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Modelling 'contracts' between a terrorist group and a government in a sequential game

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In this paper, we apply a sequential game to study the possibility of 'contracts' (or at least mutually beneficial arrangements) between a government and a terrorist group. We find equilibrium solutions for complete and incomplete information models, where the government defends and/or provides positive rent, and the terrorist group attacks. We also study the sensitivities of equilibria as a function of both players' target valuations and preferences for rent. The contract option, if successful, may achieve (partial) attack deterrence, and significantly increase the payoffs not only for the government, but also for some types of terrorist groups. Our work thus provides some novel insights in combating terrorism.

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1. Introduction

Defending against intentional threats such as terrorism is fundamentally different from defending against unintentional threats such as natural disaster because terrorists are intelligent and adaptive, while natural disasters are not (Bier, 2005; Zhuang and Bier, 2007). Thus, more other strategies are possible when dealing with intelligent and adaptive terrorists than when dealing with natural disasters.

To counter terrorism, the governments' strategies include: defending critical infrastructures (Powell, 2006), attacking terrorists' resources (Arce and Sandler, 2005) and offering positive incentives (that is, rent) on politics and finance (Sandler, 2000; Andreoni *et al*, 2003; Bier and Hausken, 2011). Offering rent to terrorists may not only deter terrorists with relatively low costs, but also increase social benefits by avoiding direct conflicts with terrorists (LaFree and Dugan, 2009). The recent news (Roston, 2009) depicts how the US government funds suspected insurgents in Afghanistan to safely transport the US military supplies.

Other counterterrorism methods have been explored. Sandler and Siqueira (2006) analyze the deterrence and pre-emption games for the anti-terrorism policy making at home and abroad, and conclude that leadership in global anti-terrorism policy might be helpful for deterrence but not for pre-emption. Frey (2004) discusses four possible anti-terrorism strategies: deterrence (stick), positive incentives (carrot), decentralization of targets to reduce vulnerability, and diffusing media attention for terrorist activities. Hausken and Levitin (2009) and Levitin and Hausken (2010) study the defender's trade-off among target redundancy, protection, and creation of false targets when a defender is facing a strategic attacker.

In this paper, we consider a sequential game, where the government moves first by deciding the defense effort and rent offered to the terrorist. Then the terrorist observes the defense and rent, and decides his attack effort. The next section of this paper introduces the notation and assumptions, and formulates the problem as a sequential game with both incomplete and complete information, where in the incomplete information model, the government is uncertain about the terrorist's target valuation and/or preference for rent. Section 3 studies the terrorist's best-response function. Section 4 presents the analytical solution and numerical sensitivity analysis for the complete information model with four illustrative examples. Section 5 provides an algorithm for solving the incomplete information model and four numerical illustrations. Section 6 concludes this paper and provides some future research directions. Finally, the appendix provides proofs for two propositions in the paper.

2. The model

2.1. Notation, assumptions, and sequence of moves

We consider a terrorist group and a government in a single-target, leader–follower sequential game. Table 1 summarizes the notation used in this paper. Figure 1 shows the sequence of moves. We assume that the terrorist group and government both know the rules of the game and want to

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Table 1 Notation that is used in this paper

Notation	*Explanation*
$a \geqslant 0$	Terrorist's effort spent on attacking the target
$d \geqslant 0$	Government's investment in defending the target
$r \geqslant 0$	Rent provided by the government to the terrorist
$\theta \geqslant 0$	Terrorist's preference coefficient for rent
$\beta \geqslant 0$	Government's preference coefficient for rent
$v \geqslant 0$	Terrorist's target valuation
$V \geqslant 0$	Government's target valuation
$c \geqslant 0$	Inherent defense level
$P_a \geqslant 0$	Penalty to the terrorist for launching attacks
$P_d \geqslant 0$	Cost for the government to penalize the terrorist's attack
$U_a(a, d, r \mid \theta, v)$	Terrorist's utility
$U_d(a, d, r)$	Government's utility
$\hat{a}(d, r \mid \theta, v)$	Terrorist's best response; that is, $\hat{a}(d, r \mid \theta, v) \equiv \arg\max_{a \geqslant 0} U_a(a, d, r \mid \theta, v)$
$Ea(d, r \mid \theta, v)$	Terrorist's expected attack effort in incomplete information model
$EU_d(a(d, r \mid \theta, v), d, r)$	Government's expected utility
F_i	Feasibility set for case i, $i = 1, \ldots, 7$
O_i	Optimality set for case i, $i = 1, \ldots, 7$
$f(\theta, v)$	Joint PMF or PDF of θ and v
$f_\theta(\theta)$	PMF or PDF of θ
$f_v(v)$	PMF or PDF of v

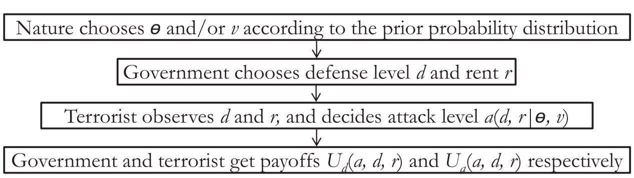

Figure 1 Sequence of moves of the sequential game.

maximize their total expected utilities. The terrorist is assumed to know the attributes of the government. In the complete information model, the government knows the terrorist's attributes such as the preference for rent, and/or target valuation. While for the incomplete information model, we assume that the government does not know the terrorist's preference for rent, and/or target valuation, in which case the nature will first choose the terrorist's preference for rent (θ), and/or target valuation (v), according to a prior (joint) probability mass function (PMF) or probability density function (PDF) $f(\theta, v)$.

2.2. Terrorist's optimization problem

For a given defense level d and rent r, the terrorist of type (θ, v) chooses the attack effort a to maximize his utility as follows:

$$\max_{a \geqslant 0} U_a(a, d, r \mid \theta, v) = -\underbrace{a}_{\text{attack effort}} + \underbrace{\frac{a}{a+d+c} v}_{\text{expected damage}} + \underbrace{\theta r}_{\text{received rent}} - \underbrace{\theta a r}_{\substack{\text{penalty when receiving}\\ \text{rent and attack}}} - \underbrace{P_a \cdot \mathbf{1}_{\{a>0\}}}_{\substack{\text{penalty when}\\ \text{launching attack}}} \quad (1)$$

There are five components in the terrorist's utility function specified in Equation (1). The first component is the effort of launching an attack, a, which the terrorist wants to minimize. The second part is the expected damage (that is, the product of the terrorist's target valuation v and the contest success probability), which the terrorist wants to maximize. In this paper, we consider a simple ratio form of the contest success probability, $a/(a+d+c)$, where $c \geqslant 0$ is the inherent defense level, which is commonly used in the literature on attacker–defender games (see, eg, Zhuang and Bier, 2007). For more general contest success functions, see Skaperdas (1996). The third part of the terrorist's utility function is the rent r, weighted by the coefficient θ, received through the contract with the government. The fourth one is the penalty ar, weighted by the coefficient θ, accruing to the terrorist if he launches an attack of level a after he received a rent r. When $a = 0$ or $r = 0$, this penalty is zero; if both a and r are positive, this penalty is positive and increases in both a and r. The purpose of incorporating the penalty ar into the terrorist's utility function is to discourage the terrorist from launching an attack after receiving rent. Without modelling such penalty, in sequential games the terrorist would just ignore any rent he has already received in making his attack decision, which voids the purpose of rent. Finally, the fifth part is the penalty of launching attack, P_a, when the terrorist attacks, regardless of receiving rent or not. We use the following indicator function:

$$\mathbf{1}_{\{a>0\}} = \begin{cases} 1 & \text{if } a > 0 \\ 0 & \text{otherwise} \end{cases} \quad (2)$$

We could also allow the terrorist to decline the rent; however, this would complicate the model by adding a binary decision variable without adding too many new qualitative insights.

2.3. Government's optimization with incomplete information

In this subsection, we model a game of incomplete information, where the government does not know the terrorist's preference for rent, θ, and/or the target valuation, v. For a given attack effort a, the government chooses the defense d and rent r to maximize her utility U_d as follows:

$$\max_{d\geqslant 0, r\geqslant 0} U_d(a,\, d,\, r) = -\underbrace{d}_{\text{defense investment}} - \underbrace{\frac{a}{a+d+c}V}_{\text{expected damage}} - \underbrace{\beta r}_{\text{rent given to the terrorist}} - \underbrace{P_d \cdot \mathbf{1}_{\{a>0\}}}_{\substack{\text{effort to penalize}\\ \text{the terrorist when he attacks}}} \quad (3)$$

There are four components in the government's utility function, Equation (3), that she wants to minimize: the defense investment cost d, the government's expected damage (given by the contest success function multiplied by the government's target valuation V), the rent r to be given to the terrorist (weighted by the coefficient β, which reflects the government's preference for rent), and the effort, P_d, to penalize the terrorist when he attacks. In a sequential game with incomplete information, the government is assumed to maximize her expected utility knowing that all terrorist types with attributes θ and v will implement their best-response strategies $\hat{a}(d,r|\theta,v) \equiv \arg\max_{a\geqslant 0} U_a(a,d,r|\theta,v)$. Thus, incorporating the terrorist's best response $\hat{a}(d,r|\theta,v)$ and taking expectation with respect to the random variables θ and v, Equation (3) becomes

$$\begin{aligned}
&\max_{d\geqslant 0, r\geqslant 0} EU_d[\hat{a}(d,r|\theta,v),d,r] \\
&= -d - E\left[\frac{\hat{a}(d,r|\theta,v)V}{\hat{a}(d,r|\theta,v)+d+c}\right] \\
&\quad - \beta r - P_d \cdot \mathbf{1}_{\{a>0\}} \\
&= \begin{cases} -d - \sum_{\theta,v} U_d(\hat{a}(d,r|\theta,v),d,r)f(\theta,v) \\ \quad -\beta r - P_d \cdot \mathbf{1}_{\{a>0\}} \qquad \text{if } \theta \text{ and } v \text{ are discrete} \\ -d - \int_\theta \int_v U_d(\hat{a}(d,r|\theta,v),d,r)f(\theta,v)\mathrm{d}\theta\mathrm{d}v \\ \quad -\beta r - P_d \cdot \mathbf{1}_{\{a>0\}} \qquad \text{if } \theta \text{ and } v \text{ are continuous} \end{cases}
\end{aligned} \quad (4)$$

where $E[\cdot]$ denotes taking expectation regarding θ and v; $f(\theta,v)$ is the PMF or PDF for discrete and continuous random variables θ and v, respectively. In Section 5, we consider the special case where θ and v are independent; that is, $f(\theta,v)=f_\theta(\theta)f_v(v)$. (However, we do not expect any difficulty in studying the non-independent case, at least numerically.) Combining the terrorist's and government's optimization problems, Equations (1) and (4), in a sequential game, we define the equilibrium as the following:

Definition 1 We call a collection of strategy $(a^*(\theta,v),d^*,r^*)$ a subgame perfect Nash equilibrium, or 'equilibrium', if and only if both Equations (5) and (6) are satisfied

$$\begin{aligned} a^*(\theta,v) &= \hat{a}(d^*,r^*|\theta,v) \\ &= \arg\max_{a\geqslant 0} U_a(a,d^*,r^*|\theta,v), \quad \forall \theta, v \end{aligned} \quad (5)$$

$$d^*, r^* = \arg\max_{d\geqslant 0,\, r\geqslant 0} EU_d[\hat{a}(d,r|\theta,v),d,r] \quad (6)$$

The model is solved with backward induction; that is, we solve for the terrorist's best-response attack strategy (in Section 3), and then solve for the government's optimal defense and rent strategies, given the terrorist's best response (in Section 4).

2.4. Government's optimization with complete information

The incomplete information model degenerates to a complete information model when the government knows the terrorist's preference for rent, r, and target valuation, v. In particular, the government's optimization problem, Equation (4), degenerates to

$$\begin{aligned} \max_{d\geqslant 0, r\geqslant 0} U_d(a,d,r) = -d - \frac{\hat{a}(d,r)V}{\hat{a}(d,r)+d+c} \\ - \beta r - P_d \cdot \mathbf{1}_{\{a>0\}} \end{aligned} \quad (7)$$

Similarly, the definition of equilibrium of complete information model degenerates from Definition 1 to the following:

Definition 2 We call a collection of strategy (a^*, d^*, r^*) a subgame perfect Nash equilibrium, or 'equilibrium', for the complete information game, if and only if both Equations (8) and (9) are satisfied

$$a^* = \hat{a}(d^*,r^*) = \arg\max_{a\geqslant 0} U_a(a,d^*,r^*) \quad (8)$$

$$d^*, r^* = \arg\max_{d\geqslant 0, r\geqslant 0} U_d[\hat{a}(d,r),d,r] \quad (9)$$

3. Terrorist's best response to defense and rent

In this section, we study the terrorist's best response $\hat{a}$ to defense d and rent r.

Proposition 1 *The solution to the terrorist's optimization problem* (1)*, that is, the terrorist's best-response function, is given by*

$$\hat{a}(d,r|\theta,v) = \begin{cases} \sqrt{\frac{v(d+c)}{\theta r+1}} - d - c & \text{if } d < \frac{v}{\theta r+1} - c, \text{ and } \\ & \sqrt{v} - \sqrt{(\theta r+1)(d+c)} > \sqrt{P_a} \\ 0 & \text{otherwise} \end{cases} \quad (10)$$

Remark First, note that the terrorist's best-response function $\hat{a}(d, r|\theta, v)$ does not depend directly on the government's target valuation V and preference for rent β. (However, in Section 4, we will show that the equilibrium attack effort might depend on V or β). Second, note that $\hat{a}(d, r|\theta, v)$ always weakly decreases in r and θ, and weakly increases in v. Third, Equation (10) implies that there are three possible cases (see the proof in Appendix A.1).

A. If the terrorist's target valuation is low, $v \leqslant \left(\sqrt{(\theta r+1)(d+c)}+\sqrt{P_a}\right)^2$, then the terrorist will not attack (that is, being completely deterred).
B. If the terrorist's target valuation is intermediate, $\left(\sqrt{(\theta r+1)(d+c)}+\sqrt{P_a}\right)^2 < v \leqslant 4c(\theta r+1)$, then $\hat{a}(d, r)$ will initially decrease in d for $0 \leqslant d < v/(\theta r+1)-c$, and then go to zero for $d \geqslant v/(\theta r+1)-c$, at which point the terrorist will be completely deterred.
C. If the terrorist's target valuation is high, $v > \max\{4c(\theta r+1), \left(\sqrt{(\theta r+1)(d+c)}+\sqrt{P_a}\right)^2\}$, then $\hat{a}(d, r)$ will initially increase in d for $0 \leqslant d < v/(4(\theta r+1))-c$, then decrease in d for $v/(4(\theta r+1))-c < d < v/(\theta r+1)-c$, and finally go to zero for $d \geqslant v/(\theta r+1)-c$, at which point the terrorist will be completely deterred.

