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Modeling Arbitrary Layers of Continuous-Level Defenses in Facing with Strategic Attackers

Mohsen Golalikhani[1] and Jun Zhuang[1,*]

We propose a novel class of game-theoretic models for the optimal assignment of defensive resources in a game between a defender and an attacker. Compared to the other game-theoretic models in the literature of defense allocation problems, the novelty of our model is that we allow the defender to assign her continuous-level defensive resources to any subset (or arbitrary layers) of targets due to functional similarity or geographical proximity. We develop methods to solve for equilibrium, and illustrate our model using numerical examples. Compared to traditional models that only allow for individual target hardening, our results show that our model could significantly increase the defender's payoff, especially when the unit cost of defense is high.

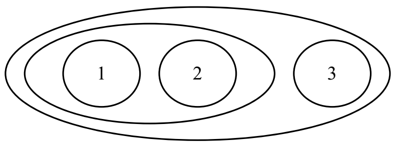
KEY WORDS: Defender optimization; game theory; homeland security; resource allocation

1. INTRODUCTION

In this article, we propose a novel game-theoretic model for assigning defensive resources to protect a set of targets against intentional threats such as terrorist attacks. The novelty of our model is that we allow the defender to assign her defensive resources to any subset (arbitrary layers) of targets due to functional similarity or geographical proximity, while previous models in the literature could only prescribe defensive investment on either individual targets or all targets together (e.g., border security). Fig. 1 shows two examples of arbitrary layers of protection against three threats. In Example 1, the defender could protect against three types of attacks (chemical, biological, and explosive terrorism) either individually ({1}, {2}, and {3}) through specialty detectors, or collectively ({1, 2, 3}) by enhancing emergency responses. The defender could also protect against both chemical and biological terrorism jointly ({1, 2}) through public-health surveillance programs in order to facilitate early detection, but this method might not be effective for protection against explosive terrorism. Similarly, in Example 2, the defender can counter threats against three urban areas (New York, Washington, D.C., and Los Angeles) either individually ({1}, {2}, and {3}) through individual target hardening, or collectively ({1, 2, 3}) by improving U.S. border security. The defender can also protect New York and Washington, D.C. jointly ({1, 2}) due to their geographical proximity (200 miles away). This can be achieved by establishing a regional northeastern U.S. emergency response system, which would most likely not benefit the Los Angeles area (located 2,450 miles from New York) too much. To the best of our knowledge, such game-theoretic models of arbitrary layers of protection in facing with strategic attacks, as provided in Fig. 1, have not been studied in the literature.

There are many mathematical models that have considered centralized defense for protecting

[1]Department of Industrial and Systems Engineering, University at Buffalo, Buffalo, NY 14260, USA.
*Address correspondence to Dr. Jun Zhuang, Department of Industrial and Systems Engineering, University at Buffalo, The State University of New York, 403 Bell Hall, Buffalo, NY 14260, USA; tel: 1-716-645-4707; fax: 1-716-645-3302; jzhuang@buffalo.edu.



Threat	Example 1: Functional Similarity	Example 2: Geographical Proximity
1	Chemical Terrorism	New York
2	Biological Terrorism	Washington, D.C.
3	Explosive Terrorism	Los Angeles

Fig. 1. Examples of arbitrary layers of protection.

individual targets. For example, Azaiez and Bier[1] studied optimal resource allocation in reliability systems with both parallel and series subsystems. Hausken *et al.*[2] and Zhuang and Bier[3] used a game-theoretic model to study target protection against both terrorism and natural disasters. Powell[4] compared optimal defenses against strategic and nonstrategic attackers. Golany *et al.*[5] provided a model for allocating limited resources to defend sites that face either probabilistic risk or strategic risk. Zhuang and Bier[6] and Zhuang *et al.*[7] provided a game-theoretic model that considers secrecy and deception in an attacker-defender resource-allocation and signaling game. Levitin and Hausken[8] proposed a mathematical model for resource distribution in multiple attacks against a single target. They considered two attacker objectives, that is, to maximize the target vulnerability or to minimize the expected attacker resource expenditure.

Overarching and collective protections have also been studied in literature. Many of these studies focused on reliability maximization in multistate systems. For example, Levitin[9] proposed a nongame-theoretic model for multilevel protection, and considered both target hardening and overarching protection in the context of both series and parallel systems. Korczak and Levitin[10] proposed a nongame-theoretic model for multilevel protection against multiple destructive factors in multistate series-parallel systems. Levitin and Hausken[11] considered a two-period game where the defender distributes her resources between the deployment of redundant elements and protecting them from attacks. Inspection games could also be considered as a special case for overarching or collective protection. Recent models of inspection games for border security and the control of nuclear weapons, nuclear material, or other weapons of mass destruction include Avenhaus and Canty,[12] Boros *et al.*,[13] Bier and Haphuriwat,[14] and McLay *et al.*[15]

This article is intended as a natural extension of the above-mentioned literature in the overarching and collective protection in attacker-defender games. Our model allows the defender to allocate the continuous-level resources to any subset (or arbitrary layer) of targets, and thus obtain better payoffs. This has not yet been studied in the literature. The rest of the article is organized as follows. Section 2 provides the general model formulation and three theorems for solving the model. Section 3 provides specific forms for the functions of the model and some analytical results for such model. Section 4 illustrates the model using numerical examples, and Section 5 provides conclusions and future research directions. Finally, the Appendix provides the proofs to the four theorems presented in Sections 2 and 3.

2. THE MODEL

2.1. Notation

Throughout this article, we use the following notations:

- T: The set of all targets.
- $|T|$: Cardinality of set T.
- S_k: Nonempty subset defined on the set T (i.e., $S_k \subset T$ and $S_k \neq \phi$, $k = 1, 2, \ldots, 2^{|T|} - 1$). Note that we do not consider the subset ϕ since it would be meaningless to assign any defensive resource to protect a subset consisting of no target.
- V_i: Value of target $i, i \in T$. For simplicity, we assume that the attacker and defender have the same target valuation V_is.
- (X_i, Y_i): Two-dimensional coordinates of target $i, i \in T$.
- D_{ij}: The distance between targets i and j. That is: $D_{ij} = \sqrt{(X_j - X_i)^2 + (Y_j - Y_i)^2}$, $i, j \in T$. (In this article we focus on geographical proximity as illustrated in Example 2, Fig. 1. However, functional similarity, as illustrated in Example 1, Fig. 1, could be similarly defined as a function of the attributes of the threats.)
- $R_{S_k}(\{D_{ij} : i, j \in S_k\})$: Defense efficiency coefficient on subset S_k, as a function of the distances

D_{ij}s between targets i and j, both belonging to the subset S_k.
- $d_{S_k} \geq 0$: Defender's decision variable of defensive investment for protecting subset S_k.
- $d \equiv \{d_{S_k} : S_k \subset T, S_k \neq \phi\}$: The set of decision variables of the defender.
- $a_i \in \{0, 1\}$: Attacker's decision variable to attack target i ($a_i = 1$) or not ($a_i = 0$). Note that in this article for simplicity, we only consider binary attack choice as studied in Konrad,[16] Bier *et al.*,[17,18] Dighe *et al.*,[19] Zhuang and Bier,[6] Zhuang,[20] Zhuang *et al.*,[7] and Bier and Haphuriwat.[14] This might be relevant in some high-level strategic decision-making situations, concerning which targets are likely to be attacked (rather than the level of attack effort on each targets). However, we acknowledge that in principle, attack effort may be different among attacked targets and future works could consider continuous-level attack, as studied in Zhuang and Bier[3] and Levitin and Hausken.[8,11]
- $a \equiv \{a_i : i \in T\}$: The set of attacker's decision variables.
- $u_A(a, d)$ and $u_D(a, d)$: Attacker's and defender's total expected payoffs, respectively.
- $\hat{a}(d)$: Attacker's best response function; i.e., $\hat{a}(d) = \arg\max_a u_A(a, d)$.
- T_a and T_n: Set of targets that are attacked and not attacked, respectively (i.e., $T_a = \{i : a_i = 1\}$ and $T_n = T - T_a = \{i : a_i = 0\}$).
- $P(R_{S_k} d_{S_k}, a_i)$: Probability that protection S_k fails to protect target i when target i is attacked. We assume that such probability decreases in $R_{S_k} d_{S_k}$ and increases in a_i.
- $\prod_{S_k \ni i} P(R_{S_k} d_{S_k}, a_i)$: Probability that all layers of protection S_k (that cover target i) fail to protect target i.
- C: Cost of attack per target.
- B: Unit cost of defense.

