Chapter III

Plasmas

1. INTRODUCTION

The discussion in Chapter II was concerned primarily with the behavior and properties of individual particles in a partially ionized gas. We wish now to turn our attention to the macroscopic behavior of collections of charged particles. These considerations will lead to the introduction of two related fundamental parameters associated with the electrical properties of a partially ionized gas, namely, the Debye length and the plasma frequency. As noted in Sec. II 8, the collective behavior of neighboring charged particles during a collision between two charged particles plays an essential role in the calculation of the charged particle momentum transfer collision cross section. The notion of shielding involved here also enters the description of the ionized gas region, called a sheath, immediately adjacent to a solid surface.

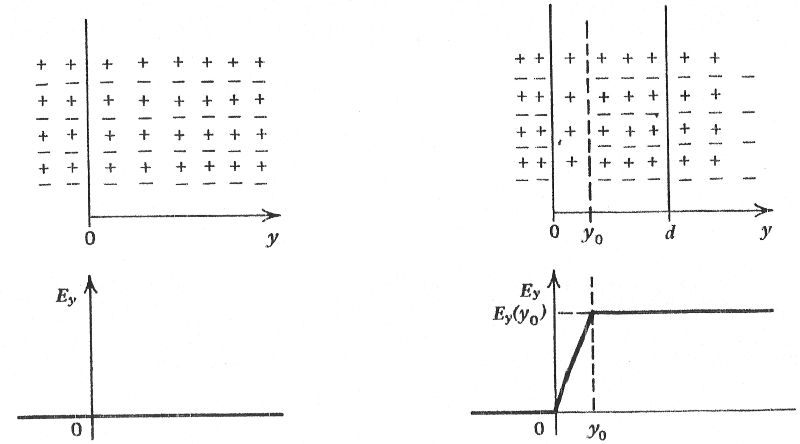
The last three sections of this chapter are concerned with several topics which involve applications of the fundamental concepts introduced earlier. We discuss first the classical theory of electrostatic probes and their use in making measurements of the properties of low-pressure ionized gases. We then discuss some of the concepts involved in the description of collision-dominated ionized gases adjacent to solid surfaces. Finally, we discuss the elementary theory of the propagation of electromagnetic radiation through an ionized gas and how diagnostic information about ionized gases can be inferred from experiments which employ electromagnetic waves.

2. ELECTRICAL NEUTRALITY—THE DEBYE LENGTH

A basic property of a partially ionized gas is its tendency towards electrical neutrality. If over a macroscopic volume the magnitudes of the charge densities of the negative and positive particles differed just slightly, very large electrostatic forces would exist, for which the potential energy per

particle would enormously exceed the mean thermal energy. Unless very special mechanisms were involved to support such large potentials, the charged particles would move rapidly in such a way as to reduce these potential differences and thereby restore electrical neutrality.

To obtain a quantitative estimate of the dimensions over which deviations from charge neutrality may occur, let us consider the following simplified model. Let us suppose the gas is initially electrically neutral and that the electrons and ions are uniformly distributed throughout space, as indicated schematically in Fig. 1(a). Initially, the electron and ion number



(a) Initial condition (b) Intermediate condition

Figure 1. Work necessary to create a region of net positive charge.

densities have the common value $n = n_e = n_i$. We wish to calculate the work necessary to displace the electrons bodily to the right through some distance, d. This will then be the work necessary to create a region of net positive charge density $\rho^c = ne$ between the planes $y = 0$ and $y = d$.

Let us suppose that y_0 represents some intermediate displacement of the electrons, as shown in Fig. 1(b). The electric field **E** set up by a distribution of charge density ρ^c is determined quite generally by Gauss' equation

$$\nabla \cdot \mathbf{E} = \frac{\rho^c}{\epsilon_0}. \tag{2.1}$$

For the one dimensional distribution in Fig. 1(b), we have

$$y \le 0, \frac{dE_y}{dy} = 0 \Rightarrow E_y = 0, \tag{2.2a}$$

$$0 \le y \le y_0, \frac{dE_y}{dy} = \frac{ne}{\epsilon_0} \Rightarrow E_y = \frac{ne}{\epsilon_0} y, \tag{2.2b}$$

$$y_0 \le y, \frac{dE_y}{dy} = 0 \Rightarrow E_y = E_y(y_0) = \frac{ne}{\epsilon_0} y_0 . \tag{2.2c}$$

In obtaining the results expressed by equations (2.2), we have assumed that the electric field is zero prior to the displacement and that E_y is continuous.

In the region to the right of y_0, the electric field acting on each electron is $E_y(y_0)$. The work necessary to move each electron an additional distance dy_0 is

$$dW = eE_y(y_0)\, dy_0 .$$

Therefore, the total work that must be done on each electron to produce a total charge separation of distance d is

$$W = \int_0^d eE_y(y_0)\, dy_0 = \frac{ne^2}{\epsilon_0}\frac{d^2}{2}. \tag{2.3}$$

In particular, if this energy is to be derived from the mean thermal energy in the y-direction $kT/2$, then the corresponding distance $d = \lambda_D$ is called the *Debye length* and is given by

$$\frac{ne^2}{\epsilon_0}\frac{\lambda_D^2}{2} = \frac{kT}{2},$$

or

$$\boxed{\lambda_D = \left(\frac{\epsilon_0 kT}{ne^2}\right)^{1/2}} \tag{2.4a}$$

In MKS units,

$$\lambda_D = 69.0\left(\frac{T}{n}\right)^{1/2} \text{m}. \tag{2.4b}$$

As an example, for the characteristic conditions in an MHD generator $T = 2500°K$ and $n = 10^{20}$ m^{-3}, we have $\lambda_D \simeq 3.4 \times 10^{-7}$ m. This distance may be compared to the value of the electron mean free path $l_e \simeq 1.3 \times 10^{-6}$ m, for the conditions specified in Exercise II 8.2. In this case, therefore, the Debye length is about a factor of five less than the electron mean free path. Values of the Debye length for various other conditions of interest are shown in Fig. 12 of Chapter II.