Figure 2 shows the possible terrorist's best responses when $c=1$. Figures 2(a–d), (e–f), and (g–h) show the cases when $P_a=0$, $P_a=0.1$, and $P_a=2$, respectively. In particular, Figures 2(a, c, e, g) show that when $\theta=0$, the terrorist's best responses do not depend on the rent. When the terrorist's valuation of the target is intermediate ($v=2$), satisfying the condition $d+c=\left(\sqrt{(\theta r+1)(d+c)}+\sqrt{P_a}\right)^2 < v \leqslant 4(\theta r+1)=4$, Figure 2(a) shows that $\hat{a}(d, r)$ will initially decrease in d for $0 \leqslant d < v/(\theta r+1)-c=1$, and then zero for $d \geqslant 1$, at which point the terrorist will be completely deterred. By contrast, when the terrorist's target valuation is high ($v=9$), satisfying $v > 4(\theta r+1)=4$ and $v > \left(\sqrt{(\theta r+1)(d+c)}+\sqrt{P_a}\right)^2 = d+c$, Figure 2(c) shows that the terrorist's best response $\hat{a}(d, r)$ will initially increase in d for $0 \leqslant d = v/(4(\theta r+1))-1 < 1.25$, then decrease in d for $1.25 < d < v/(\theta r+1)-c=8$, and finally zero for $d \geqslant 8$, at which point the terrorist will be completely deterred. By contrast, Figures 2(b) and (d) show when $\theta=1$ (that is, the terrorist values rent and damage equally), the terrorist's best responses $\hat{a}(d, r)$ weakly decrease in the rent r. In Figure 2(b), since the target valuation is intermediate satisfying $(r+1)(d+c) < v \leqslant 4(r+1), \hat{a}(d, r)$ will initially decrease in d for $0 \leqslant d < (2/(r+1))-1$, and then zero for $d \geqslant 2/(r+1)-1$ or $r \geqslant 1$, at which point the terrorist will be completely deterred. In Figure 2(d), since the target valuation is high, the terrorist's best response $\hat{a}(d, r)$ will initially increase in d for $0 \leqslant d < 9/(4(r+1))-1$, then decrease in d for $9/(4(r+1))-1 < d < 9/(r+1)-1$, and finally zero for $d \geqslant 9/(r+1)-1$, at which point the terrorist will be completely deterred. Comparing Figures 2(a–d) with (e–h), respectively, we note that when P_a is larger, the attacker effort is smaller.

4. Analytical solution and numerical illustration for complete information model

4.1. Analytical solution

To find the equilibria of this sequential game, we insert the terrorist's best-response function (10), into the government's optimization problem (7), and solve for the optimal defense investment d^* and rent r^* to maximize the government's expected utility:

$$\max_{d \geqslant 0, r \geqslant 0} U_d[\hat{a}(d, r), d, r] = \min_{d \geqslant 0, r \geqslant 0} d + \frac{\hat{a}(d, r)V}{\hat{a}(d, r)+d+c} + \beta r + P_d \cdot \mathbf{1}_{\{a>0\}} \tag{11}$$

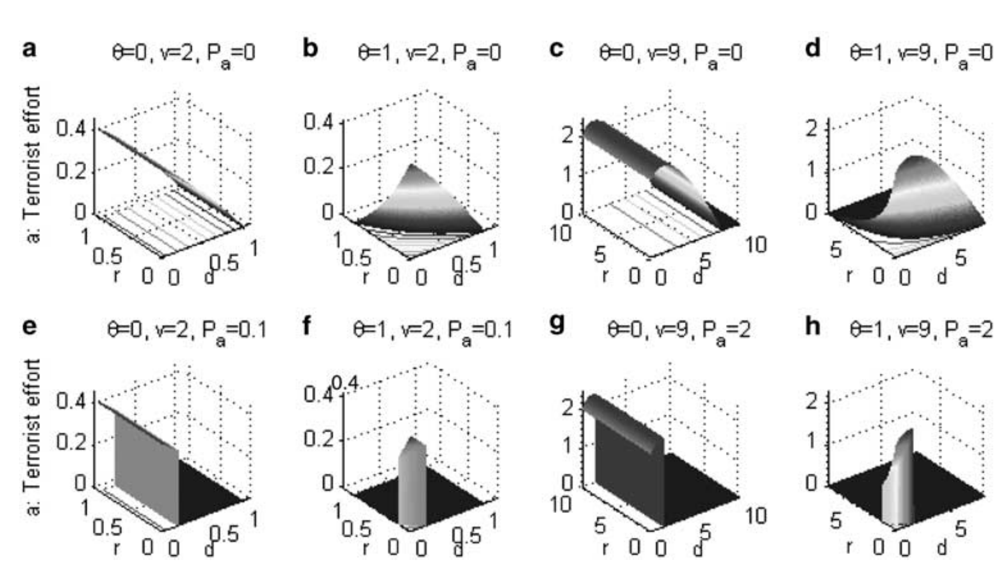

Figure 2 Possible terrorist's best-response functions when $c=1$.

$$= \min_{d\geqslant 0, r\geqslant 0} \begin{cases} d + \dfrac{\left(\sqrt{\frac{v(d+c)}{\theta r+1}} - d - c\right)V}{\sqrt{\frac{v(d+c)}{\theta r+1}}} + \beta r + P_d & \text{if } d < \frac{v}{\theta r+1} - c, \text{ and } \\ & \sqrt{v} - \sqrt{(\theta r+1)(d+c)} > \sqrt{P_a} \\ d + \beta r & \text{otherwise} \end{cases} \tag{12}$$

Proposition 2 *For any fixed collection of parameter values $(\theta, v, \beta, V, c, P_a, P_d)$, there exists a unique equilibrium (a^*, d^*, r^*), as shown in* Table 2 *for the case $P_a = 0$. Moreover*

a. The government's equilibrium utility U_d^* (weakly) increases in the terrorist's preference for rent θ. The terrorist's equilibrium effort a^* (weakly) decreases in θ.
b. The government's equilibrium utility U_d^* and equilibrium rent r^* (weakly) decrease in the government's preference for rent β. The terrorist's equilibrium effort a^* (weakly) increases in β.

Remark In principle, there exist eight possible optimal solutions: Case 1: $a^* = d^* = r^* = 0$; Case 2: $a^* = d^* = 0$, $r^* > 0$; Case 3: $a^* = r^* = 0$, $d^* > 0$; Case 4: $a^* = 0$, $d^* = r^* > 0$; Case 5: $a^* > 0$, $d^* = r^* = 0$; Case 6: $a^* > 0$, $d^* > 0$, $r^* = 0$; Case 7: $a^* > 0$, $r^* > 0$, $d^* = 0$; and Case 8 : $a^* > 0$, $d^* > 0$, $r^* > 0$. When $P_a = 0$, the first seven cases are possible and the Case 8 is not possible (see the proof in Appendix A.2.2).[1] (For $P_a > 0$, the solution becomes too complicated and intractable, and we omit it for simplicity. However Section 4.6 provides numerical illustration when $P_a \geqslant 0$.)

In Table 2, the first seven possible equilibria are provided. Figures 3(a–d) show four typical cases of equilibrium solutions with contours representing the government's utility level, corresponding to ($d^* = 0$, $r^* > 0$), ($d^* > 0, r^* > 0$), ($d^* > 0$, $r^* = 0$), and ($d^* = 0, r^* = 0$), respectively. In subsections 4.2–4.5, we will study the typical cases when ($v = V = 2$), ($v = 2, V = 9$), ($v = 9, V = 2$), and ($v = V = 9$), respectively, given $c = 1$, $P_a = P_d = 0$. In subsection 4.6, we will study the effect of general values of $c \geqslant 0$, $P_a \geqslant 0$, $P_d \geqslant 0$ to the equilibrium solution.

4.2. Low terrorist's and low government's target valuations

When $v = V = 2$, both the terrorist's and government's target valuations are low. Table 2 degenerates to Table 3 below, containing three equilibrium Cases 2, 5, and 7.

Figure 4(a) shows the three possible cases of equilibria as a function of θ and β. Note that no defense is needed for all three cases. For a fixed value of $\beta = 30$, Figure 4(b) shows the sensitivity of equilibrium as a function of θ. As Proposition 2a predicts the government's utility U_d^* always weakly increases in θ. When the value of θ is low ($0 \leqslant \theta < 30\sqrt{2} = 42.4$), implying that the terrorist does not value monetary gain too much, we have Case 5, where the terrorist's attack effort, government's and terrorist's utilities equal to constants: $a^* = \sqrt{2} - 1 = 0.41$, $U_d^* = \sqrt{2} - 2 = 0.59$, and $U_a^* = 3 - 2\sqrt{2} = 0.17$. When the value of θ is moderate ($42.4 \leqslant \theta < 2\beta = 60$), the terrorist has a moderate preference for rent, and we have Case 7, where the attack effort $a^* = 60/\theta - 1$, decreases in θ; the rent $r^* = \theta/1800 - 1/\theta$, the government's utility, $U_d^* = \theta/60 - 2 + 30/\theta$, and terrorist's utility, $U_a^* = \theta^2/900 - \theta/15 + 1$ all increase in θ. When $\theta \geqslant 60$, where the terrorist strongly prefers rent to damage, we have Case 2. The terrorist is deterred ($a^* = 0$) by a modest amount of rent. And the rent $r^* = 1/\theta$ converges to 0 when θ goes to infinity. The government's utility, $U_d^* = -30/\theta$, increases in θ while the terrorist's utility, $U_a^* = 1$, is a constant.

For a fixed value of $\theta = 30$, Figure 4(c) shows the sensitivity of equilibrium as a function of the government's preference for rent β. As Proposition 2b predicts, the government's utility U_d^* always weakly decreases in β. At low values of β, $0 \leqslant \beta < (1/2)\theta = 15$, where the government does not care much about the cost of providing rent, the government deters the terrorist by rent alone (Case 2), $r^* = 1/\theta \approx 0.03$; the terrorist's utility $U_a^* = 1$ is a constant over this interval, but the government's utility $U_d^* = -\beta/30$ decreases in β as she begins to care more about the rent she provides to the terrorist. At moderate values of β, $15 \leqslant \beta < (\sqrt{2}/2)\theta = 21.2$, the government deters the terrorist by rent (Case 7), $r^* = 15/\beta^2 - 1/30$, which

[1] Although Table 2 provides technical solution for this theoretical paper, and one of the contribution of this paper is to provide some novel insights in combating terrorism, instead of summarizing real-world scenarios, we still observe some real-world examples corresponding to each of the seven cases in Table 2: (Case 1): since the desert has little valuation to both the terrorist and government, we observe no attack, defense, and rent for desert targets; (Case 2): in many developing countries receiving foreign aid (rent), there is no attack or defense activities; (Case 3): after the 11 September 2001 attacks to the World Trade Center, there was no attack from Al Qaeda following the original attack, and no rent provided to Al Qaeda from the US government, although the US government is still defending from Al Qaeda; (Case 4): before the recent 2010 Korean peninsula crisis, South Korea provided rent to, and defended from, North Korea, and no attack happened; (Case 5): the government may have no incentive to defend from, and provide rent to, terrorists that are active in other countries; (Case 6): there is no (known) rent provided from the US government to Taliban recently, but the two players are still attacking and defending from each other in Afghanistan; and (Case 7): there might be no direct defense for oversea targets, while rent is still provided and attacks still occur.

For Case 8, which is not possible at equilibrium for complete information model, we still observe multiple real-world examples. For the Iraq example, we simultaneously observe insurgent attacks, US defense, and US rent to the Iraq government. For the North Korea case over the past 10 years, South Korea provided rent to, defended from, and was recently attacked by North Korea.

Note that as a building block, this paper focuses on a one-government-one-terrorist, one-period, contract game; extensions to multiple-government, multiple-terrorist, multiple-period models could be developed in the future, to better model the complex real-world scenarios.

Table 2 Equilibrium solutions for complete information model (when $P_a = 0$)

No. i	*Feasible set* F_i	*Optimal set* O_i	a^*	d^*	r^*	U_d^*	U_a^*
1	$\{v \leqslant c\}$	$\{F_1\}$	0	0	0	0	0
2	$\{v > c\}$	$\{F_2$ and $\beta/\theta < c$, $\beta v/(\theta c) - 2\sqrt{\beta v/\theta} + c < 0$, $\beta/\theta < c/(c-v)(\sqrt{c/v}V - V - P_d)$, $\beta/\theta < c/(c-v)(V^2/4v + c - V - P_d)$, $\beta/\theta < c/(c-v)(\theta V^2 c/(4\beta v) + \beta/\theta - V - P_d)\}$	0	0	$1/\theta(v/c-1)$	$\beta/\theta(1-v/c)$	$v/c-1$
3	$\{v > c\}$	$\{F_3$ and $\beta/\theta > c$, $\beta/\theta - 2\sqrt{\beta v \theta} + v < 0$, $\sqrt{c/v}V - V + v - c < P_d$, $\theta V^2 c/(4\beta v) + \beta/\theta - V - c + v < P_d$, $V^2/(4v) - V + v - c < P_d\}$	0	$v-c$	0	$c-v$	0
4	$\{c^2/v < \beta/\theta < v\}$	$\{F_4$ and $\beta v/(\theta c) - 2\sqrt{\beta v/\theta} + c > 0$, $\beta/\theta - 2\sqrt{\beta v/\theta} + v > 0$, $\beta/\theta - 2\sqrt{\beta v/\theta} - \sqrt{c/v}V + V + P_d + c > 0$, $\beta/\theta - 2\sqrt{\beta v/\theta} + V + P_d > V^2/(4v)$, $\theta V^2 c/(4\beta v) + 2\sqrt{\beta v/\theta} < V + P_d + c\}$	0	$\sqrt{v\beta/\theta} - c$	$1/\theta(\sqrt{\theta v/\beta} - 1)$	$\beta/\theta - 2\sqrt{v\beta/\theta} + c$	$\sqrt{\theta v/\beta} - 1$
5	$\{v > c\}$	$\{F_5$ and $\beta/\theta - \beta v/(\theta c) < \sqrt{c/v}V - V - P_d$, $\sqrt{c/v}V - V + v - c > P_d$, $\sqrt{c/v}V - V - P_d > \beta/\theta - 2\sqrt{\beta v/\theta} + c$, $\sqrt{c/v}V > V^2/(4v) + c$, $\sqrt{c/v}V > \theta V^2 c/(4\beta v) + \beta/\theta\}$	$\sqrt{vc} - c$	0	0	$\sqrt{c/v}V - V - P_d$	$v + c - 2\sqrt{vc}$
6	$\{2\sqrt{vc} < V < 2v\}$	$\{F_6$ and $V^2/(4v) + \beta/\theta(v/c-1) - V + c > P_d$, $V^2/(4v) - V + v > P_d$, $V^2/(4v) - V - \beta/\theta + 2\sqrt{\beta v/\theta} > P_d$, $V^2/(4v) + c > \sqrt{c/v}V$, $V^2/(4v) + c > \theta V^2 c/(4\beta v) + \beta/\theta\}$	$V/2 - V^2/(4v)$	$V^2/(4v) - c$	0	$V^2/(4v) + \mathrm{c} - V - P_d$	$V^2/(4v) + v - V$
7	$\{cV/(2v) < \beta/\theta < (V/2)\sqrt{c/v}\}$	$\{\mathrm{F}_7$ and $\theta V^2 c/(4\beta v) + \beta v/(\theta c) - V > P_d$, $\theta V^2 c/(4\beta v) + \beta/\theta - V - c + v > P_d$, $\theta V^2 c/(4\beta v) + 2\sqrt{\beta v/\theta} - V - c > P_d$, $\theta V^2 c/(4\beta v) + \beta/\theta > \sqrt{c/v}V$, $\theta V^2 c/(4\beta v) + \beta/\theta > V^2/(4v) + c\}$	$2\beta v/(\theta V) - c$	0	$\theta V^2 c/(4\beta^2 v) - 1/\theta$	$\theta V^2 c/(4\beta^2 v) - V + \beta/\theta - P_d$	$(c+1)\theta^2 V^2/(4\beta^2 v) - \theta V c/\beta + v - 1$