2.2. Attacker's and Defender's Optimization Problems

We assume that the attacker desires to maximize the total expected damage across all targets, subtracting the total attack costs. In other words:

$$\max_a u_A(a, d) = \sum_{i=1}^{|T|} V_i \left[\prod_{S_k \ni i} P\big(R_{S_k} d_{S_k}, a_i\big) \right] - C \sum_{i=1}^{|T|} a_i$$
$$s.t. \quad a_i = 0 \text{ or } 1, \quad \forall i = 1, \ldots, |T|. \tag{1}$$

Similarly, we assume that the defender maximizes the total expected value of nondamaged targets, subtracting the total defense costs. That is:

$$\max_d u_D(a, d) = \sum_{i=1}^{|T|} V_i \left[1 - \prod_{S_k \ni i} P(R_{S_k} d_{S_k}, a_i) \right] - B \sum_{S_k \subset T, S_k \neq \phi} d_{S_k}$$
$$s.t. \quad d_{S_k} \geq 0, \quad \forall k = 1, 2, \ldots, 2^{|T|} - 1. \tag{2}$$

2.3. Subgame Perfect Nash Equilibrium (SPNE)

Following Powell[4] and Bier *et al.*,[17] we consider a sequential game in which the defender is the first mover. We define a strategy pair (a^*, d^*) as a SPNE if and only if:

$$a^* = \hat{a}(d^*) = \arg\max_a u_A(a, d^*) \tag{3}$$

and

$$d^* = \arg\max_d u_D(\hat{a}(d), d). \tag{4}$$

In this article, we assume that at SPNE the attacker will not attack if he is indifferent to either attacking and not attacking.

2.4. General Analytical Results for SPNE

In order to solve for the SPNE defined by Equations (3) and (4) introduced in Section 2.3, we first study the attacker optimization problem (Equation (1)) and solve for the attacker's best responses $\hat{a}(d)$ in relation to each feasible defender's strategy d.

THEOREM 1: *At SPNE $a_i = 1$ if and only if $\prod_{S_k \ni i} P(R_{S_k} d_{S_k}, a_i = 1) > \frac{C}{V_i}$.*

Remarks: Theorem 1 tells us that, for a given level of defenses: (a) the attacker decision could be decentralized (i.e., optimization problem (1) could degenerate to $|T|$ separate suboptimization problems); and (b) attacks are more likely to happen to target i if the effective defenses ($R_{S_k} d_{S_k}$), or the attack cost (C), decrease; or if the target valuation (V_i) increases.

Using Theorem 1, we provide Theorem 2 for finding the SPNE:

THEOREM 2: *The SPNE defined by Equations (3) and (4) can be equally obtained by solving the following optimization problem:*

$$\max_{T_a \subset T} \left\{ \right. \tag{5}$$

$$\max_d U_D(a,d) = \sum_{i \in T_a} V_i \left[1 - \prod_{S_k \ni i} P(R_{S_k} d_{S_k}, a_i = 1) \right] + \sum_{i \in T_n} V_i - B \sum_{S_k \subset T, S_k \neq \phi} d_{S_k} \tag{6}$$

subject to

$$\prod_{S_k \ni i} P(R_{S_k} d_{S_k}, a_i = 1) > \frac{C}{V_i}, \quad \forall i \in T_a \tag{7}$$

$$\prod_{S_k \ni i} P(R_{S_k} d_{S_k}, a_i = 1) \leq \frac{C}{V_i}, \quad \forall i \in T_n \tag{8}$$

$$\left. d_{S_k} \geq 0; \quad \forall S_k \subset T, \quad S_k \neq \phi \right\}.$$

Remarks: Theorem 2 provides an equivalent definition for the SPNE. In particular, using Theorem 2, we can always calculate for SPNE by considering all possible $2^{|T|}$ combinations of targets that could be attacked ($T_a \subset T$), solving the inner constrained optimization problem (Equations (6)–(8)) for each of these $2^{|T|}$ combinations, and then comparing these $2^{|T|}$ optimal objective values in Equation (5) in order to obtain the SPNE. However, an important deficiency for Theorem 2 is that, for some combinations of targets T_a, we may face an "open set" problem in which the feasible space for the set of inner optimization problem (Equations (6)–(8)) does not contain any optimum solution. This situation may occure when Equation (7) makes the set of feasible solutions an open set, so that this set is not compact. To solve this deficiency, we provide Theorem 3.

THEOREM 3: *In the inner optimization problem defined by Equations (6)–(8), we can replace the strict Equation (7) with the weak Equation (9), below, without changing the SPNE:*

$$\prod_{S_k \ni i} P(R_{S_k} d_{S_k}, a_i = 1) \geq \frac{C}{V_i}, \quad \forall i \in T_a. \tag{9}$$

Remark: Theorem 3 implies the existence of SPNE for our model. To see this, first note that the solution to an inner optimization problem defined by Equations (6)–(8) and (9) must exist because the feasible region of the set of constraints defined by Equations (8) and (9) is compact, and the inner objective function of Equation (6) is continuous in d. Second, note that the feasible region for the outer maximization problem of Equation (5) is always nonempty and finite. Therefore, we can always find a SPNE by comparing the optimal objective values from all possible inner optimization problems using Equation (5).

3. SPECIFIC FORMS FOR THE FUNCTIONS OF THE MODEL

We assume specific forms for the following two functions: the defense efficiency coefficient (R_{S_k}) and the probability of damage (P).

The first function, $R_{S_k}(\{D_{ij} : i, j \in S_k\})$, is a function of the distances D_{ij}s between the targets i and j, both belonging to the subset S_k, and representing the efficiency reduction of investment on the subset S_k. To understand this function, consider an investment made on emergency facilities in order to jointly protect a subset consisting of two cities. If these two cities are far away from each other, this joint protection would be inefficient. In the rest of this article, we consider the following functional forms for R_{S_k}:

$$R_{S_k}(\{D_{ij} : i, j \in S_k\}) = 1 - \frac{\max_{i,j \in S_k} D_{ij}}{1 + \max_{i,j \in T} D_{ij}}. \tag{10}$$

Note that R_{S_k} in Equation (10) depends on D_{ij}s only through the maximum distance between the targets belonging to the subset S_k. However, other types of functional forms for R_{S_k} could also be reasonable; e.g., a function of the total distances between the targets belonging to the subset S_k:

$$R'_{S_k}(\{D_{ij} : i, j \in S_k\}) = 1 - \frac{\sum_{i,j \in S_k} D_{ij}}{1 + \sum_{i,j \in T} D_{ij}}. \tag{11}$$

Regarding the function of failure in protection (P), we follow the exponential function in Bier *et al.*,[17] and customize this function for our model as follows:

$$P(R_{S_k} d_{S_k}, a_i) = \begin{cases} 0 & \text{if } a_i = 0 \\ \exp(-R_{S_k} d_{S_k}) & \text{if } a_i = 1. \end{cases} \tag{12}$$

For more general contest success functions, see Skaperdas.[21]

COROLLARY 1: *Inserting Equation (12) in Theorem 2, and replacing Equation (7) with*

Equation (9) according to Theorem 3, we obtain the optimization problem for finding the SPNE as follows:

$$\max_{T_a \subset T} \left\{ \right. \tag{13}$$

$$\max_{d} U_D(a, d) = \sum_{i \in T_a} V_i \left[1 - \exp\left(- \sum_{S_k \ni i} R_{S_k} d_{S_k} \right) \right] + \sum_{i \in T_n} V_i - B \sum_{S_k \subset T, S_k \neq \phi} d_{S_k} \tag{14}$$

subject to

$$\sum_{S_k \ni i} R_{S_k} d_{S_k} \leq -\ln\left(\frac{C}{V_i}\right), \quad \forall i \in T_a \tag{15}$$

$$\sum_{S_k \ni i} R_{S_k} d_{S_k} \geq -\ln\left(\frac{C}{V_i}\right), \quad \forall i \in T_n \tag{16}$$

$$\left. d_{S_k} \geq 0; \quad \forall S_k \subset T, \quad S_k \neq \phi \right\}.$$

Remark: According to Corollary 1, and as mentioned in the remark following Theorem 2, in order to find the SPNE, we need to consider all possible $2^{|T|}$ combinations of targets that could be attacked ($T_a \subset T$). This method is always feasible, but one concern is the computational time, which increases exponentially in $|T|$. To address this concern, Theorem 4 below provides a method to significantly reduce the number of inner optimization problems to be solved, without missing the SPNE.

