor.2013.01.029>.

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