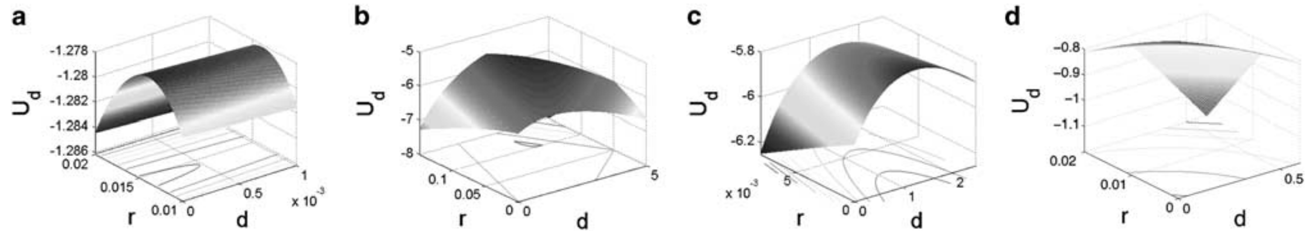

Figure 3 Government's utilities as a function of defense and rent in the complete information model.

Table 3 Equilibrium solution when $v=V=2$, $c=1$, $P_a=P_d=0$

No. i	F_i	O_i	a^*	d^*	r^*	U_d^*	U_a^*
2	$v>1$	$0<\beta/\theta<1/2$	0	0	$1/\theta$	$-\beta/\theta$	1
5	$v>1$	$\beta/\theta>\sqrt{2}/2$	$\sqrt{2}-1$	0	0	$\sqrt{2}-2$	$3-2\sqrt{2}$
7	$v>1,\ 1/2<\beta/\theta<\sqrt{2}/2$	$v>1,\ 1/2<\beta/\theta<\sqrt{2}/2$	$2\beta/\theta-1$	0	$\theta/(2\beta^2)-1/\theta$	$\theta/(2\beta)-2+\beta/\theta$	$(\theta/\beta-1)^2$

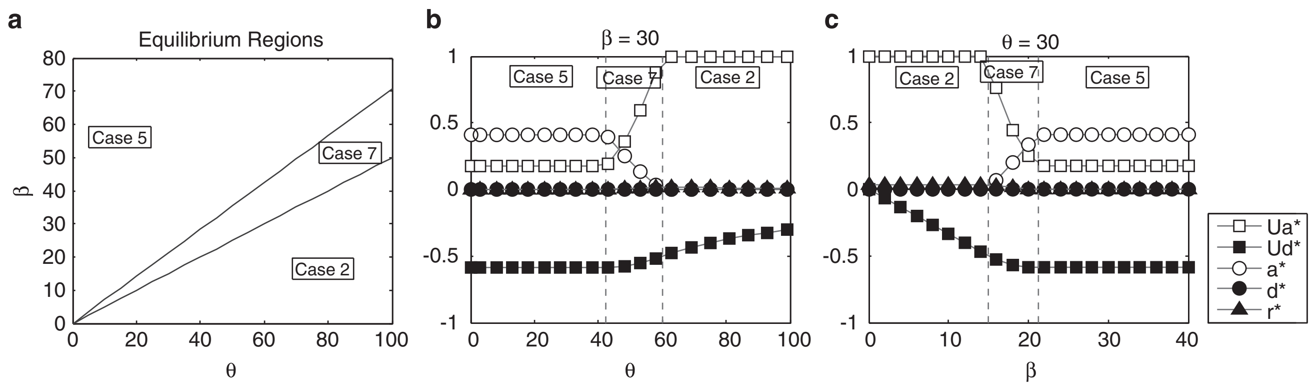

Figure 4 Equilibrium of the complete information model when $v=2$, $V=2$, $c=1$, $P_a=P_d=0$.

decreases in β; both the government's and terrorist's utilities, $U_d^*=15/\beta+\beta/30-2$ and $U_a^*=(30/\beta-1)^2$, decrease in β. At high values of β, $\beta\geqslant 21.2$, rent becomes too costly to the government; therefore, the government does not defend and offer rent (Case 5), $d^*=r^*=0$; and both the government's the terrorist's utilities stay constant over this range.

4.3. Low terrorist's and high government's target valuations

When $v=2$ and $V=9$, the terrorist's target valuation is relatively low, while the government's target valuation is relatively high. Table 2 degenerates to Table 4 below, containing three possible equilibria Cases 2, 3, and 4.

Figure 5(a) shows the three possible equilibria as a function of θ and β. Note that in all three cases we have no attacks at equilibrium, which implies that the terrorist is fully deterred due to the relative low target valuation. Specifically, the terrorist is deterred by rent only in Case 2, by defense only in Case 3, and by a combination of defense and rent in Case 4. For a fixed value of $\beta=30$, Figure 5(b) shows the sensitivity of equilibrium as a function of θ. When the value of θ is low ($0\leqslant\theta<\beta/2=15$), implying that the terrorist does not value monetary gain too much, we have Case 3, where the government provides no rent, but uses defense effort $d^*=1$ to fully deter terrorists; both the terrorist's attack effort and utility are zero. At moderate values of θ, $15\leqslant\theta<2\beta=60$, where the terrorist more prefers rent, we have Case 4, where the terrorist is completely deterred by a combination of rent and defense; defense effort $d^*=\sqrt{60/\theta}-1$ decreases in θ; rent $r^*=1/\theta$ $(\sqrt{\theta/15}-1)$ also decreases in θ; and both the government's utility $U_d^*=30/\theta-\sqrt{240/\theta}+1$ and the terrorist's utility $U_a^*=\sqrt{\theta/15}-1$ increase in θ. At high values of $\theta\geqslant 60$, where the terrorist strongly prefers rent to damage, we have Case 2, where the terrorist is completely deterred by only a modest amount of rent, $r^*=1/\theta\in(0,1/60)=(0,0.017)$; the government's utility $U_d^*=-30/\theta$ increases in θ; while the terrorist's utility is constant $U_a^*=1$.

Table 4 Equilibrium solutions when $v=2$, $V=9$, $c=1$, $P_a=P_d=0$

No. i	F_i	O_i	a^*	d^*	r^*	U_d^*	U_a^*
2	$v>1$	$v>1$, $\beta/\theta<1/2$	0	0	$1/\theta$	$-\beta/\theta$	1
3	$v>1$	$v>1$, $\beta/\theta>2$	0	1	0	-1	0
4	$v>1$	$v>1$, $1/2<\beta/\theta<2$	0	$\sqrt{2\beta/\theta}-1$	$1/\theta(\sqrt{2\theta/\beta}-1)$	$(\beta/\theta-\sqrt{2})^2-1$	$\sqrt{2\theta/\beta}-1$

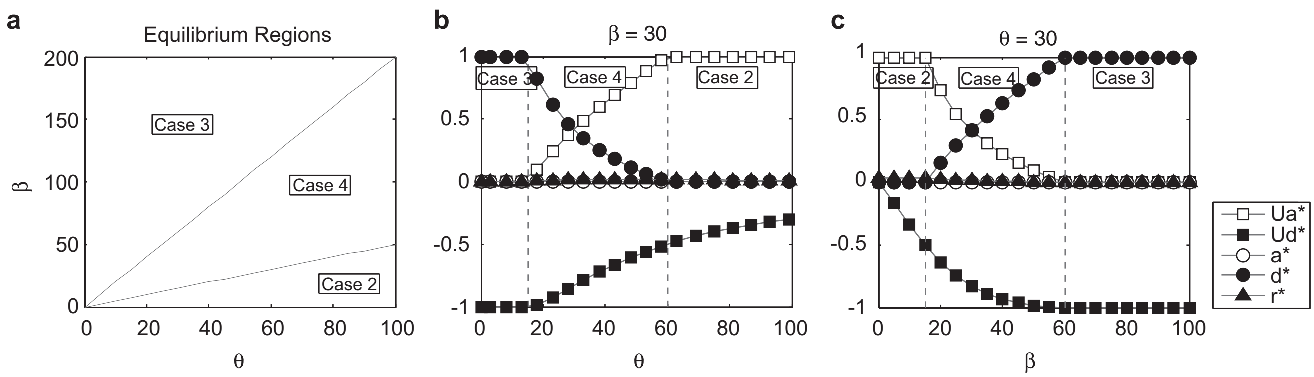

Figure 5 Equilibrium of the complete information model when $v=2$, $V=9$, $c=1$, $P_a=P_d=0$.

For a fixed value of $\theta=30$, Figure 5(c) shows the sensitivity of equilibrium as a function of the government's preference for rent β. At low values of β, $0\leqslant\beta<\theta/2=15$, where the government does not care too much about the cost of providing rent, the government deters the terrorist by rent alone, $r^*=1/\theta=0.033$ (Case 2); the government's utility, $U_d^*=-\beta/30$, decreases in β; but the terrorist's utility, $U_a^*=1$, stays constant over this interval, as she begins to care more about the rent provided to the terrorist. At moderate values of β, $15\leqslant\beta<2\theta=60$, the government deters the terrorist by a combination of defense and rent (Case 4), where $d^*=\sqrt{\beta/15}-1$ increases in β, $r^*=\sqrt{2/\beta}-1/30$ decreases in β; and both the government's and terrorist's utilities, $U_d^*=(\beta/30-\sqrt{2})^2-1$ and $U_a^*=\sqrt{60/\beta}-1$, decrease in β. At high values of $\beta, \beta\geqslant 60$, the government is unlikely to offer rent and deters the terrorist by defense alone, $d^*=1$, $a^*=r^*=0$ (Case 3). Both the government's and terrorist's utilities become constant: $U_d^*=-1$ and $U_a^*=0$.

4.4. *High terrorist's and low government's target valuations*

Now we provide an example for which the terrorist's target valuation is relatively high, $v=9$; while the government's target valuation is relatively low, $V=2$. Table 2 degenerates to Table 5, containing equilibria Cases 2, 5, and 7. Figure 6(a) shows the three possible equilibria as a function of θ and β. Note that in all three cases here we have no defense at equilibrium, because of the low government target valuation. Rent and attack effort can be either positive or zero at equilibrium.

For a fixed value of $\beta=7$, Figure 6(b) shows the sensitivity of equilibrium as a function of the terrorist's preference for rent, θ. When the value of θ is relatively low ($0\leqslant\theta<3\beta=21$), implying that the terrorist does not value the rent too much, we have Case 5, in which the government does not defend or offer rent, $d^*=r^*=0$, because of the low target valuation and her utility is $U_d^*=-4/3$; while the terrorist's equilibrium effort is $a^*=2$ and he enjoys an equilibrium utility $U_a^*=4$. When $21\leqslant\theta<9\beta=63$, implying that the terrorist moderately values the rent, we observe that the government uses the rent only to deter the terrorist (Case 7), $r^*=\theta/441-1/\theta$ increases in θ, and her utility $U_d^*=\theta/63+7/\theta-2$ increases in θ; while the terrorist reduces attack effort in return to receive some modest amount of rent, where $a^*=63/\theta-1$ decreases in θ; it is interesting to observe that the terrorist's utility $U_a^*=2\theta^2/441-2\theta/7+8$ decreases in θ when $21\leqslant\theta<9\beta/2=31.5$ (the first-order condition $\partial U_a^*/\partial\theta=0\Rightarrow\theta=9\beta/2$), due to the penalty of receiving rent and attacking; and increases in θ when $31.5\leqslant\theta<9\beta=63$. When $\theta\geqslant 63$, which implies the terrorist prefers rent very much, so he does not attack and receives a modest amount of rent, $r^*=8/\theta$, which decreases in θ; and his utility $U_a^*=8$; the government's utility $U_d^*=-56/\theta$ (Case 2) increases in θ as the Proposition 2a predicts.