THEOREM 4: *For any target j satisfying $V_j - C\exp(\frac{C}{B}) \leq 0$, the attacker will not attack target j in SPNE. Therefore, we can neglect those inner optimization problems in Theorem 2 for which $T_a \ni j$. In other words, using the notation $J \equiv \{j : V_j - C\exp(\frac{C}{B}) \leq 0\}$, Equation (13) can be simplified to Equation (17) below without missing the SPNE:*

$$\max_{T_a \subset T - J} \{ \tag{17}$$

Furthermore, the computational time will be reduced by $100(1 - 2^{-|J|})\%$.

Remark: To see the benefits of Theorem 4, consider a problem in which $|J| = 3$. For finding the SPNE by using Theorem 4, we only need to solve $2^{|T|-3}$ inner optimization problems, which reduces the computational time by $100(1 - 2^{-3})\% = 87.5\%$ as compared to the method that does not use Theorem 4.

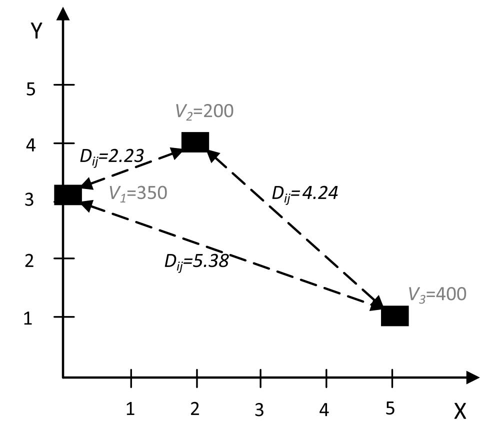


Fig. 2. An example with three targets.

4. ILLUSTRATIVE EXAMPLES

4.1. Baseline Example

In this section, we illustrate our model using a baseline example provided in Fig. 2 in which $C = 4$ and $B = 1$. In this example we have three targets ($T = \{1, 2, 3\}$), with $V_1 = 350$, $V_2 = 200$, $V_3 = 400$, $(X_1, Y_1) = (0, 3)$, $(X_2, Y_2) = (2, 4)$, and $(X_3, Y_3) = (5, 1)$. The total $2^{|T|} - 1 = 2^3 - 1 = 7$ defense investment options d_{S_k}s are associated with the subsets $S_1 = \{1\}$, $S_2 = \{2\}$, $S_3 = \{3\}$, $S_4 = \{1, 2\}$, $S_5 = \{1, 3\}$, $S_6 = \{2, 3\}$, and $S_7 = \{1, 2, 3\}$. The distances are $D_{12} = 2.23$, $D_{13} = 5.38$, and $D_{23} = 4.42$, as shown in Fig. 2.

Inserting the above-mentioned distances into Equation (10), we get the defense efficiency coefficients R_{S_k}s as follows: $R_{S_1} = 1$, $R_{S_2} = 1$, $R_{S_3} = 1$, $R_{S_4} = 1 - \frac{2.23}{1+5.38} = 0.65$, $R_{S_5} = 1 - \frac{5.35}{1+5.38} = 0.16$, $R_{S_6} = 1 - \frac{4.24}{1+5.38} = 0.34$, and $R_{S_7} = 1 - \frac{5.38}{1+5.38} = 0.16$. Therefore, given the values of defensive investment d_{S_k}s, the attacker optimization problem of Equation (1) becomes:

$$\begin{aligned} \max_{a} u_A(a, d) = {} & 350\left[a_1 e^{-d_{S_2}} e^{-0.65 d_{S_4}} e^{-0.16 d_{S_5}} e^{-0.16 d_{S_7}}\right] \\ & + 200\left[a_2 e^{-d_{S_2}} e^{-0.65 d_{S_4}} e^{-0.34 d_{S_6}} e^{-0.16 d_{S_7}}\right] \\ & + 400\left[a_3 e^{-d_{S_3}} e^{-0.16 d_{S_5}} e^{-0.34 d_{S_6}} e^{-0.16 d_{S_7}}\right] \\ & - 4[a_1 + a_2 + a_3] \\ a_i = {} & 0 \text{ or } 1, \quad \forall i = 1, 2, 3. \end{aligned}$$

Similarly, the defender optimization problem of Equation (2) becomes:

$$\max_d u_D(a,d)$$
$$= 350\left[1 - a_1 e^{-d_{S_2}} e^{-0.65 d_{S_4}} e^{-0.16 d_{S_5}} e^{-0.16 d_{S_7}}\right]$$
$$+ 200\left[1 - a_2 e^{-d_{S_2}} e^{-0.65 d_{S_4}} e^{-0.34 d_{S_6}} e^{-0.16 d_{S_7}}\right]$$
$$+ 400\left[1 - a_3 e^{-d_{S_3}} e^{-0.16 d_{S_5}} e^{-0.34 d_{S_6}} e^{-0.16 d_{S_7}}\right]$$
$$- \sum_{k=1}^{7} d_{S_k}$$
$$d_{S_k} \geq 0, \quad \forall k = 1, \ldots, 7.$$

4.2. Finding the SPNE for the Baseline Example

According to Corollary 1, when finding the SPNE we need to consider all $2^{|T|} = 2^3 = 8$ combinations of targets $T_a \subset T$ that can be attacked, and solve for these eight corresponding inner optimization problems. However, according to Theorem 4, we know that target 2 will not be attacked in SPNE because $V_2 - C\exp(\frac{C}{B}) = -18 \leq 0$. Therefore, we only need to consider $2^{|T|-1} = 2^{3-1} = 4$ combinations of targets, and solve their corresponding inner optimization problems as defined by Equations (14)–(16), as shown in the following four cases (without involving target 2).

Case 1: $T_a = \{1\}$

$$\max_d u_D(a,d) = 350\left[1 - e^{-\left(d_{S_1} + 0.65 d_{S_4} + 0.16 d_{S_5} + 0.16 d_{S_7}\right)}\right]$$
$$+ 200 + 400 - \sum_{k=1}^{7} d_{S_k}$$

subject to:

$$d_{S_1} + 0.65 d_{S_4} + 0.16 d_{S_5} + 0.16 d_{S_7} \leq 4.47$$
$$d_{S_2} + 0.65 d_{S_4} + 0.34 d_{S_6} + 0.16 d_{S_7} \geq 3.91$$
$$d_{S_3} + 0.16 d_{S_5} + 0.34 d_{S_6} + 0.16 d_{S_7} \geq 4.60$$
$$d_{S_k} \geq 0, \forall k = 1, \ldots, 7.$$

Using any standard optimization software (e.g., LINGO, MATLAB optimization toolbox), the optimal solution for this case is: $d_{S_1} = 0.56$; $d_{S_3} = 4.60$; $d_{S_4} = 6.02$; $d_{S_k} = 0.00$ for $k = 2, 5, 6, 7$; and $u_D(a,d) = 934.8$.

Case 2: $T_a = \{3\}$

$$\max_d u_D(a,d) = 350 + 200$$
$$+ 400\left[1 - e^{-(d_{S_3} + 0.16 d_{S_5} + 0.34 d_{S_6} + 0.16 d_{S_7})}\right] - \sum_{k=1}^{7} d_{S_k}$$

subject to

$$d_{S_1} + 0.65 d_{S_4} + 0.16 d_{S_5} + 0.16 d_{S_7} \geq 4.47$$
$$d_{S_2} + 0.65 d_{S_4} + 0.34 d_{S_6} + 0.16 d_{S_7} \geq 3.91$$
$$d_{S_3} + 0.16 d_{S_5} + 0.34 d_{S_6} + 0.16 d_{S_7} \leq 4.60$$
$$d_{S_k} \geq 0, \quad \forall k = 1, \ldots, 7.$$

The optimal solution for this case is: $d_{S_1} = 0.56$; $d_{S_3} = 4.60$; $d_{S_4} = 6.02$; $d_{S_k} = 0.00$ for $k = 2, 5, 6, 7$; and $u_D(a,d) = 934.8$.

Case 3: $T_a = \{1, 3\}$

$$\max_d u_D(a,d) = 350\left[1 - e^{-(d_{S_1} + 0.65 d_{S_4} + 0.16 d_{S_5} + 0.16 d_{S_7})}\right]$$
$$+ 200 + 400\left[1 - e^{-(d_{S_3} + 0.16 d_{S_5} + 0.34 d_{S_6} + 0.16 d_{S_7})}\right]$$
$$- \sum_{k=1}^{7} d_{S_k}$$

subject to

$$d_{S_1} + 0.65 d_{S_4} + 0.16 d_{S_5} + 0.16 d_{S_7} \leq 4.47$$
$$d_{S_2} + 0.65 d_{S_4} + 0.34 d_{S_6} + 0.16 d_{S_7} \geq 3.91$$
$$d_{S_3} + 0.16 d_{S_5} + 0.34 d_{S_6} + 0.16 d_{S_7} \leq 4.60$$
$$d_{S_k} \geq 0, \quad \forall k = 1, \ldots, 7.$$

The optimal solution for this case is: $d_{S_1} = 0.56$; $d_{S_3} = 4.60$; $d_{S_4} = 6.02$; $d_{S_k} = 0.00$ for $k = 2, 5, 6, 7$; and $u_D(a,d) = 930.8$.