When $\theta=100$, which means the terrorist cares rent very much, Figure 6(c) illustrates the sensitivity of equilibrium as a function of β. While $0\leqslant\beta<\theta/9=11.1$, the government fully deters the terrorist by a modest amount of rent, where

Table 5 Equilibrium solutions when $v=9$, $V=2$, $c=1$, $P_a=P_d=0$

No. i	F_i	O_i	a^*	d^*	r^*	U_d^*	U_a^*
2	$v>1$	$v>1$, $\beta/\theta<1/9$	0	0	$8/\theta$	$-8\beta/\theta$	8
5	$v>1$	$v>1$, $\beta/\theta>1/3$	2	0	0	$-4/3$	4
7	$v>1$, $1/9<\beta/\theta<1/3$	$v>1$, $1/9<\beta/\theta<1/3$	$9\beta/\theta-1$	0	$\theta/(9\beta^2)-1/\theta$	$\theta/(9\beta)+\beta/\theta-2$	$2\theta^2/(9\beta^2)-2\theta/\beta+8$

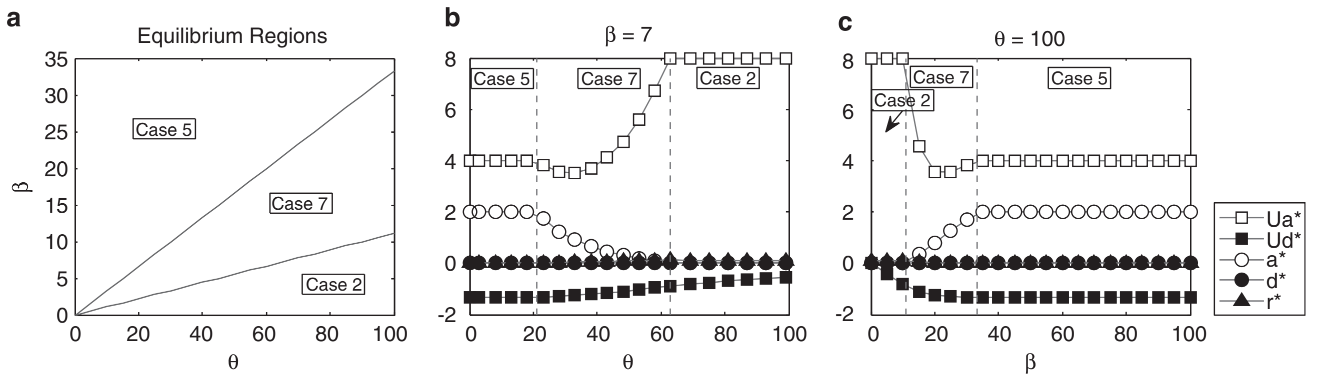

Figure 6 Equilibrium of the complete information model when $v=9$, $V=2$, $c=1$, $P_a=P_d=0$.

Table 6 Equilibrium solutions when $v=V=9$, $c=1$, $P_a=P_d=0$

No. i	F_i	O_i	a^*	d^*	r^*	U_d^*	U_a^*
2	$v>1$	$v>1$, $\beta/\theta<1/9$	0	0	$8/\theta$	$-8\beta/\theta$	8
4	$v>1$, $1/9<\beta/\theta<9$	$v>1$, $1/9<\beta/\theta<9/4$	0	$3\sqrt{\beta/\theta}-1$	$1/\theta(3\sqrt{\theta/\beta}-1)$	$\beta/\theta-6\sqrt{\beta/\theta}+1$	$3\sqrt{\theta/\beta}-1$
6	$v>1$, $6<V<18$	$v>1$, $\beta/\theta>9/4$	9/4	5/4	0	$-23/4$	9/4

$r^*=8/\theta=0.08$, $U_d^*=-0.08\beta$ decreases in β, and $U_a^*=8$ (Case 2). When $11.1 \leqslant \beta<\theta/3=33.3$, the government moderately values the rent and therefore she provides smaller and smaller amount of rent, which cannot fully deter the terrorist (Case 7); note that again the terrorist's utility, $U_a^*=2\times10^4/(9\beta^2)-200/\beta+8$, initially decreases in $11.1\leqslant\beta<2\theta/9=22.2$ (the first-order condition $\partial U_a^*/\partial\beta=0\Rightarrow\beta=2\theta/9=22.2$) due to the binding effect of the smaller amount of received rent, and when $22.2\leqslant\beta<33.3$, U_a^* increases due to the relaxation of the binding of decreasing rent; the government's utility, $U_d^*=100/(9\beta)+\beta/100-2$, decreases in β. When the value of β is high ($\beta\geqslant 33.3$), the government values the rent very much and still has a low target valuation; as a result no defense and rent provided to the terrorist, $d^*=r^*=0$; and a positive attack effort exists, $a^*=2$; both the government's and terrorist's utilities are constant, $U_d^*=-4/3$ and $U_a^*=4$ (Case 5).

4.5. High terrorist's and high government's target valuations

At last, we provide an example of high terrorist's and government's target valuation, where $v=9$ and $V=9$. Table 2 degenerates to Table 6, which includes the equilibrium Cases 2, 4, and 6. Figure 7(a) shows the three equilibria as a function of θ and β.

For a fixed value of $\beta=7$, Figure 7(b) shows the sensitivity of equilibrium as a function of θ. When $0\leqslant\theta<4\beta/9=3.1$, the terrorist does not value the rent too much and therefore the equilibrium rent is zero, while both attack effort and defense effort are positive constants at equilibria (Case 6), $a^*=9/4$, $d^*=5/4$, to compete for this valuable target. When $3.1\leqslant\theta<9\beta=63$, the terrorist moderately values the rent so he is deterred by both rent and defense effort (Case 4), where $d^*=3\sqrt{7/\theta}-1$ decreases in θ, $r^*=1/\theta(3\sqrt{\theta/7}-1)$ decreases in θ,

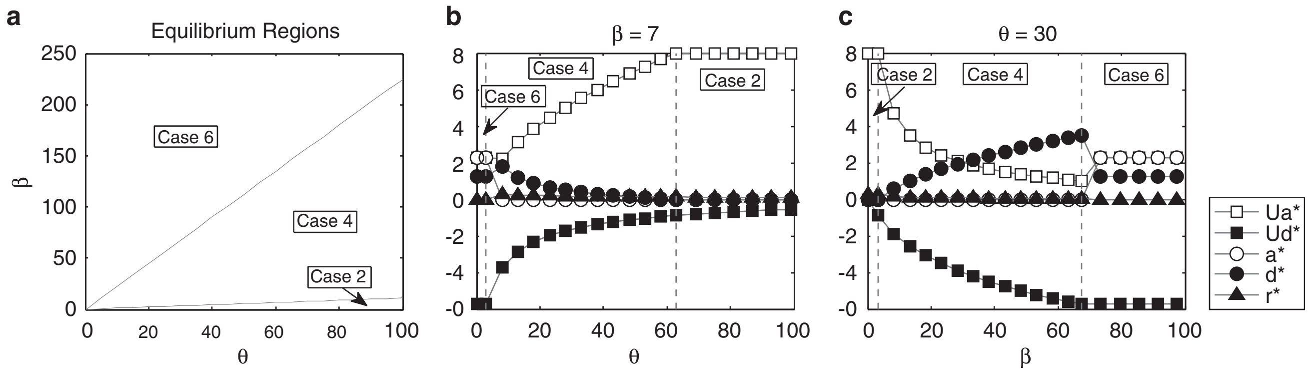

Figure 7 Equilibrium of the complete information model when $v=9$, $V=9$, $c=1$, $P_a=P_d=0$.

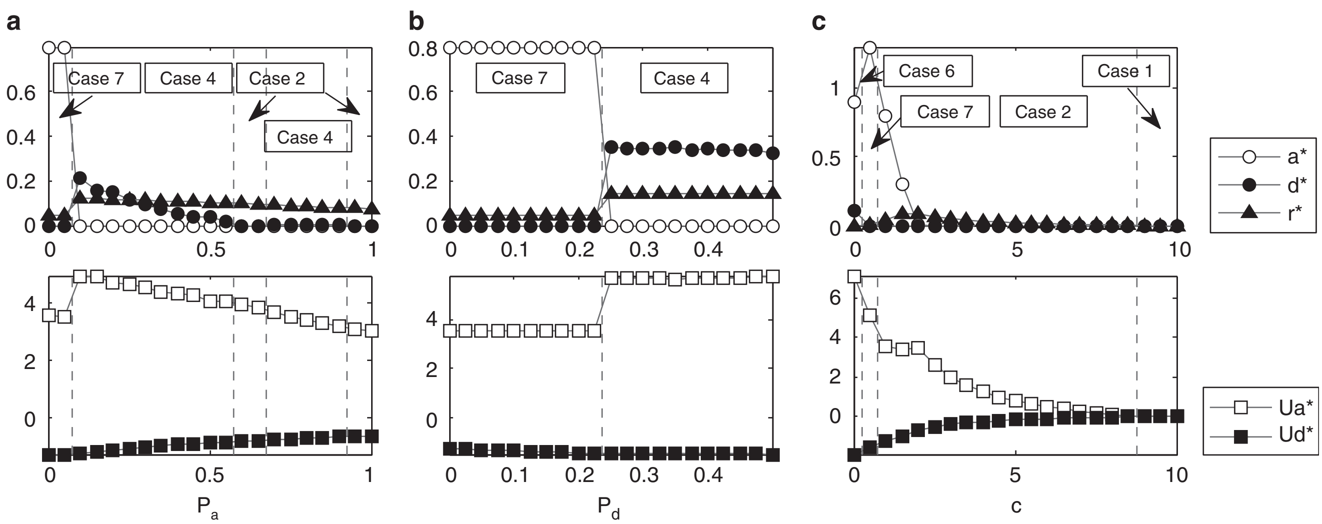

Figure 8 Sensitivity analysis of P_a, P_d, and c.

$U_d^*=7/\theta-6\sqrt{7/\theta}+1$ slowly increases in θ, and $U_a^*=3\sqrt{\theta/7}-1$ increases in θ. When $\theta\geqslant 63$, the terrorist values the rent sufficiently so that he does not attack and receives a modest amount of rent, $r^*=8/\theta$ decreases in θ, and $U_a^*=8$; the government does not defend, $U_d^*=-56/\theta$ increases in θ (Case 2).

For a fixed value of $\theta=30$, Figure 7(c) shows the equilibrium sensitivity as a function of β and there are three possible cases. When $0\leqslant\beta<\theta/9=3.3$, the government dose not value the rent too much and therefore generously use a positive rent, $r^*=8/30$, to fully deter the terrorist without using defense, $d^*=0$; the government's utility, $U_d^*=-8\beta/30$ decreases in β; the terrorist's utility is constant, 8 (Case 2). When $3.3\leqslant\beta<9\theta/4=67.5$, the government moderately valuates the rent and therefore uses positive rent and defense, to deter the terrorist (Case 4), where $d^*=3\sqrt{\beta/30-1}$ increases in β, $r^*=1/30(3\sqrt{30/\beta}-1)$ decreases in β, and $U_d^*=\beta/30-6\sqrt{\beta/30}+1$ decreases in β. When $\beta\geqslant 67.5$, the government values the rent very much, and therefore she stops using the rent, and the terrorist increases the attack effort. Rent becomes zero at equilibria and $d^*=5/4$, $a^*=9/4$, $U_d^*=-23/4$, $U_a^*=9/4$ (Case 6).

4.6. Sensitivity analysis of c, P_a, and P_d

In this section, we analyse how the government's and terrorist's equilibria strategies a^*, d^*, r^*, and equilibrium utilities U_a^*, U_d^* change with respect to the parameters P_a, P_d, and c. The baseline parameters' values are $v=9$, $V=2$, $\beta=8$, $\theta=40$, $c=1$, $P_a=0$, $P_d=0$, for Figure 8, and $v=2$, $V=2$, $\beta=10$, $\theta=10$, $c=1$, $P_a=0$, $P_d=0$, or Figure 9, respectively. With such baseline values, Figure 8 contains the Cases 1, 2, 4, 6 and 7, and Figure 9 contains Cases 1, 3–6; thus these two figures contain all the possible seven cases as mentioned in the remark after Proposition 2.

It is interesting to observe that: (1) The optimal defense level d^* weakly decreases in the terrorist's penalty P_a (Case 4 of Figure 8(a), and Case 3 of Figure 9(a)), and inherent defense c (Case 6 of Figure 9(c)); (2) the equilibrium attack effort a^* (slightly) increases in c when c

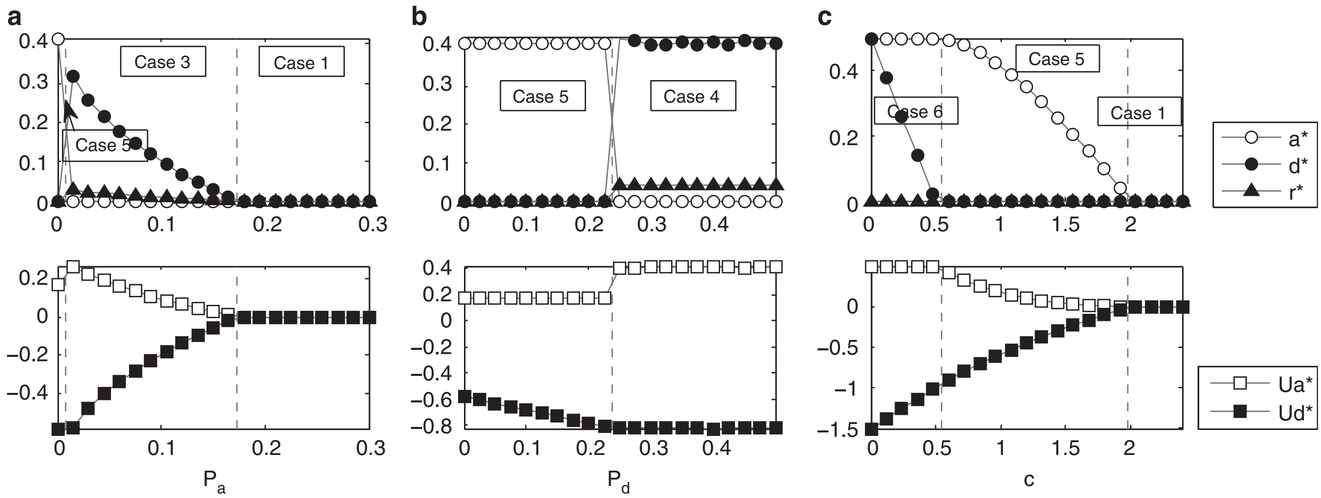

Figure 9 Sensitivity analysis of P_a, P_d, and c.

is small (Cases 6 of Figures 8(c) and 9(c)), and decreases in c when c is large (Cases 2 and 7 in Figure 8(c) and Case 5 in Figure 9(c)), due to the deterring effectiveness of marginal increase of c to the attack effort based on Equation (1); (3) the rent r^* increases in P_a when P_a is small (Case 7 in Figure 8(a) and Case 5 in Figure 9(a)), and decreases in P_a when P_a is large (Case 4 in Figure 8 and Case 3 in Figure 9); and (4) the government's equilibrium utility U_d^* increases in c and P_a and decreases in P_d, while the terrorist's equilibrium utility U_a^* is not monotonic since the terrorist is the second mover.

5. Algorithm and numerical illustration for incomplete information model

In this section, we consider the equilibrium of incomplete information model as defined in Definition 1 in Section 2.3. Suggested by the complex solution for complete information model provided in Table 2, we expect an almost intractable solution for the incomplete information model. As a result we focus on the numerical solutions. For simplicity, in this section, we consider the cases that $P_a = P_d = 0$, and $c = 1$. Section 5.1 provides an algorithm, Sections 5.2–5.5 provide four numerical illustrations.

5.1. Algorithm

We consider both discrete and continuous probability distributions that θ and v could follow. Note that at optimality to the government, the cost of defense and rent must not be greater than the target valuation; that is, $d + \beta r \leqslant V$, therefore we have $d \leqslant V \equiv d_{\max}$ and $r \leqslant (V-d)/\beta \equiv r_{\max}$. For each possible values of (d, r), we calculate the terrorist's best response using Equation (10), and then calculate the equilibrium defense level and rent using Equation (4). The algorithm is described as following:

(1) Given θ and v, enumerate d from 0 to $d_{\max}$ by δ_d at each step: $d = 0, \delta_d, 2\delta_d, \ldots, d_{\max}$, and r from 0 to $r_{\max}$ by δ_r at each step, $r = 0, \delta_r, 2\delta_r, \ldots, r_{\max}$. For each pair of (d, r), calculate the best response of each terrorist's type (θ, v), $\hat{a}(d, r|\theta, v)$ using Equation (1).
(2) Calculate the government's utility $U_d(\hat{a}(d, r|\theta, v), d, r)$ using Equation (3).
(3) Calculate the expected value of the government's utility using Equation (4).
$EU_d(\hat{a}(d, r|\theta, v), d, r) = \sum_{\theta, v} f(\theta, v) U_d(\hat{a}(d, r|\theta, v), d, r)$.
(4) Compare the government's utility values $EU_d(\hat{a}(d, r|\theta, v), d, r)$ over all pairs of (d, r) and denote the maximum as U_d^*. Then locate the corresponding (d^*, r^*), the corresponding terrorist's attack effort $\hat{a}(d^*, r^*|\theta, v)$, and the terrorist's expected attack effort $E_a(d^*, r^*) = \sum_{\theta, v} f(\theta, v)\hat{a}(d^*, r^*|\theta, v)$.

If θ and v follow continuous probability distribution on the interval $[\theta_1, \theta_2]$ and $[v_1, v_2]$ according to a PDF $f(\theta, v)$, we approximate the intervals using Riemann summation (Bartle and Sherbert, 2007). The algorithm is as following. Consider the integration point, $\theta = \theta_1 + (\delta_\theta/2), \theta_1 + (3\delta_\theta/2), \ldots, \theta_2 - (\delta_\theta/2)$, and $v = v_1 + (\delta_v/2)$, $v_1 + (3\delta_v/2)$, $\ldots$, $v_2 - (\delta_v/2)$. Then using the similar procedure above for θ, v following discrete distribution to compute the expected value of government's utility $EU_d(\hat{a}(d, r|\theta, v), d, r) = \sum_{\theta, v} f(\theta, v) U_d(\hat{a}(d, r|\theta, v), d, r)\delta_\theta \delta_v$. Compare all the values of EU_d, denote the maximum value as U_d^*. Then locate the corresponding (d^*, r^*) and calculate the expected attack effort $Ea(d^*, r^*) = \sum_{\theta, v} f(\theta, v)\hat{a}(d, r|\theta, v)\delta_\theta\delta_v$.

In the following subsections 5.2–5.5, we use the algorithm provided to study four numerical examples when θ or v follows Bernoulli or uniform distributions.

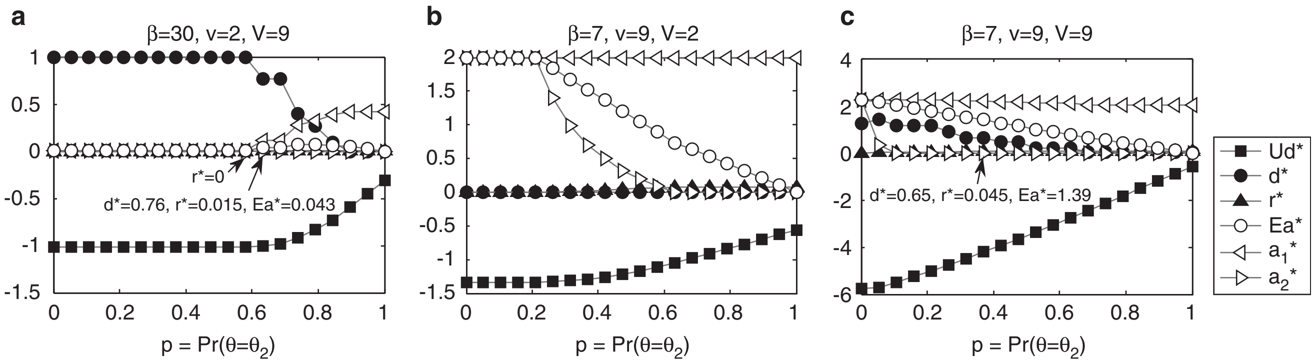

Figure 10 Equilibrium when θ follows Bernoulli distribution, $\theta_1 = 0$, $\theta_2 = 100$, $c = 1$, $P_a = P_d = 0$.

5.2. Numerical illustration when θ follows Bernoulli distribution

Here we study the scenario that θ follows Bernoulli distribution with the following PMF:

$$f(\theta) = \begin{cases} p & \text{if } \theta = 100 \\ 1-p & \text{if } \theta = 0 \\ 0 & \text{otherwise} \end{cases} \tag{13}$$

We study the equilibrium behaviour as a function of $p = \Pr(\theta = 100)$. When p is large the terrorist is more likely prefer rent. The terrorist's attack levels are a_1^* and a_2^* corresponding to $\theta = 0$ and $\theta = 100$, respectively.

Figure 10(a) shows that, when $0 \leqslant p \leqslant 0.58$, rent $r^* = 0$ and defense effort $d^* = 1$. It is interesting to observe the same attack effort of two types of terrorists ($a_1^* = a_2^* = E_a^* = 0.005$) when $r^* = 0$. When $p > 0.58$, r^* is positive, the defense level drops drastically, the attack effort of a_1^* increases, and the government's utility increases; this is because when the rent is positive, the terrorist type with high preference for rent does not attack ($a_2^* = 0$), and the type with zero preference for rent increases his attack effort (a_1^*) due to the decreased defense effort. So offering some amount of rent (possibly a little when the government's preference for rent is high) increases the government's utility significantly when p increases. Note that the terrorist's expected attack effort E_a^* increases in p for $0.63 < p < 0.73$, then decreases in p to zero for $p > 0.73$. It is interesting to observe that there exist equilibria such that $d^* > 0$, $r^* > 0$, $E_a^* > 0$ as marked in Figure 10(a) and (c) with ($d^* = 0.76$, $r^* = 0.015$, $E_a^* = 0.043$), and ($d^* = 0.65$, $r^* = 0.045$, $E_a^* = 1.39$), respectively. This is a different from complete information model in Table 2, which does not allow the case that ($d^* > 0$, $r^* > 0$, $a^* > 0$). In other words, in incomplete information model, the government might use both defense and rent even she expects a positive attack effort. The boundary conditions, $p = 0$ and $p = 1$ of Figures 10(a–c), match well with the boundary conditions ($\theta = 0, 100$) of Figures 5(b), 6(b), and 7(b), respectively. Note the government's expected utility weakly increases in the probability of terrorists having high preference for rent as shown in Figures 10(a–c).

5.3. Numerical illustration when θ follows uniform distribution

In this subsection, we analyse how the equilibrium changes when the terrorist's preference for rent θ varies according to a uniform distribution on the interval $[0, \theta_m]$. In other words, the PDF of θ is

$$f_\theta(\theta) = \begin{cases} \frac{1}{\theta_m} & \text{if } 0 \leqslant \theta \leqslant \theta_m \\ 0 & \text{if } \theta < 0 \text{ or } \theta > \theta_m \end{cases} \tag{14}$$

Comparing Figures 11(a–c) with Figure 5(b), Figure 6(b), Figure 7(b), respectively, we see that the equilibrium dynamics when θ follows uniform distribution has a similar shape to the equilibrium of corresponding complete information model. When the upper bound θ_m goes to infinity, the government's utility converges to zero. We observe that there exist equilibria such that $d^* > 0$, $r^* > 0$, $E_a^* > 0$ as marked in Figure 11(a) and (c) with ($d^* = 0.8$, $r^* = 0.003$, $E_a^* = 0.019$), and ($d^* = 1.36$, $r^* = 0.14$, $E_a^* = 0.3$), respectively. Note also Figure 13(a), (b), and (c) suggests the government's expected utility weakly increases in θ_m.

5.4. Numerical illustration when v follows Bernoulli distribution

In this subsection, we study the equilibrium when v follows two types of Bernoulli distribution. The PMF of v is

$$f(v) = \begin{cases} p & \text{if } v = 100 \\ 1-p & \text{if } v = 2 \\ 0 & \text{otherwise} \end{cases}$$

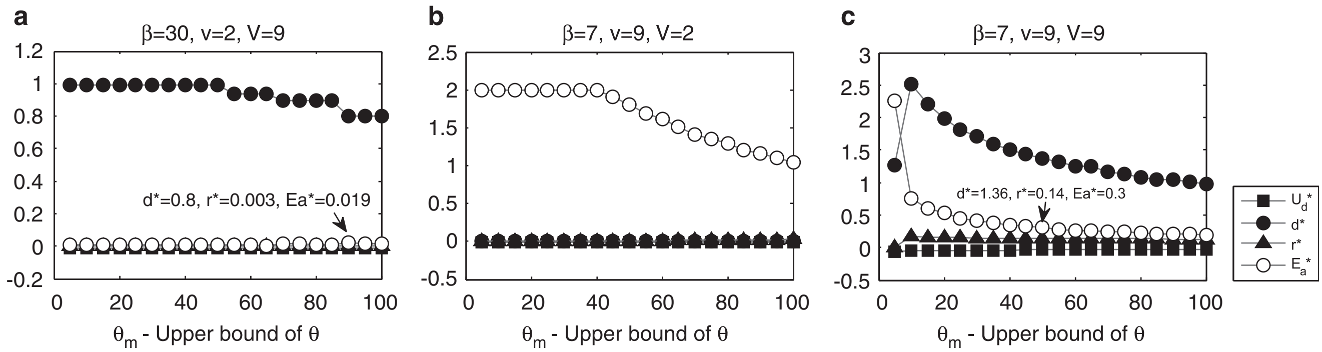

Figure 11 Equilibrium when θ follows uniform distribution, $\theta \in [0, \theta_m]$, $c=1$, $P_a=P_d=0$.

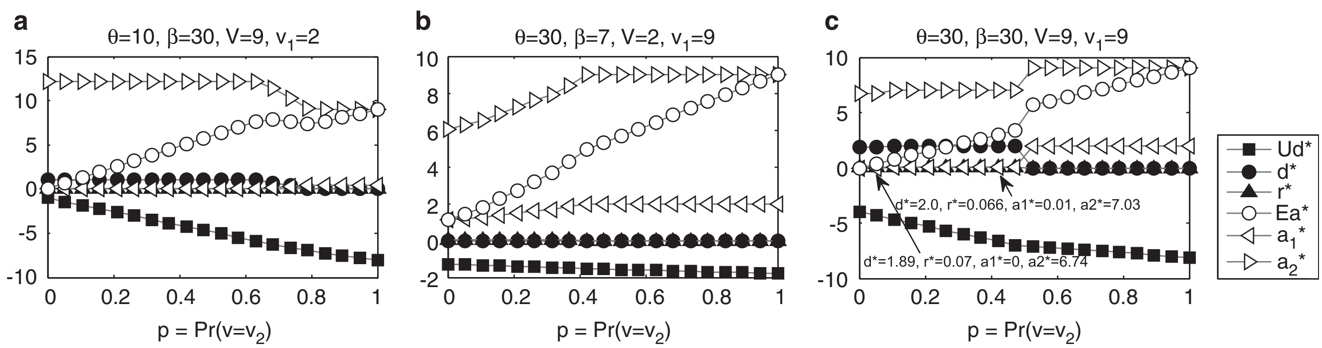

Figure 12 Equilibrium when v follows Bernoulli distribution, $v_2=100$, $c=1$, $P_a=P_d=0$.

or

$$f(v) = \begin{cases} p & \text{if } v = 100 \\ 1-p & \text{if } v = 9 \\ 0 & \text{otherwise} \end{cases} \tag{15}$$

The terrorist's attack levels a_1^* and a_2^* correspond to $v_1=2$, $v_2=100$ in Figure 12(a); and $v_1=9$ and $v_2=100$, respectively, in Figure 12(b–c). Figure 12(c) shows the existence of equilibrium when $d^*>0$, $r^*>0$, $E_a^*>0$. For example, when $p=0.05$, $d^*=1.89$, $r^*=0.07$, $a_1^*=0$, $a_2^*=6.74$, $E_a^*=0.36$; when $p=0.42$, $d^*=2.0$, $r^*=0.066$, $a_1^*=0.01$, $a_2^*=7.03$, $E_a^*=2.97$. Note that the government's expected utility weakly decreases in the probability of terrorists having high target valuation.

Comparing Figure 12(a–c), when $p=0$ and $p=1$ to Figures 5(b), 6(b), and 7(b), respectively, we found the boundary conditions of incomplete information model (when $p=0$) match well to the complete information model.

5.5. *Numerical illustration when v follows uniform distribution*

We also study when the terrorist's target valuation is uniformly distributed. Here we assume v is uniformly distributed on the interval $[0, v_m]$. The PDF of v is

$$f_v(v) = \begin{cases} \frac{1}{v_m} & \text{if } 0 \leqslant v \leqslant v_m \\ 0 & \text{if } v<0 \text{ or } v>v_m \end{cases} \tag{16}$$

There exist equilibria such that $d^*>0$, $r^*>0$, $E_a^*>0$, as marked on Figure 13(a) and (c) with ($d^*=1.92$, $r^*=0.063$, $E_a^*=1.02$), and ($d^*=0.75$, $r^*=0.22$, $E_a^*=0.048$), respectively. As we can see in Figures 13(a), (b), and (c), the government's expected utility weakly decreases in v_m. When v_m is larger, the government is more uncertain about the information of terrorists, then the optimal defense and rent offer may deviate far from the optimal value if the government knew the terrorist's information. For example, the government may defend too much on a target valued low by the terrorist.

6. Conclusion and future research directions

In this paper, we study the possibility of 'contracts' between a terrorist group and a government in a sequential game model with both complete and incomplete information. We allow the government to make two choices: defense effort and rent. The terrorist observes the defense and rent, and then decides on the attack effort.