Case 4: $T_a = \phi$

$$\max_d u_D(a,d) = 150 + 100 + 200 - \sum_{k=1}^{7} d_{S_k}$$

subject to

$$d_{S_1} + 0.65 d_{S_4} + 0.16 d_{S_5} + 0.16 d_{S_7} \geq 4.47$$
$$d_{S_2} + 0.65 d_{S_4} + 0.34 d_{S_6} + 0.16 d_{S_7} \geq 3.91$$
$$d_{S_3} + 0.16 d_{S_5} + 0.34 d_{S_6} + 0.16 d_{S_7} \geq 4.60$$
$$d_{S_k} \geq 0, \quad \forall k = 1, \ldots, 7.$$

The optimal solution for this case is: $d_{S_1} = 0.56$; $d_{S_3} = 4.60$; $d_{S_4} = 6.02$; $d_{S_k} = 0.00$ for $k = 2, 5, 6, 7$; and $u_D(a,d) = 938.8$.

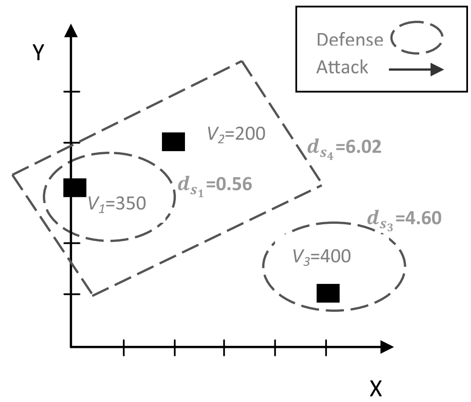


Fig. 3. SPNE for the illustrative example.

By comparing the defender's payoffs $u_D(a, d)$ in the above four cases (934.8, 934.8, 930.8, and 938.8), we obtain the SPNE for this problem in Case 4 in which the defender's optimal payoff is 938.8, the defender's strategy is $d_{S_1} = 0.56; d_{S_3} = 4.60; d_{S_4} = 6.02; d_{S_k} = 0.00$ for $k = 2, 5, 6, 7$, and the attacker's best responses to the defenses are: $a_1 = 0; a_2 = 0; a_3 = 0$. This solution is illustrated in Fig. 3. In this baseline example the defender chooses to protect targets 1 and 2 collectively (due to the geographical closeness), and to protect targets 1 and 3 individually (due to the high target valuation).

4.3. Sensitivity Analysis for Varying Parameters *B* and *C*

Note that, in the baseline example provided in Sections 4.1 and 4.2, the unit cost of defense ($B = 1$) is so small that the defender highly protects all targets in order to fully deter the attacker in the SPNE. In this section we first study the role of parameter *B*. Fig. 4 shows the impact of parameter *B* on the total payoffs of the defender and attacker as well as the values of the defender's decision variables in

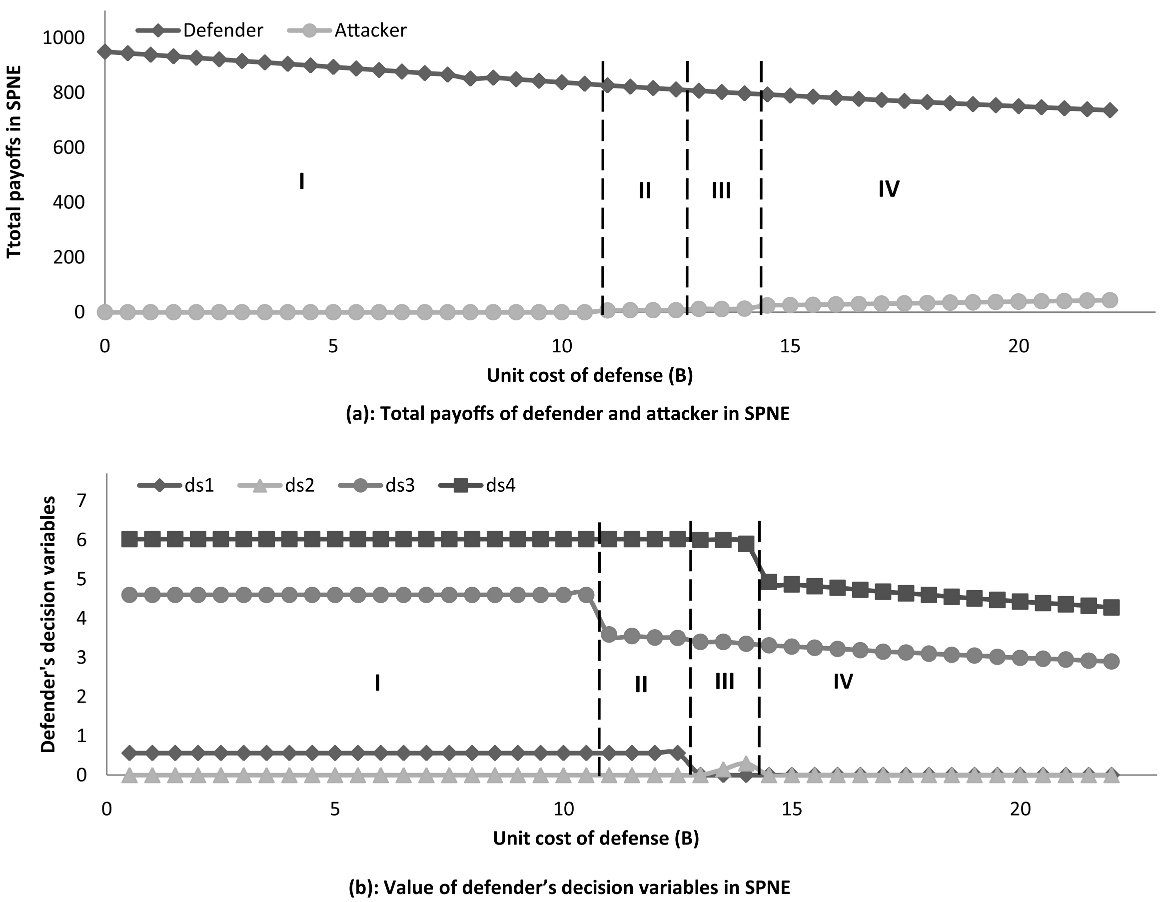


Fig. 4. SPNE of the numerical example for various values of parameter *B*.

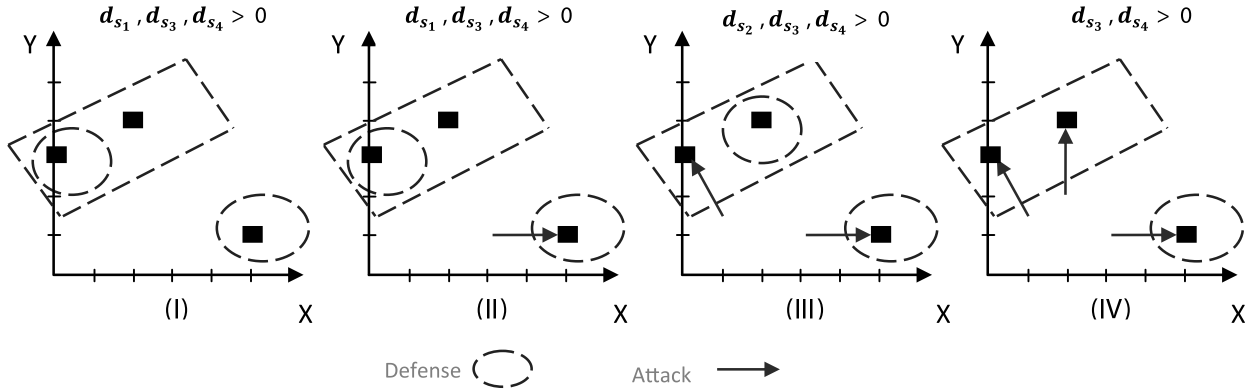


Fig. 5. Four classes of SPNE strategies.