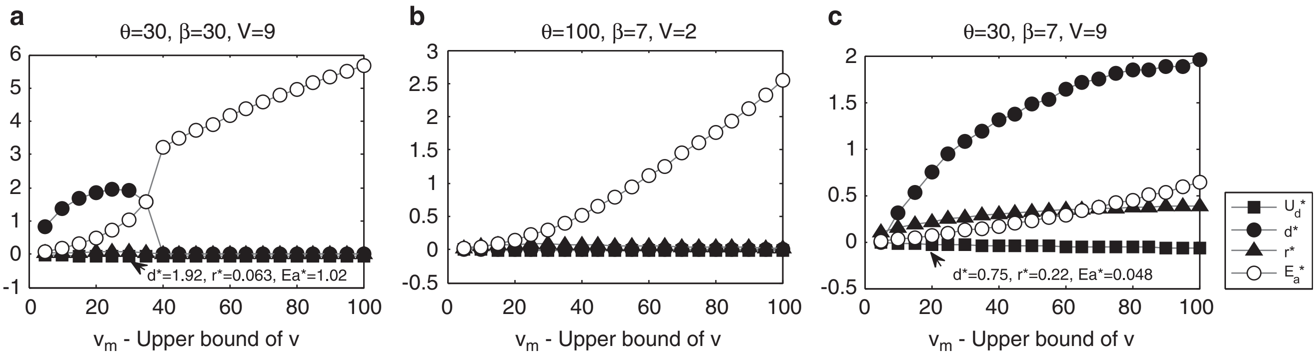

Figure 13 Equilibrium when v follows uniform distribution, $v \in [0, v_m]$, $c = 1$, $P_a = P_d = 0$.

Our analytical results in the complete information model suggest that the government's equilibrium utility always (weakly) increases in the terrorist's preference for rent, while the terrorist's equilibrium attack effort always (weakly) decreases in his preference for rent. Analogously, we find that the government's equilibrium utility and rent (weakly) decrease in her preference for rent, while the terrorist's equilibrium attack effort (weakly) increases in the government's preference for rent. We also find that it will not be optimal for the government to use both rent and defense, if the terrorist is going to launch an attack.

We numerically illustrate complete and incomplete information models. In both cases, we observe that at equilibrium the level of rent is small. This is because, if the terrorist does not value rent highly, then there is no need to provide rent; while if the terrorist does value the rent highly, then only a modest amount of rent will sufficiently satisfy the terrorist.

In contrast to the complete information model, the incomplete information model possesses the equilibrium where the government uses both defense and rent, while expecting some terrorist types (low preference for rent, or high target valuation) to attack. One corresponding real-world situation is that Iraq attacks while US defends and offers rent. Furthermore, our numerical analysis of the incomplete information model suggests that the government's expected utility weakly increases in the probability of terrorists having high preference for rent, and in the upper bounds of terrorists' preference for rent. Similarly, the government's expected utility weakly decreases in the probability of terrorists having high target valuation, and in the upper bounds of terrorists' target valuation.

Possible future research directions include modelling a multi-period contract game with incomplete information, where credibility is more important than in one-period games (see Zhuang *et al*, 2010). We also recognize that the functional forms (eg, contest success function and penalty) used in this paper are simplistic, and it is interesting to explore more realistic (and of course more complex) functions. We might forbid the terrorist to receive future rent if he attacks after receiving rent in the previous periods. We might allow the terrorist to have an option to decline an offered rent, which may eliminate some unrealistic results that the terrorist's attack effort always decreases in the offered rent.

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Appendix

A.1. Proof of Proposition 1

Based on the terrorist's utility specified in Equation (1), there are two possibilities:

(i) If $a>0$ then $U_a=-a+av/(a+d+c)+\theta r-\theta ar-P_a$, $\partial U_a/\partial a=\partial/\partial a(-a+av/(a+d+c)+\theta r-\theta ar-p_a)=0$, we have $a=\sqrt{v(d+c)/(\theta r+1)}-d-c$. In order to satisfy $a>0$, $\sqrt{v(d+c)/(\theta r+1)}-d-c>0 \Rightarrow d<v/(\theta r+1)-c$.
(ii) If $a=0$, then $U_a=\theta r$, the terrorist would not attack and will have the rent if it is offered.

Note that the terrorist will attack if and only if $U_a(\sqrt{v(d+c)/(\theta r+1)}-d-c)>U_a(0)$, which becomes

$$-a+\frac{a}{a+d+c}v+\theta r-\theta ar-P_a>\theta r \tag{A.1}$$

where $a=\sqrt{(v(d+c))/(\theta r+1)}-d-c$. Simplify this inequality to, $\sqrt{v}-\sqrt{(\theta r+1)(d+c)}>\sqrt{P_a}$. We also check that the attacker's utility, Equation (1), should be positive when the terrorist launches attack, that is, $a=\sqrt{v(d+c)/(\theta r+1)}-d-c$, which turns out to be $(\sqrt{v}-\sqrt{(\theta r+1)(d+c)})^2+\theta r>P_a$, which is redundant when inequality (A.1) is satisfied. In summary, we have the best response of the terrorist as the following:

$$\hat{a}(d,r|\theta,v)=\begin{cases}\sqrt{\frac{v(d+c)}{\theta r+1}}-d-c & \text{if } d<\frac{v}{\theta r+1}-c, \text{ and} \\ & \sqrt{v}-\sqrt{(\theta r+1)(d+c)}>\sqrt{P_a} \\ 0 & \text{otherwise}\end{cases}$$

This is Equation (10) shown in Section 3. Considering the case when the government offers the rent and defends, and the terrorist chooses the attack effort to maximize his utility, set the first derivative of $\hat{a}$ with respect to d equal to zero: $\partial\hat{a}/\partial d=\partial/\partial d(\sqrt{v(d+c)/(\theta r+1)}-d-c)=v/(4(d+c)(\theta r+1))-1=0$. Then we have $d=v/(4(\theta r+1))-c$. If $v/(4(\theta r+1))-c\leqslant 0$, or equivalently, $v\leqslant 4c(\theta r+1)$, the terrorist's attack level $\hat{a}$, is decreasing in d until fully deterred. When $v>4c(\theta r+1)$, the government defends and the terrorist responds with positive attack effort. Note that $\partial^2\hat{a}/\partial d^2=\partial^2/\partial d^2(\sqrt{v(d+c)/(\theta r+1)}-d-c)=-v/(4(\theta r+1)(d+c)^2)<0$, $\hat{a}$ will increase in d when $v/(4(d+c)(\theta r+1))-1>0 \Leftrightarrow d<v/(4(\theta r+1))-c$, and obtain the maximum value at $d=v/(4(d+c)(\theta r+1))-1$, then decrease in d when $v/(4(d+c)(\theta r+1))-1<0 \Leftrightarrow d>v/(4(\theta r+1))-c$.

A.2. Proof of Proposition 2

In order to prove Proposition 2, first, Section A.2.1 introduces Lemma 1 to prove the solutions listed in Table 2, which are mutually exclusive and collectively exhaustive; second, Section A.2.2 uses Lemma 1 to prove the equilibrium solutions; and finally Section A.2.2 proves the monotonic properties.

A.2.1. The completeness of the solutions of complete information model

Lemma 1 *The feasible set of parameter values $(\theta, v, \beta, V, c, P_d)$ is denoted as F_i, and the government's utility for case i is U_i, $i\in I$, where $I=\{1\},\{1,2\},\ldots,\{1,2,\ldots,7\}$. Denote the optimal set of parameters $(\theta, v, \beta, V, c, P_d)$ as*

$$O_i \equiv \bigcap_{\substack{i,j\in I\\ j\neq i}}\{\{U_i>U_j\}\cap F_i\cap F_j\}\cup\{F_i\cap\bar{F}_j\}$$

Then we have $\cup_{i\in I} O_i=\cup F_i$ and $\cap_{i\in I} O_i=\emptyset$.

Proof First we consider the case of $I=\{1,2\}$. Given feasible condition sets F_i, and equilibrium sets U_i, $i=1$, 2. Let $O_i=\cap_{\substack{j=1,2\\ j\neq i}}\{\{U_i>U_j\}\cap F_i\cap F_j\}\cup\{F_i\cap\bar{F}_j\}$. Then we want to show $\cup_{i=1,2} O_i=\cup F_i$, and $\cap_{i=1,2} O_i=\emptyset$.

In this case we have $O_1=\{\{U_1>U_2\}\cap F_1\cap F_2\}\cup\{F_1\cap\bar{F}_2\}$ and $O_2=\{\{U_2>U_1\}\cap F_2\cap F_1\}\cup\{F_2\cap\bar{F}_1\}$.

Let $T_1=\{U_1>U_2\}$. Then we have $O_1=\{T_1\cap F_1\cap F_2\}\cup\{F_1\cap\bar{F}_2\}$, and $O_2=\{\bar{T}_1\cap F_2\cap F_1\}\cup\{F_2\cap\bar{F}_1\}$ Then $O_1\cup O_2=\{T_1\cap F_1\cap F_2\}\cup\{\bar{T}_1\cap F_1\cap F_2\}\cup\{F_1\cap\bar{F}_2\}\cup\{\bar{F}_1\cap F_2\}=\{F_1\cap F_2\}\cup\{F_1\cap\bar{F}_2\}\cup\{\bar{F}_1\cap F_2\}$.

Denote the universal set, $\Omega\triangleq\cup F_i$, $i\in I$. Then $\forall x\in\Omega$, we must have either $x\in\{F_1\cap F_2\}$, or $x\in\{F_1\cap\bar{F}_2\}$, or $x\in\{\bar{F}_1\cap F_2\}$.

So we have $O_1\cup O_2=\cup F_i$, $i=1$, 2. $O_1\cap O_2=\{T_1\cap F_1\cap F_2\}\cap\{\bar{T}_1\cap F_1\cap F_2\}\cap\{F_1\cap\bar{F}_2\}\cap\{\bar{F}_1\cap F_2\}=$

$\{F_1 \cap F_2\} \cap \{F_1 \cap \bar{F}_2\} \cap \{\bar{F}_1 \cap F_2\}$. For all $x \in \Omega$, if $x \in \{F_1 \cap F_2\}$, then $x \notin \{F_1 \cap \bar{F}_2\}$; if $x \in \{F_1 \cap \bar{F}_2\}$, then $x \notin \{\bar{F}_1 \cap F_2\}$; if $x \in \{\bar{F}_1 \cap F_2\}$, then $x \notin \{F_1 \cap \bar{F}_2\}$.

So we have shown $O_1 \cap O_2 = \emptyset$. Therefore we have proved Lemma 1 for $I = 1, 2$. The case of $I = \{1, \ldots, 7\}$ could be similarly proved.

A.2.2. Proof of the analytical equilibrium solution for complete information model in Table 2

The analytical solutions of complete information model when $P_a = 0$ are derived by considering the following eight cases. For each case, we specifically denote the government's equilibrium utility as U_d^{*i}, $i = 1, \ldots, 8$.

Case 1: $(a^* = 0,\ d^* = 0,\ r^* = 0)$

According to Equation (7), the government's utility $U_d^{*1}(\hat{a}, d, r) = 0$. According to Equation (1), the terrorist's utility $U_a^* = 0$.

Case 2: $(a^* = 0,\ d^* = 0,\ r^* > 0)$ In order to satisfy $a^* = 0$, based on Equation (10), we have $d \geqslant v/(\theta r + 1) - c$ or $(\sqrt{v} - \sqrt{(\theta r + 1)(d + c)})^2 \leqslant 0$.

- If $d \geqslant v/(\theta r + 1) - c$, then $v/(\theta r + 1) - c \leqslant 0 \Rightarrow r \geqslant 1/\theta(v/c - 1)$. Based on Equation 7, $r^* = 1/\theta(v/c - 1)$, and $U_d^{*2}(\hat{a}, d, r) = -\beta r = \beta/\theta(1 - v/c)$ according to Equation (11), $U_a^* = \theta r = v/c - 1$ according to Equation (1).
- If $(\sqrt{v} - \sqrt{(\theta r + 1)(d + c)})^2 \leqslant 0$, which equals to $|\sqrt{(\theta r + 1)(d + c)} - \sqrt{v}| = 0$. So $r^* = 1/\theta(v/c - 1)$, and $U_d^{*2}(\hat{a}, d, r) = (\beta/\theta)(1 - v/c)$, and $U_a^* = v/c - 1$.

In summary, we have $r^* = 1/\theta(v/c - 1)$, $U_d^{*2}(\hat{a}, d, r) = \beta/\theta(1 - v/c)$, and $U_a^* = v/c - 1$.

Case 3: $(a^* = 0,\ d^* > 0,\ r^* = 0)$ In order to satisfy $a^* = 0$, based on Equation (10), we have $d \geqslant v - c$, or $(\sqrt{(d + c)} - \sqrt{v})^2 \leqslant 0$.

- If $d \geqslant v - c$, U_d^{*3} is maximized when $d^* = v - c$.
- If $(\sqrt{(d + c)} - \sqrt{v})^2 \leqslant 0$, which turns out to be $d = v - c$, then $U_d^{*3}(\hat{a}, d, r) = c - v$, and $U_a^* = 0$. So $d^* = v - c$, $U_d^{*3}(\hat{a}, d, r) = c - v$, and $U_a^* = 0$.

Case 4: $(a^* = 0,\ d^* > 0,\ r^* > 0)$ From $a^* = 0$, we have $d \geqslant v/(\theta r + 1) - c$, or $(\sqrt{v} - \sqrt{(\theta r + 1)(d + c)})^2 \leqslant 0$.

- If $d \geqslant v/(\theta r + 1) - c$, then solve the following minimization problem, based on Equation (7):

$$\begin{aligned} &\min_{d,r}\ d + \beta r \\ &\text{s.t. } d \geqslant \frac{v}{\theta r + 1} - c, \\ &d, r > 0 \end{aligned}$$

Using Karush–Kuhn–Tucker method (KKT) (Mokhtar and Shetty, 2006) the optimal problem is solved. Let $f(d, r) = d + \beta r$, $g(d, r) = v/(\theta r + 1) - c - d$, we have

$$\nabla f = \begin{pmatrix} 1 \\ \beta \end{pmatrix}, \quad \nabla g = \begin{pmatrix} -1 \\ -\dfrac{\theta v}{(\theta r + 1)^2} \end{pmatrix}$$

Then $\exists \mu \geqslant 0$, such that

$$\begin{cases} 1 - \mu = 0 \\ \beta - \mu \frac{\theta v}{(\theta r + 1)^2} = 0 \\ \mu(\frac{v}{\theta r + 1} - c - d) = 0 \end{cases}$$

Solving the above equations, we have $r^* = 1/\theta(\sqrt{\theta v/\beta} - 1)$, and $d^* = \sqrt{v\beta/\theta} - c$, also $U_d^{*4}(\hat{a}, d, r) = \beta/\theta - 2(\sqrt{v\beta/\theta} + c)$, $U_a^* = \theta v/\beta - 1$.

- If $(\sqrt{v} - \sqrt{(\theta r + 1)(d + c)})^2 \leqslant 0$, which equals to $(\sqrt{v} - \sqrt{(\theta r + 1)(d + c)})^2 = 0 \Rightarrow d = v/(\theta r + 1) - c$, then $U_d = -d - \beta r = -v/(\theta r + 1) + c - \beta r$, from Equation (11). Let $\mathrm{d}U_d/\mathrm{d}r = 0 \Rightarrow r^* = 1/\theta(\sqrt{\theta v/\beta} - 1)$, and $d^* = \sqrt{v\beta/\theta} - c$, also $U_d^{*4}(\hat{a}, d, r) = \beta/\theta - 2\sqrt{v\beta/\theta} + c$, $U_a^* = \theta v/\beta - 1$. In summary, we have $d^* = \sqrt{v\beta/\theta} - c$, $r^* = 1/\theta(\sqrt{\theta v/\beta} - 1)$, $U_d^{*4}(\hat{a}, d, r) = \beta/\theta - 2\sqrt{v\beta/\theta} + c$, and $U_a^* = \theta v/\beta - 1$.

Case 5: $(a^* > 0,\ d^* = 0,\ r^* = 0)$ In order to have $a^* > 0$, we have $d < v/(\theta r + 1) - c$, and $(\sqrt{v} - \sqrt{(\theta r + 1)(d + c)})^2 > 0$, which turns out to be $v > c$. From Equation (10), we have $a^* = \sqrt{vc} - c$, $U_d^{*5}(\hat{a}, d, r) = (c/v - 1)V - P_d$, and $U_a^* = v - 2\sqrt{vc} + c$.

Case 6: $(a^* > 0,\ d^* > 0,\ r^* = 0)$ Since $a^* > 0$, from Equation (10), we have $d < v - c$, and $(\sqrt{v} - \sqrt{(\theta r + 1)(d + c)})^2 > 0$, which equals to $d < v - c$. Then $a^* = \sqrt{v(d + c)} - d - c$. The equilibrium problem, Equation (11), turns out to be

$$\begin{aligned} &\min_{d > 0}\ d + V - \sqrt{\frac{d + c}{v}}V + P_d \\ &\text{s.t. } d < v - c \\ &\qquad d > 0 \end{aligned}$$

Solve to get, $d^* = V^2/(4v) - c$. So $a^* = V/2 - V^2/(4v)$, $U_d^{*6}(\hat{a}, d, r) = V^2/(4v) + c - V - P_d$, and $U_a^* = V^2/(4v) + v - V$.

Case 7: $(a^* > 0,\ d^* = 0,\ r^* > 0)$ From $a^* > 0$, we have $d < v/(\theta r + 1) - c$, and $(\sqrt{v} - \sqrt{(\theta r + 1)(d + c)})^2 > 0$, which gives $r < \left(\frac{v}{c} - 1\right)\frac{1}{\theta}$. Based on

Equation (10), $a=\sqrt{vc/(\theta r+1)}-c$. The equilibrium problem, Equation (11), turns out to be

$$\begin{aligned}&\min_{r>0}\ V-\sqrt{\frac{c(\theta r+1)}{v}}V+\beta r+P_d\\ &\text{s.t. } r<\left(\frac{v}{c}-1\right)\frac{1}{\theta}\\ &r>0\end{aligned}$$

Solve to find, $r^*=V^2c\theta/(4\beta^2v)-1/\theta$. So $a^*=2\beta v/(V\theta)-c$, $U_d^{*7}(\hat{a},d,r)=\theta V^2c/(4\beta v)-V+\beta/\theta-P_d$, and $U_a^*=(c+1)\theta^2V^2/(4\beta^2v)-\theta Vc/\beta+v-1$.

Case 8: ($a^*>0$, $d^*>0$, $r^*>0$) Since $a^*>0$, we have $d<v/(\theta r+1)-c$, and $(\sqrt{v}-\sqrt{(\theta r+1)(d+c)})^2>0$. Then $a=\sqrt{v(d+c)/(\theta r+1)}-d-c$. The equilibrium problem, Equation (11), turns out to be

$$\begin{aligned}&\min_{d,r>0}\ d+V-\sqrt{\frac{(d+c)(\theta r+1)}{v}}V+\beta r+P_d\\ &\text{s.t. } d<\frac{v}{\theta r+1}-c\end{aligned}$$

Using KKT method to solve the optimal problem. Let $f(d,r)=d+V-\sqrt{(d+c)(\theta r+1)/v}\,V+\beta r+P_d$, $g(d,r)=d-v/(\theta r+1)+c$, then

$$\nabla f=\begin{pmatrix}1-\frac{V(\theta r+1)}{2\sqrt{v(d+c)(\theta r+1)}}\\ \beta-\frac{V(d+c)\theta}{2\sqrt{v(d+c)(\theta r+1)}}\end{pmatrix},$$

$$\nabla g=\begin{pmatrix}-1\\ \frac{\theta v}{(\theta r+1)^2}\end{pmatrix}$$

Then $\exists\mu\geqslant 0$, such that

$$\begin{cases}1-\frac{V(\theta r+1)}{2\sqrt{v(d+c)(\theta r+1)}}+\mu=0\\ \beta-\frac{V(d+c)\theta}{2\sqrt{v(d+c)(\theta r+1)}}+\mu\frac{\theta v}{(\theta r+1)^2}=0\\ \mu\left(d-\frac{v}{\theta r+1}+c\right)=0\end{cases}$$

Solving the above equations, we find $d=\sqrt{\beta v/\theta}-c$ and $r=-1/\theta+v/\sqrt{\beta v\theta}$. Then $a=\sqrt{v(d+c)/(\theta r+1)}-d-c=0$, which contradicts to the assumption that $a>0$. *So there is no feasible solution for this case*. So far, we have find the optimal solutions for a^*, d^*, r^*, U_d^*, and U_a^* for each case.

In the following, we discuss how to obtain the optimal regions O_i of the parameters $(\theta, v, \beta, V, c, P_d)$ given $P_a=0$. For case i ($i=1,\ldots,7$) to be optimal over other cases, we must have

$$U_d^{*i}\geqslant U_d^{*j},\quad j=1,\ldots,7, j\neq i$$

Then we obtain the ranges of parameters $(\theta, v, \beta, V, c, P_d)$ as the following.

Case 1: ($a^*=0$, $d^*=0$, $r^*=0$)

In order to satisfy $a^*=0$, $d^*=0$, $r^*=0$, we have $v\leqslant c$, or $P_a\geqslant v-c$, or $2\sqrt{vc}+P_a\geqslant v+c$ from Equation (10). So the feasible set for Case 1 is $F_1=\{v\leqslant c, \text{or } P_a\geqslant v-c, \text{or } 2\sqrt{vc}+P_a\geqslant v+c\}$. Based on $U_d^{*1}=0$, observing $U_d\leqslant 0$ from Equation (3), we know U_d^{*1} must be the maximum value of U_d. And we do not need to compare the U_d^{*i}, $i=2,\ldots,7$ to U_d^{*1}. So the optimal set for Case 1 is $O_1=\{v\leqslant c, \text{or } P_a\geqslant v-c, \text{or } 2\sqrt{vc}+P_a\geqslant v+c\}$.

Case 2: ($a^*=0$, $d^*=0$, $r^*=1/\theta(v/c-1)>0$)

In order to satisfy $r^*=1/\theta(v/c-1)>0$, which is obtained from above, we have $v>c$. Furthermore, $a^*=0$ requires that $v\leqslant c$ (contradiction), or $P_a>0$, or $P_a>v/c-1$ from Equation (10). So the feasible set for Case 2 is $F_2=\{v>c; P_a>v/c-1\}$. Since $F_1\cap F_2=\emptyset$, Case 1 is infeasible for Case 2. So we do not need to compare between Cases 1 and 2. In fact we do not need to compare between Case 1 and all other cases because $F_1\cap F_j=\emptyset$, $j=2,\ldots,7$. To obtain the optimality of Case 2 over all other cases, we have $U_d^{*2}>U_d^{*j}$ ($j=3,\ldots,7$). It gives us a system of equations

$$\begin{cases}\frac{\beta}{\theta}\left(1-\frac{v}{c}\right)>c-v\\ \frac{\beta}{\theta}\left(1-\frac{v}{c}\right)>\frac{\beta}{\theta}-2\sqrt{\frac{v\beta}{\theta}}+c\\ \frac{\beta}{\theta}\left(1-\frac{v}{c}\right)>\sqrt{\frac{c}{v}}V-V-P_d\\ \frac{\beta}{\theta}\left(1-\frac{v}{c}\right)>\frac{V^2}{4v}+c-V-P_d\\ \frac{\beta}{\theta}\left(1-\frac{v}{c}\right)>\frac{\theta V^2c}{4\beta v}+\frac{\beta}{\theta}-V-P_d\end{cases}$$

which can be simplified to

$$\begin{cases}\frac{\beta}{\theta}<c\\ \frac{\beta v}{\theta c}-2\sqrt{\frac{\beta v}{\theta}}+c<0\\ \frac{\beta}{\theta}<\frac{c}{c-v}\left(\sqrt{\frac{c}{v}}V-V-P_d\right)\\ \frac{\beta}{\theta}<\frac{c}{c-v}\left(\frac{V^2}{4v}+c-V-P_d\right)\\ \frac{\beta}{\theta}<\frac{c}{c-v}\left(\frac{\theta V^2c}{4\beta v}+\frac{\beta}{\theta}-V-P_d\right)\end{cases}$$

So the optimal set for Case 2 is $O_2=\{v>c, \text{or } P_a>v/c-1, \beta/\theta<c, \beta v/(\theta c)-2\sqrt{\beta v/\theta}+c<0, \beta/\theta<c/(c-v)(\sqrt{c/v}V-V-P_d), \beta/\theta<c/(c-v)(V^2/(4v)+c-V-P_d), \beta/\theta<c/(c-v)(\theta V^2c/(4\beta v)+\beta/\theta-V-P_d)\}$.

Case 3: ($a^*=0$, $d^*=v-c>0$, $r^*=0$)

From $d^*=v-c>0$, we have $v>c$. Furthermore, from $a^*=0$, we have $d\geqslant v-c$ or $P_a\geqslant 0$ based on Equation (10).

So the feasible set for Case 3 is $F_3=\{v>c\}$. Based on $U_d^{*3}>U_d^{*j}$ $(j=2,4,5,6,7)$, we have the following equations:

$$\begin{cases} c-v>\frac{\beta}{\theta}\left(1-\frac{v}{c}\right) \\ c-v>\frac{\beta}{\theta}-2\sqrt{\frac{v\beta}{\theta}}+c \\ c-v>\sqrt{\frac{c}{v}}V-V-P_d \\ c-v>\frac{V^2}{4v}+c-V-P_d \\ c-v>\frac{\theta V^2c}{4\beta v}+\frac{\beta}{\theta}-V-P_d \end{cases}$$

Note $v>c$, we have the above becomes

$$\begin{cases} \frac{\beta}{\theta}>c \\ \frac{\beta}{\theta}-2\sqrt{\frac{\beta v}{\theta}}+c<0 \\ \sqrt{\frac{c}{v}}V+v-V-P_d<c \\ \frac{V^2}{4v}+v-V-P_d<c \\ \frac{\theta V^2c}{4\beta v}+\frac{\beta}{\theta}+v-V-c<P_d \end{cases}$$

So the optimal set for Case 3 is $O_3=\{v>c, \beta/\theta>c, \beta/\theta-2\sqrt{\beta v/\theta}+v<0, \sqrt{c/v}V+v-V-P_d<c, V^2/(4v)+v-V-P_d<c, \theta V^2c/(4\beta v)+\beta/\theta+v-V-c<P_d\}$.

Case 4: $(a^*=0, \quad d^*=\sqrt{v\beta/\theta}-c>0, \quad r^*=1/\theta(\sqrt{\theta v/\beta}-1)>0)$

From $d^*=\sqrt{v\beta/\theta}-c>0$ and $r^*=1/\theta(\sqrt{\theta v/\beta}-1)>0$, we have $c^2/v<\beta/\theta<v$. Furthermore, from $a^*=0$, we have (i): $v>c(\theta r+1)=c\sqrt{\theta v/\beta}$ and $\sqrt{v\beta/\theta}-c\geqslant v/(\theta r+1)-c=v/\sqrt{\theta v/\beta}-c=\sqrt{\beta v/\theta}-c\Rightarrow\beta/\theta>c^2/v$; (ii): $P_a\geqslant 0$; (iii): $P_a\geqslant\theta v/\beta-1$. Therefore, the feasible set for Case 4 is $F_4=\{c^2/v<\beta/\theta<v;\ P_a\geqslant\theta v/\beta-1\}$. According to $U_d^{*4}>U_d^{*j}$ $(j=2,3,5,6,7)$, we have the following equations:

$$\begin{cases} \frac{\beta}{\theta}-2\sqrt{\frac{v\beta}{\theta}}+c>\frac{\beta}{\theta}\left(1-\frac{v}{c}\right) \\ \frac{\beta}{\theta c}-2\sqrt{\frac{v\beta}{\theta}}+c>c-v \\ \frac{\beta}{\theta}-2\sqrt{\frac{v\beta}{\theta}}+c>\sqrt{\frac{c}{v}}V-V-P_d \\ \frac{\beta}{\theta}-2\sqrt{\frac{v\beta}{\theta}}+c>\frac{V^2}{4v}+c-V-P_d \\ \frac{\beta}{\theta}-2\sqrt{\frac{v\beta}{\theta}}+c>\frac{\theta V^2c}{4\beta v}+\frac{\beta}{\theta}-V-P_d \end{cases}$$

Simplifying the above equations, we have

$$\begin{cases} \frac{\beta v}{\theta c}-2\sqrt{\frac{v\beta}{\theta}}+c>0 \\ \frac{\beta}{\theta}-2\sqrt{\frac{v\beta}{\theta}}+v>0 \\ \frac{\beta}{\theta}-2\sqrt{\frac{v\beta}{\theta}}-\sqrt{\frac{c}{v}}V+V+P_d+c>0 \\ \frac{\beta}{\theta}-2\sqrt{\frac{v\beta}{\theta}}+V+P_d>\frac{V^2}{4v} \\ \frac{\theta V^2c}{4\beta v}+2\sqrt{\frac{v\beta}{\theta}}<V+P_d+c \end{cases}$$

So the optimal set for Case 4 is $O_4=\{c^2/v<\beta/\theta<v;\ P_a\geqslant\theta v/\beta-1;\ \beta v/\theta c-2\sqrt{v\beta/\theta}+c>0,\ \beta/\theta-2\sqrt{v\beta/\theta}+v>0,\ \beta/\theta-2\sqrt{v\beta/\theta}-\sqrt{c/v}V+V+P_d+c>0,\ \beta/\theta-2\sqrt{v\beta/\theta}+V+P_d>V^2/(4v),\ \theta V^2c/(4\beta v)+2\sqrt{v\beta/\theta}<V+P_d+c\}$.