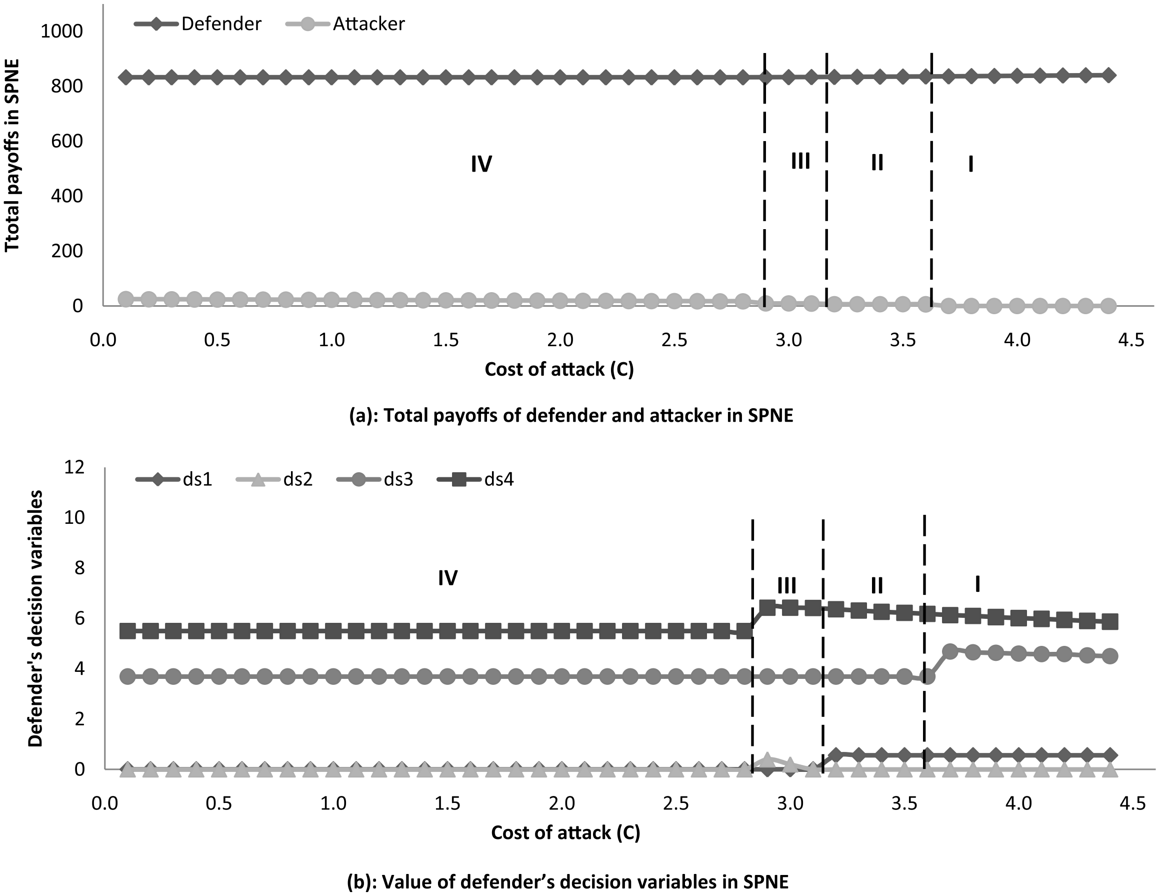


Fig. 6. SPNE of the numerical example for various values of parameter *C*.

the SPNE. Four different types of SPNE strategies (I, II, III, and IV) in Fig. 4 are explained in Fig. 5. According to Figs. 4(b) and 5, as the unit cost of defense (B) increases, in SPNE the defender generally invests less effort to protect the targets (except when $13.0 \leq B \leq 14.0$ where d_{S_3} and d_{S_4} decrease in B, but d_{S_2} slightly increases in B); as a result, more targets are attacked (from Case I with no attack, to Case IV with three attacks). Moreover, Fig. 4(b) shows that, when B goes to infinity, all defense levels go to zero.

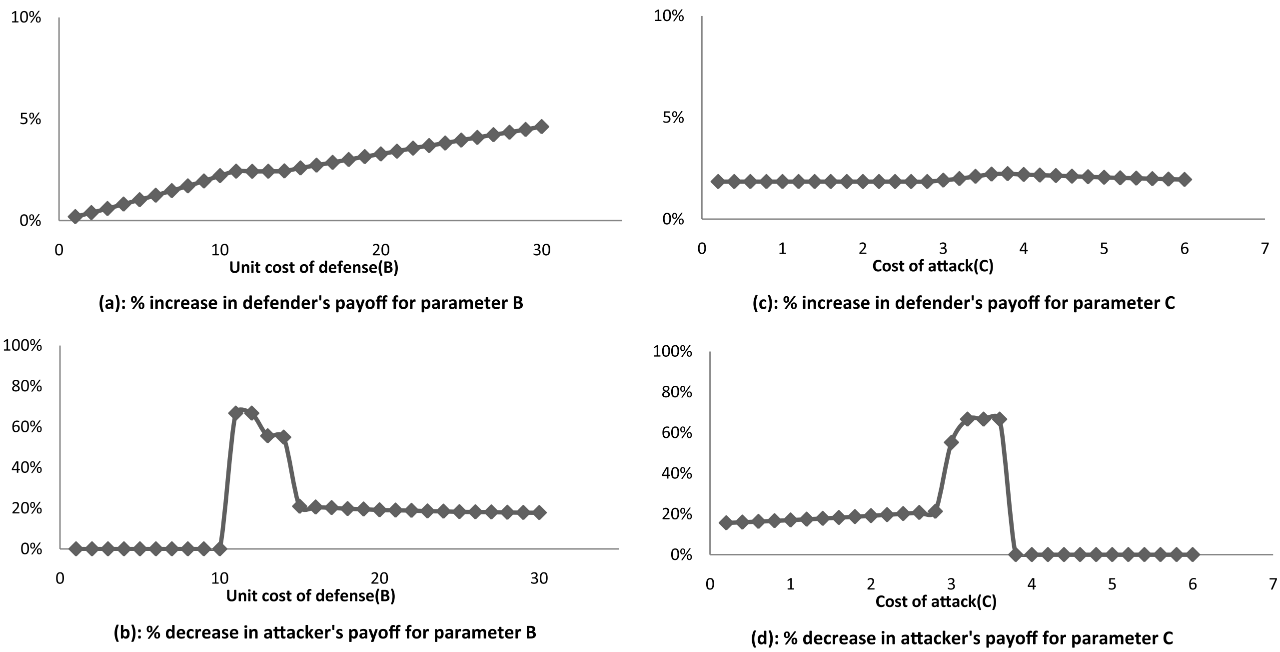


Fig. 7. Percentage increase in defender's payoff and the percentage decrease in attacker's payoffs when the ALD model is used compared to the ITH model.

Finally, Fig. 4(a) shows that, when B increases, the defender's total payoff decreases, and the attacker's total payoff increases.

Similarly, Fig. 6 shows the impact of the cost of attack (parameter C) on the total payoffs of the defender and attacker, as well as the value of the defender's decision variables in SPNE. From Figs. 5 and 6, we see that when C increases, fewer targets are attacked in SPNE; and the attacker's total payoff decreases while the defender's total payoff slightly increases. Moreover, when C goes to infinity, no attack happens and all defense levels go to zero.

By comparing Figs. 4 and 6, it is important to note that all defense levels go to zero, both when parameter B goes to infinity (too costly to defend) and when parameter C goes to infinity (too expensive to attack; therefore, no need to defend exists). Moreover, it is worth noting that $d_{S_5} = d_{S_6} = d_{S_7} = 0$ in all cases. This is because the targets in subsets S_5, S_6, and S_7 are so far away from each other that the defense efficiency coefficients on these subsets are sufficiently small ($R_{S_5} = R_{S_7} = 0.1$ and $R_{S_6} = 0.34$, compared to $R_{S_1} = R_{S_2} = R_{S_3} = 1$ and $R_{S_4} = 0.65$). Finally, we see that the attacker's and defender's total payoffs are more sensitive to parameter B than to parameter C. This suggests that, in order for the defender to increase her total payoff in SPNE, it would be more efficient to develop methods (if possible) to decrease the unit cost of defense, rather than to increase the cost of attack.

4.4. Arbitrary Layers of Defense (ALD) Versus Individual Target Hardening (ITH)

In this section we compare our numerical examples from the model of ALD in Sections 4.1–4.3 with a model only allowing ITH that was studied in the literature. The ITH model is easier to implement than the ALD model because: (1) there is no need to calculate the R_{S_k}s; and (2) the number of decision variables will decrease from 7 in ALD to 3 in the ITH model (i.e., d_{S_4}, d_{S_5}, d_{S_6}, and d_{S_7} are forced to be zero). Therefore, we have only three decision variables d_{S_1}, d_{S_2}, and d_{S_3} in the ITH compared to seven in the ALD. (However, this is not important because, in general, the computational time mainly depends on the number of constraints rather than the number of decision variables; the ITH model has the same number of constraints as in ALD model.)