Case 5: $(a^*=\sqrt{vc}-c>0,\ d^*=0,\ r^*=0)$

From $a^*=\sqrt{vc}-c>0, \rightarrow v>c$; $v>c$, $0<v-c$; or $P_a<v-c$, or $P_a<c+v-2\sqrt{vc}$, we have the feasible set for Case 5, $F_5=\{v>c;\ P_a<v-c, \text{ or } P_a<c+v-2\sqrt{vc}\}$. Based on $U_d^{*5}>U_d^{*j}$ $(j=2,3,4,6,7)$, the following equations are satisfied:

$$\begin{cases} \sqrt{\frac{c}{v}}V-V-P_d>\frac{\beta}{\theta}\left(1-\frac{v}{c}\right) \\ \sqrt{\frac{c}{v}}V-V-P_d>c-v \\ \sqrt{\frac{c}{v}}V-V-P_d>\frac{\beta}{\theta}-2\sqrt{\frac{v\beta}{\theta}}+c \\ \sqrt{\frac{c}{v}}V-V-P_d>\frac{V^2}{4v}+c-V-P_d \\ \sqrt{\frac{c}{v}}V-V-P_d>\frac{\theta V^2c}{4\beta v}+\frac{\beta}{\theta}-V-P_d \end{cases}$$

Solving the above equations gives us

$$\begin{cases} \frac{\beta}{\theta}-\frac{\beta v}{\theta c}<\sqrt{\frac{c}{v}}V-V-P_d \\ \sqrt{\frac{c}{v}}V-V+v-c>P_d \\ \sqrt{\frac{c}{v}}V-V-P_d>\frac{\beta}{\theta}-2\sqrt{\frac{v\beta}{\theta}}+c \\ \sqrt{\frac{c}{v}}V>\frac{V^2}{4v}+c \\ \sqrt{\frac{c}{v}}V>\frac{\theta V^2c}{4\beta v}+\frac{\beta}{\theta} \end{cases}$$

So the optimal set for Case 5 is $O_5=\{v>c;\ P_a<v-c$, or $P_a<c+v-2\sqrt{vc};\ \beta/\theta-\beta v/(\theta c)<\sqrt{vc}V-V-P_d,\ \sqrt{c/v}V-V+v-c>P_d,\ \sqrt{c/v}V-V-P_d>\beta/\theta-2\sqrt{v\beta/\theta}+c,\ \sqrt{c/v}V>V^2/(4v)+c,\ \sqrt{c/v}V>\theta V^2c/(4\beta v)+\beta/\theta\}$.

Case 6: $(a^*=V/2-V^2/(4v)>0,\ d^*=V^2/(4v)-c>0,\ r^*=0)$

From $a^*=(V/2)-V^2/(4v)>0$ and $d^*=V^2/(4v)-c>0$, we have $2\sqrt{vc}<V<2v$, or $P_a<v-V^2/(4v)$, or $P_a<V^2/(4v)+v-V$, so $F_6=\{2\sqrt{vc}<V<2v;\ P_a<v-V^2/(4v)$, or $P_a<V^2/(4v)+v-V\}$. According to $U_d^{*6}>U_d^{*j}$ $(j=2,3,4,5,7)$, we have the following equations:

$$\begin{cases} \frac{V^2}{4v}+c-V-P_d>\frac{\beta}{\theta}\left(1-\frac{v}{c}\right) \\ \frac{V^2}{4v}+c-V-P_d>c-v \\ \frac{V^2}{4v}+c-V-P_d>\frac{\beta}{\theta}-2\sqrt{\frac{v\beta}{\theta}}+c \\ \frac{V^2}{4v}+c-V-P_d>\sqrt{\frac{c}{v}}V-V-P_d \\ \frac{V^2}{4v}+c-V-P_d>\frac{\theta V^2c}{4\beta v}+\frac{\beta}{\theta}-V-P_d \end{cases}$$

Solving the above equations, we have

$$\begin{cases} \frac{V^2}{4v} + c - V + \frac{\beta}{\theta}\left(1 - \frac{v}{c}\right) > P_d \\ \frac{V^2}{4v} + v - V > P_d \\ \frac{V^2}{4v} - V - \frac{\beta}{\theta} + 2\sqrt{\frac{v\beta}{\theta}} > P_d \\ \frac{V^2}{4v} + c > \sqrt{\frac{c}{v}}V \\ \frac{V^2}{4v} + c > \frac{\theta V^2 c}{4\beta v} + \frac{\beta}{\theta} \end{cases}$$

so the optimal set for Case 6 is $O_6 = \{2\sqrt{vc} < V < 2v;\ P_a < v - V^2/(4v)$, or $P_a < V^2/(4v) + v - V;\ V^2/(4v) + c - V + \beta/\theta(1 - v/c) > P_d,\ V^2/(4v) + v - V > P_d,\ V^2/(4v) - V - \beta/\theta + 2\sqrt{v\beta/\theta} > P_d,\ V^2/(4v) + c > \sqrt{c/v}V,\ V^2/(4v) + c > \theta V^2 c/(4\beta v) + \beta/\theta\}$.

Case 7: $(a^* = 2\beta v/(\theta V) - c > 0,\ d^* = 0,\ r^* = \theta V^2 c/(4\beta^2 v) - 1/\theta > 0)$

From $a^* = 2\beta v/(\theta V) - c > 0$ and $r^* = \theta V^2 c/(4\beta^2 v) - 1/\theta > 0$, we have $F_7 = cV/(2v) < \beta/\theta < V/2\sqrt{c/v}$ and $\theta Vc/\beta - 4\beta v/(\theta V) + P_a < \theta^2 V^2 c^2/(4\beta^2 v) - 2c + v$, or $\theta Vc/\beta + P_a < \theta^2 V^2 c^2/(2\beta^2 v) + v - 1$. According to $U_d^{*7} > U_d^{*j}$ $(j = 2, \ldots 7, j \neq 7)$, reverting the comparison result shown above, we have the optimal set $O_7 = \{cV/(2v) < \beta/\theta < V/2\sqrt{c/v};\ \theta Vc/\beta - 4\beta v/\theta V + P_a < \theta^2 V^2 c^2/(4\beta^2 v) - 2c + v$, or $\theta Vc/\beta + P_a < \theta^2 V^2 c^2/(2\beta^2 v) + v - 1;\ \theta V^2 c/(4\beta v) + \beta v/(\theta c) - V > P_d,\ \theta V^2 c/(4\beta v) + \beta/\theta - V - c + v > P_d,\ \theta V^2 c/(4\beta v) + 2\sqrt{\beta v/\theta} - V - c > P_d,\ \theta V^2 c/(4\beta v) + \beta/\theta > \sqrt{c/v}V,\ \theta V^2 c/(4\beta v) + \beta/\theta > V^2/(4v) + c\}$.

A.2.3. Proof of monotonic property

To check if the utility function U_d^*, a^*, r^* in Table 2 are monotonic functions of θ, and β, we evaluate the first derivative of U_d^{*i}, a^{*i}, r^{*i} $(i = 1, \ldots, 7)$ with respect to θ and β, respectively.

- First, we consider U_d^* with respect to θ.

Case 1: $\partial U_d^{*1}/\partial\theta = 0$.
Case 2: $\partial U_d^{*2}/\partial\theta = \partial/\partial\theta(\beta(1 - v/c)/\theta) = -\beta(1 - v/c)/\theta^2 \geq 0$, since $\beta \geq 0$ and $v > c$ from F_2; that is, $\partial U_d^{*2}/\partial\theta \geq 0$ as long as the feasibility set F_2 is true.
Case 3: $\partial U_d^{*3}/\partial\theta = \partial/\partial\theta(c - v) = 0$.
Case 4: $\partial U_d^{*4}/\partial\theta = \partial/\partial\theta(\beta/\theta - 2\sqrt{v\beta/\theta} + c) = \sqrt{\beta}(\sqrt{v\theta} - \sqrt{\beta})/\theta^2 \geq 0$, since from F_4, $\beta/\theta < v \Leftrightarrow \sqrt{\beta} < \sqrt{v\theta}$, that is, $\partial U_d^{*4}/\partial\theta \geq 0$ as long as the feasibility set F_4 is true.
Case 5: $\partial U_d^{*5}/\partial\theta = \partial/\partial\theta(V/\sqrt{v} - V - P_d) = 0$.
Case 6: $\partial U_d^{*6}/\partial\theta = \partial/\partial\theta(V^2/(4v) - V + c - P_d) = 0$.
Case 7: $\partial U_d^{*7}/\partial\theta = \partial/\partial\theta(\theta V^2 c/(4\beta v) - V + \beta/\theta - P_d) = V^2 c/(4\beta v) - \beta/\theta^2 > 0$, since from F_7, $\beta/\theta < V/2\sqrt{c/v} \Leftrightarrow V^2 c/(4\beta v) > \beta/\theta^2$. So $\partial U_d^{*7}/\partial\theta > 0$ as long as the feasibility set F_7 is true.

In conclusion, $\partial U_d^*/\partial\theta \geq 0$. So U_d^* is a weakly increasing function in θ.

- Similarly, considering U_d^* with respect to β as follows:

Case 1: $\partial U_d^{*1}/\partial\beta = 0$.
Case 2: $\partial U_d^{*2}/\partial\beta = \partial/\partial\beta(\beta(1 - v/c)/\theta) = (1 - v/c)/\theta < 0$, given $F_2 = \{v > c\}$.
Case 3: $\partial U_d^{*3}/\partial\beta = (\partial/\partial\beta(c - v)) = 0$.
Case 4: $\partial U_d^{*4}/\partial\theta = \partial/\partial\beta(\beta/\theta - 2\sqrt{v\beta/\theta} + c) = 1/\sqrt{\theta}(1/\sqrt{\theta} - \sqrt{v/\beta}) < 0$, given $\beta/\theta < v$ from $F_4 = \{c^2/v < \beta/\theta < v\}$.
Case 5: $\partial U_d^{*5}/\partial\beta = \partial/\partial\beta(V/\sqrt{v} - V - P_d) = 0$.
Case 6: $\partial U_d^{*6}/\partial\beta = \partial/\partial\beta(V^2/(4v) - V + c - P_d) = 0$.
Case 7: $\partial U_d^{*7}/\partial\beta = \partial/\partial\beta(\theta V^2 c/(4\beta v) - V + \beta/\theta - P_d) = -\theta V^2 c/(4v\beta^2) + 1/\theta < 0$, since F_7 gives $\beta/\theta < V/2\sqrt{c/v} \Leftrightarrow \theta^2/\beta^2 > 4v/(V^2 c) \Leftrightarrow -\theta V^2 c/(4v\beta^2) + 1/\theta < 0$.

In conclusion, $\partial U_d^*/\partial\beta \leq 0$. So U_d^* is a weakly decreasing function in β.

- Similarly, considering the first-order condition of the terrorist's equilibrium effort a^* with respect to θ and β as follows:

Case 1: $\partial a^{*1}/\partial\theta = 0$.
Case 2: $\partial a^{*2}/\partial\theta = 0$.
Case 3: $\partial a^{*3}/\partial\theta = 0$.
Case 4: $\partial a^{*4}/\partial\theta = 0$.
Case 5: $\partial a^{*5}/\partial\theta = \partial/\partial\theta(\sqrt{vc} - 1) = 0$.
Case 6: $\partial a^{*6}/\partial\theta = \partial/\partial\theta(V/2 - V^2/(4v)) = 0$.
Case 7: $\partial a^{*7}/\partial\theta = \partial/\partial\beta(2\beta v/(\theta V) - c) = -2\beta v/(\theta^2 V) \leq 0$, given β, $v \geq 0$.

In conclusion, $\partial a^*/\partial\theta \leq 0$. So a^* is a weakly decreasing function in θ.

- Next, we consider the first-order condition of a^* with respect to β:

Case 1: $\partial a^{*1}/\partial\beta = 0$.
Case 2: $\partial a^{*2}/\partial\beta = 0$.
Case 3: $\partial a^{*3}/\partial\beta = 0$.
Case 4: $\partial a^{*4}/\partial\beta = 0$.
Case 5: $\partial a^{*5}/\partial\beta = \partial/\partial\beta(\sqrt{vc} - 1) = 0$.
Case 6: $\partial a^{*6}/\partial\beta = \partial/\partial\beta(V/2 - V^2/(4v)) = 0$.
Case 7: $\partial a^{*7}/\partial\beta = \partial/\partial\beta(2\beta v/(\theta V) - c) = 2v/(\theta V) \geq 0$, given β, $v \geq 0$.

In conclusion, $\partial a^*/\partial\beta \geqslant 0$. So a^* is a weakly increasing function in β.

- Next we consider the monotonic property of r^* with respect to β.

Case 1: $\partial r^{*1}/\partial\beta = 0$.
Case 2: $\partial r^{*2}/\partial\beta = \partial/\partial\beta(1/\theta(v/c-1)) = 0$.
Case 3: $\partial r^{*3}/\partial\beta = 0$.
Case 4: $\partial r^{*4}/\partial\beta = \partial/\partial\beta(1/\theta\sqrt{(\theta v/\beta - 1)}) = -1/(2\beta\sqrt{\beta})\sqrt{v/\theta} < 0$.
Case 5: $\partial r^{*5}/\partial\beta = 0$.
Case 6: $\partial r^{*6}/\partial\beta = 0$.
Case 7: $\partial r^{*7}/\partial\beta = \partial/\partial\beta(\theta V^2 c/(4\beta^2 v) - 1/\theta) = -\theta V^2 c/(2\beta^3 v) \leqslant 0$.

In conclusion, $\partial r^*/\partial\beta \leqslant 0$. So r^* weakly decreases in β. Therefore we have proved the Proposition 2a and 2b.

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