On the other hand, our model enables us to study more general cases while obtaining better payoffs for the defender by avoiding suboptimums. To see this, Fig. 7 provides the percentage increase in the defender's payoffs and the percentage decrease in the

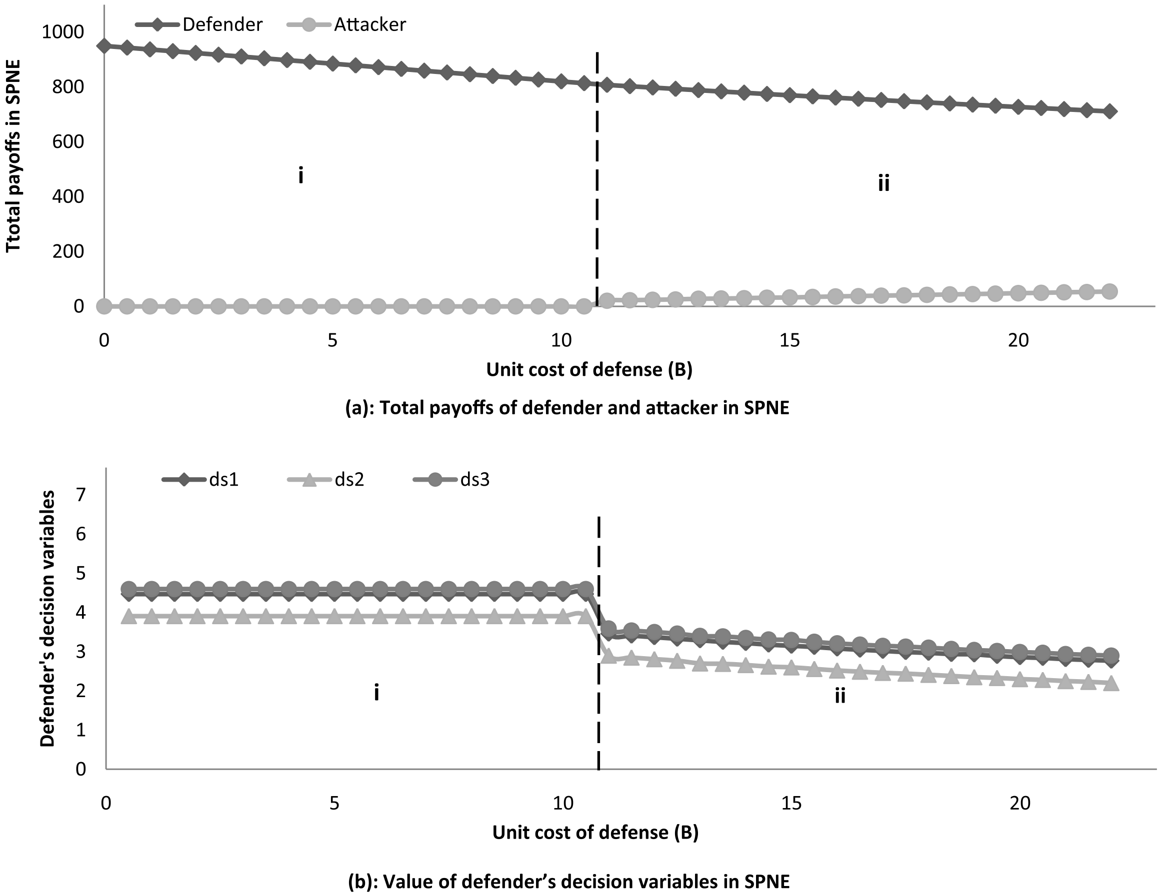


Fig. 8. SPNE of the numerical example for various values of parameter B (individual target hardening).

attacker's payoffs when the ALD model is used compared to the ITH model. We observe that for all values of parameters B and C, the defender has better payoffs (especially when parameter B is large meaning that the defense is expensive), and the attacker has worse payoffs (especially when parameter C is intermediate).

Another important observation from Fig. 7 is that the percentage decrease in the attacker's payoffs is usually more than the percentage increase in the defender's payoffs when ALD is used instead of the ITH model. In other words, ALD is a better model for disadvantaging the attacker rather than advantaging the defender. To see the details, we provide the detailed SPNE information for the ITH model in Figs. 8–10, analogous to Figs. 4–6 for the ALD model in Section 4.3. By comparing Figs. 4 and 8 where $11.0 \leq B \leq 14.0$, and Figs. 6 and 10, where $2.9 \leq C \leq 3.6$, we see that when parameters B and C are intermediate, in the ALD model the defender can assign her defensive resources more efficiently such that the attacker cannot attack all targets (as happening in the ITH model), and as a result, this makes a big decrease in attacker payoffs when ALD is used instead of ITH.

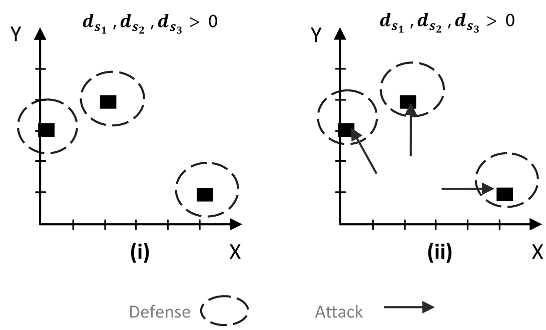


Fig. 9. Two classes of SPNE strategies (individual target hardening).

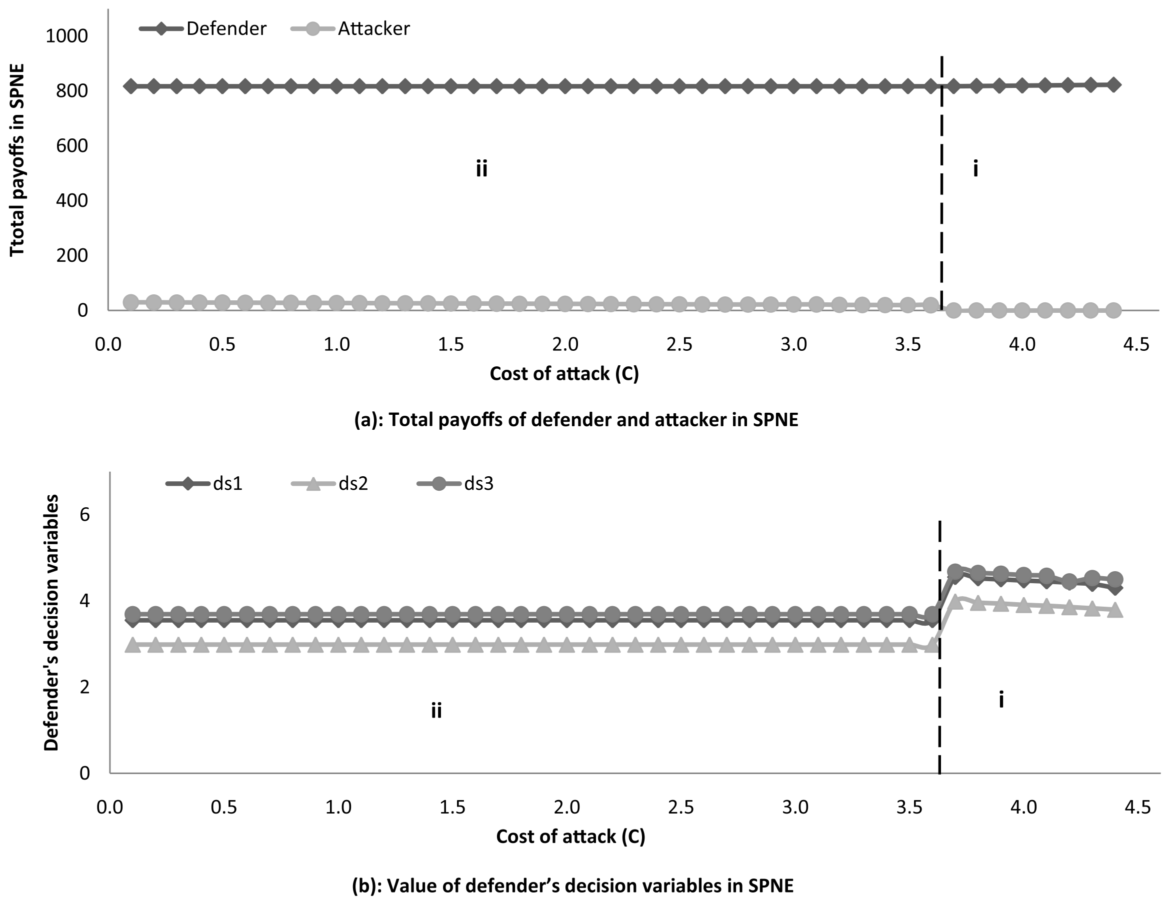


Fig. 10. SPNE of the numerical example for various values of parameter C (individual target hardening).

5. CONCLUSION AND FUTURE RESEARCH DIRECTIONS

In this article, we proposed a novel attacker-defender game-theoretic model in which the defender has the option to assign her continuous-level defensive resources to any subset (arbitrary layers) of targets, instead of choosing individual target hardening, or collective protections such as border security. We also developed an efficient method for finding the SPNE of the proposed model, and illustrated the method using numerical examples. This method for finding SPNE is based on solving optimization problems and comparing their optimal values. We noted that the number of inner optimization problems increases exponentially by the number of targets, and provided a theorem to significantly shorten the computational time by removing dominated suboptimization problems.

By comparing our model of arbitrary layers of defense with the traditional model of individual target hardening, our results show that the percentage increase in defender's payoffs are significant especially when the cost of defense is high. This suggests that our model will provide valuable solutions to governments facing strategic adversaries while the defenses are expensive and financial resources are limited.

Interesting future research directions include: (a) studying multilayers of protections due to functional similarity; (b) studying continuous-level of attack; and (c) numerically studying more functional forms for defense efficiency coefficients and probability of failures.

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APPENDIX

A.1. Proof for Theorem 1

For each strategy of the defender d, the attacker faces an optimization problem given in Equation (1). However, since we do not consider budget constraints for the attacker, the optimization problem of Equation (1) decomposes to the following $|T|$ suboptimization problems for each target $i \in T$:

$$\max_{a_i} u_A^i(a_i, d) = V_i \left(\prod_{S_k \ni i} P\big(R_{S_k} d_{S_k}, a_i\big) \right) - a_i C$$
$$a_i = 0 \text{ or } 1,$$

where $u_A^i(a_i, d)$ is the attacker's total expected payoff from target i. We must consider two possible cases:

I. $a_i = 0 \Rightarrow u_A^i(a_i = 0, d) = 0$
II. $a_i = 1 \Rightarrow u_A^i(a_i = 1, d) = V_i(\prod_{S_k \ni i} P(R_{S_k} \ d_{S_k}, a_i = 1)) - C$.

The attacker chooses $a_i = 1$ if and only if $u_A^i(a_i = 0, d) < u_A^i(a_i = 1, d)$ (recall that we assumed in Section 2.3 that, in SPNE, the attacker will not attack if he is indifferent between attacking and not attacking); or equivalently:

$$V_i \left(\prod_{S_k \ni i} P\big(R_{S_k} d_{S_k}, a_i = 1\big) \right) - C > 0$$
$$\Rightarrow \prod_{S_k \ni i} P\big(R_{S_k} d_{S_k}, a_i = 1\big) > \frac{C}{V_i}.$$

A.2. Proof for Theorem 2

Theorem 1 ensures that the attacker plays his best responses $\hat{a}(d) = \arg\max_a u_A(a, d)$ if and only if the constraints of Equations (7) and (8) are satisfied in the inner optimization problem (Equations (6)–(8)), which satisfies the requirement of Equation (3). Furthermore, the defender maximizes $u_D(\hat{a}(d), d)$ according to Equations (5) and (6), which satisfy the requirement of Equation (4). Therefore, Equations (5)–(8) satisfy Equations (3) and (4), and the SPNE defined by Equations (3) and (4) can be equivalently obtained by solving Equations (5)–(8).

A.3. Proof for Theorem 3

There are two possible cases for any inner optimization problem as defined by Equations (6)–(8) and (9). In both cases we will show that replacing strict inequality of Equation (7) with weak inequality of Equation (9) does not change the SPNE.

Case 1: There is no target $j \in T_a$, such that the constraint of Equation (9) is tight for the optimal solution of the inner optimization problem defined by Equations (6), (8), and (9). Therefore, this optimal solution is also feasible for the inner optimization problem defined by Equations (6)–(8) with a reduced feasible region and, therefore, must still be optimal. Therefore, replacing strict inequality of Equation (7) with weak inequality of Equation (9) does not change the SPNE.

Case 2: There exists at least one target $j \in T_a$, such that the constraint of Equation (9) is tight for the optimal solution of the inner optimization problem defined by Equations (6), (8), and (9). We denote this optimal solution as $d^* = (d^*_{S_1}, d^*_{S_2}, \ldots, d^*_{S_{2^{|T|}-1}})$, and

$$J = \left\{ j \in T_a : \prod_{S_k \ni j} P\left(R_{S_k} d^*_{S_k}, a_j = 1\right) = \frac{C}{V_j} \right\}.$$

Considering that the solution d^* is feasible, and that the constraint of Equation (9) is tight, the following equations must hold:

$$u_D^*(a, d^*) = \sum_{j \in J} V_j \left[1 - \prod_{S_k \ni j} P\big(R_{S_k} d^*_{S_k}, a_j = 1\big) \right] + \sum_{i \in T_a; i \notin J} V_i \left[1 - \prod_{S_k \ni i} P\big(R_{S_k} d^*_{S_k}, a_i = 1\big) \right] + \sum_{i \in T_n} V_i - B \sum_{S_k \subset T; S_k \neq \phi} d^*_{S_k}$$

$$\prod_{S_k \ni j} P\big(R_{S_k} d^*_{S_k}, a_j = 1\big) = \frac{C}{V_j}; \quad \forall j \in J \quad \text{(A.1)}$$

$$\prod_{S_k \ni i} P\big(R_{S_k} d^*_{S_k}, a_i = 1\big) \geq \frac{C}{V_i}; \quad \forall i \in T_a, i \notin J \quad \text{(A.2)}$$

$$\prod_{S_k \ni i} P\big(R_{S_k} d^*_{S_k}, a_i = 1\big) \le \frac{C}{V_i}; \quad \forall i \in T_n. \quad \text{(A.3)}$$

Now we show that there exists another combination of targets denoted as T'_a, which has a better objective function than $u^*_D(a, d^*)$ so that $u^*_D(a, d^*)$ cannot be a SPNE solution when solving for Equation (5). In particular, consider another combination of targets T'_a, which is the same as T_a, except that the targets in J are not to be attacked (i.e., $T'_a = T_a - J$). The inner optimization problem for the set T'_a is as follows, and is referred to as Problem 3.1:

$$\max_d u_D(a, d)$$

$$= \sum_{j \in J} V_j + \sum_{i \in T_a; i \notin J} V_i \left[1 - \prod_{S_k \ni i} P\left(R_{S_k} d_{S_k}, a_i = 1\right)\right] + \sum_{i \in T_n} V_i - B \sum_{S_k \subset T; S_k \neq \phi} d_{S_k}$$

subject to

$$\prod_{S_k \ni j} P\left(R_{S_k} d_{S_k}, a_j = 1\right) \le \frac{C}{V_j}; \quad \forall j \in J \quad \text{(A.4)}$$

$$\prod_{S_k \ni i} P\left(R_{S_k} d_{S_k}, a_i = 1\right) \ge \frac{C}{V_i}; \quad \forall i \in T'_a \quad \text{(A.5)}$$

$$\prod_{S_k \ni i} P\left(R_{S_k} d_{S_k}, a_i = 1\right) \ge \frac{C}{V_i}; \quad \forall i \in T_n. \quad \text{(A.6)}$$

Note that d^* introduced in the paragraph above Equation (A.1) is a feasible solution to Problem 3.1 since Equations (A.1)–(A.3) satisfy the constraints in Equations (A.4)–(A.6), respectively. We denote the objective value of Problem 3.1 as evaluated by the feasible solution d^* as $u'_D(a, d^*)$ (which is not necessarily optimum to Problem 3.1). We have

$$\sum_{j \in J} V_j \ge \sum_{j \in J} V_j \left[1 - \prod_{S_k \ni j} P\left(R_{S_k} d^*_{S_k}, a_j = 1\right)\right]$$

$$\Rightarrow \sum_{j \in J} V_j + \sum_{i \in T_a; i \notin J} V_i \left[1 - \prod_{S_k \ni i} P\left(R_{S_k} d^*_{S_k}, a_i = 1\right)\right] + \sum_{i \in T_n} V_i - B \sum_{S_k \subset T; S_k \neq \phi} d^*_{S_k}$$

$$\ge \sum_{j \in J} V_j \left[1 - \prod_{S_k \ni j} P\left(R_{S_k} d^*_{S_k}, a_j = 1\right)\right] + \sum_{i \in T_a; i \notin J} V_i \left[1 - \prod_{S_k \ni i} P\left(R_{S_k} d^*_{S_k}, a_i = 1\right)\right] + \sum_{i \in T_n} V_i - B \sum_{S_k \subset T; S_k \neq \phi} d^*_{S_k}$$

$$\Rightarrow u'_D(a, d^*) \ge u^*_D(a, d^*).$$

Therefore, a feasible solution to Problem 3.1, $u'_D(a, d^*)$, has a (weakly) better objective value than the optimum solution to our original problem $u^*_D(a, d^*)$. This means that the original inner optimization problem defined by Equations (6), (8), and (9) is weakly dominated by the inner optimization Problem 3.1 when we optimize Equation (5). In other words, the T_a corresponding to $u^*_D(a, d^*)$ will be dominated by T'_a, and will not be selected as the optimal solution of the outer optimization problem of Equation (5). (Therefore, will not be an SPNE solution.) (Case 2-Conclusion 1)

We also know that the objective value of any inner optimization problem defined by Equations (6)–(8) is always less than or equal to the one defined by Equations (6), (8), and (9), since the objective functions of these two problems are the same, while the feasible region defined by the constraints of Equations (7) and (8) is smaller than the one defined by the constraints of Equations (8) and (9). Therefore, for a given T_a, if the optimal objective value of the inner optimization problem defined by Equations (6), (8), and (9) is not selected as the SPNE solution, the optimal objective value of the optimization problem defined by Equations (6)–(8) will also not be selected as a SPNE solution. (Case 2-Conclusion 2)

By considering *Case 2-Conclusion 1* and *Case 2-Conclusion 2* simultaneously in case 2 regardless of the application of inner optimization problems defined by Equations (6)–(8) or Equations (6), (8), and (9), the corresponding solution cannot be a SPNE solution. Therefore, replacing strict inequality of Equation (7) with weak inequality of Equation (9) does not change the SPNE.

A.4. Proof for Theorem 4

In order to show that any target j satisfying $V_j - C \exp\left(\frac{C}{B}\right) \le 0$ will not be attacked in SPNE, we suppose that $j \in T_a$ (j is considered to be attacked), in an inner optimization problem defined by Equations (14)–(16); we will then show that this inner optimization problem is dominated by another inner optimization problem in which $j \notin T_a$ and, therefore, will not be a SPNE solution when we optimize Equation (13). In particular, consider an inner optimization problem defined by Equations (14)–(16) in which $j \in T_a$. We denote this problem as Problem 4.1:

$$\max_d u_D(a,d) = \left[V_j - V_j \exp\left(-\sum_{S_k \ni j} R_{S_k} d_{S_k}\right)\right] + \sum_{i \in T_a; i \neq j} V_i \left[1 - \exp\left(-\sum_{S_k \ni i} R_{S_k} d_{S_k}\right)\right] + \sum_{i \in T_n; i \neq j} V_i - B \sum_{S_k \subset T; S_k \neq \phi} d_{S_k} \tag{A.7}$$

subject to:

$$\sum_{S_k \ni j} R_{S_k} d_{S_k} \leq -\ln\left(\frac{C}{V_j}\right) \tag{A.8}$$

$$\sum_{S_k \ni i} R_{S_k} d_{S_k} \leq -\ln\left(\frac{C}{V_i}\right); \quad \forall i \in T_a, i \neq j \tag{A.9}$$

$$\sum_{S_k \ni i} R_{S_k} d_{S_k} \geq -\ln\left(\frac{C}{V_i}\right); \quad \forall i \in T_n, i \neq j. \tag{A.10}$$

Consider another inner optimization problem defined by Equations (14)–(16) in which $j \notin T_a$. We denote this problem as Problem 4.2:

$$\max_d u_D(a,d) = V_j + \sum_{i \in T_a; i \neq j} V_i \left[1 - \exp\left(-\sum_{S_k \ni i} R_{S_k} d_{S_k}\right)\right] + \sum_{i \in T_n; i \neq j} V_i - B \sum_{S_k \subset T; S_k \neq \phi} d_{S_k} \tag{A.11}$$

$$\text{subject to:} \quad \sum_{S_k \ni j} R_{S_k} d_{S_k} \geq -\ln\left(\frac{C}{V_j}\right) \tag{A.12}$$

$$\sum_{S_k \ni i} R_{S_k} d_{S_k} \leq -\ln\left(\frac{C}{V_i}\right); \quad \forall i \in T_a, i \neq j \tag{A.13}$$

$$\sum_{S_k \ni i} R_{S_k} d_{S_k} \geq -\ln\left(\frac{C}{V_i}\right); \quad \forall i \in T_n, i \neq j. \tag{A.14}$$

Now we show that the optimal objective value to Problem 4.2 is greater than or equal to the optimal objective value to Problem 4.1 when $V_j - C\exp(\frac{C}{B}) \leq 0$. To show this we denote Problem 4.3 as follows:

$$\max_d u_D(a,d) = V_j + \sum_{i \in T_a; i \neq j} V_i \left[1 - \exp\left(-\sum_{S_k \ni i} R_{S_k} d_{S_k}\right)\right] + \sum_{i \in T_n; i \neq j} V_i - B \sum_{S_k \subset T; S_k \neq \phi} d_{S_k} \tag{A.15}$$

subject to:

$$\sum_{S_k \ni j} R_{S_k} d_{S_k} \leq -\ln\left(\frac{C}{V_j}\right) \tag{A.16}$$

$$\sum_{S_k \ni i} R_{S_k} d_{S_k} \leq -\ln\left(\frac{C}{V_i}\right); \quad \forall i \in T_a, i \neq j \tag{A.17}$$

$$\sum_{S_k \ni i} R_{S_k} d_{S_k} \geq -\ln\left(\frac{C}{V_i}\right); \quad \forall i \in T_n, i \neq j. \tag{A.18}$$

Note that there is only one difference between Problems 4.1 and 4.3: the objective function of Problem 4.3, which is given by Equation (A.15), is equal to the objective function of Problem 4.1 given by Equation (A.7) plus $(V_j \exp(-\sum_{S_k \ni j} R_{S_k} d_{S_k}))$. From Equation (A.8) we have:

$$\sum_{S_k \ni j} R_{S_k} d_{S_k} \leq -\ln\left(\frac{C}{V_j}\right)$$

$$\Leftrightarrow V_j \exp\left(-\sum_{S_k \ni j} R_{S_k} d_{S_k}\right) \geq C.$$

Therefore, transitioning from Problem 4.1 to 4.3, the objective function will increase by at least the value of C (Conclusion 1).

Similarly, there is only one difference between Problems 4.2 and 4.3: the directions of the constraint of Equation (A.12) in Problem 4.2 and the constraint of Equation (A.16) in Problem 4.3 are opposite of one another. We claim that when transitioning from Problems 4.3 to 4.2, the objective function will be decreased at most by the value of $-B\ln(\frac{C}{V_j})$ (Conclusion 2).

We prove Conclusion 2 as follows. Denoting $S_j = \{j\}$, according Equation (10), we have $R_{S_j} = 1$.

Now investigating the set of constraints in Equations (A.12)–(A.14) in Problem 4.2, we know that the variable d_{S_j} appears only in constraint of Equation (A.12). Therefore, in order to satisfy the constraint of Equation (A.12), we can feasibly select the value of d_{S_j} so that it is equal to $-\ln(\frac{C}{V_j})$ while satisfying the constraint of Equation (A.12) without violating the two other constraints in Equations (A.13) and (A.14). We also know that the coefficient of d_{S_j} in the objective function of Problem 4.2 is equal to B. Therefore, transitioning from Problems 4.3 to 4.2, the objective function will be decreased at most by the value of $-B\ln(\frac{C}{V_j})$. Therefore, *Conclusion 2* above is proved.

Considering *Conclusion 1* and *Conclusion 2* simultaneously, transitioning from Problems 4.1 to 4.2, the optimal objective value to Problem 4.2 is greater than or equal to the optimal objective value in Problem 4.1 if

$$-B\ln\left(\frac{C}{V_j}\right) \leq C \Rightarrow V_j - C\exp\left(\frac{C}{B}\right) \leq 0.$$

This means that if $V_j - C\exp(\frac{C}{B}) \leq 0$, Problem 4.1 is dominated by Problem 4.2, and therefore $j \notin T_a$ in SPNE. In other words, for any target j satisfying $V_j - C\exp(\frac{C}{B}) \leq 0$, the attacker will not attack target j in SPNE.

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