In 1929, Langmuir (1961) introduced the term *plasma* for a partially ionized gas in which λ_D is small compared to other macroscopic lengths of importance (for example, the macroscopic scale of change in electron number density). Under such circumstances one may make the assumption of electrical neutrality, i.e., $n_i \simeq n_e$. The word plasma derives from a Greek word meaning "to mold" and was suggested to Langmuir by his observations of the manner in which the positive column of a glow discharge tended to mold itself to the containing tube.

Exercise 2.1. Consider a neutral plasma of charged particle number density $n_e = n_i = 10^{14}$ cm^{-3} and temperature 2500°K:

1. Suppose that in some manner all the electrons present in a sphere of radius 1 mm were suddenly removed. Calculate the resulting electric field (in volts/m) at the sphere's surface.
2. Calculate the potential difference through which an electron would need to be accelerated in order to acquire a kinetic energy corresponding to the mean thermal energy of an electron in this plasma.
3. What is the maximum *fraction* of electrons that can be removed from the sphere such that the resulting potential difference between the center and the surface of the sphere shall not exceed the potential difference calculated in part 2?
4. Calculate the Debye length for this plasma.

3. SHEATHS

One of the most important situations in which charge neutrality does not prevail is in the region of a partially ionized gas immediately adjacent to a solid surface. Such regions are referred to as *sheaths*. The relevant macroscopic scale here being the distance from the surface, we may anticipate that the assumption of charge neutrality will be violated in a region whose extent is of the order of λ_D. The detailed structure of sheaths may be quite varied, depending on many factors. The discussion of this section will be limited to the simplest of models and is aimed at bringing out the salient features of sheaths most expeditiously.

An important aspect of the sheath problem concerns the disposition of charged particles which strike the solid surface. For many situations where the surface is cooled, it is possible to regard the surface as nonemitting and *catalytic*. Under such conditions the incident charged particles are either retained on the solid surface or they recombine and are returned to the gas as neutral particles.

Let us consider the special case of a *floating* electrode suddenly immersed into a stationary plasma. We shall assume, for simplicity, that the electron and heavy particle temperatures are equal and uniform. Initially, the electron flux to the electrode will considerably exceed that of the ions because of the relation $\overline{C}_e \gg \overline{C}_i$ between the electron and ion thermal speeds. In the steady-state condition there can be no net rate of charge accumulation on the electrode. The electrode therefore quickly acquires a negative potential of such a magnitude that the reduced electron flux is exactly balanced by the ion flux. As illustrated in Fig. 2, the two major features of interest for this

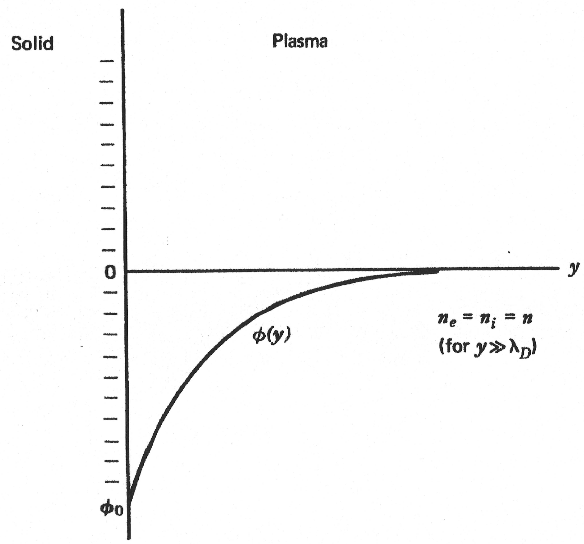


Figure 2. Steady-state potential distribution for a floating electrode.

problem are the magnitude of the *floating potential* ϕ_0 and the spatial extent of the potential distribution $\phi(y)$ associated with the sheath. We shall calculate each of these quantities separately.

To determine ϕ_0, we need to calculate first the flux density of electrons $\Gamma_e^{\uparrow}$ striking the electrode. On the average, electrons striking the electrode experience their last collision a distance approximately one mean free path l_e away from the electrode. We shall assume, somewhat arbitrarily, that the electron velocity distribution $f_e(\mathbf{C})$ at this location is Maxwellian. In addition, in order to keep the calculation simple, we shall assume that conditions are such that the sheath thickness is less than l_e. The latter assumption enables us to regard the sheath itself as collisionless and means that $n_e \simeq n_i = n$ at the location where $f_e(\mathbf{C})$ is Maxwellian. We shall see that this condition for a collisionless, or free-fall, sheath is approximately equivalent to the requirement $\lambda_D < l_e$.

With reference to Fig. 3, only those electrons with velocities directed

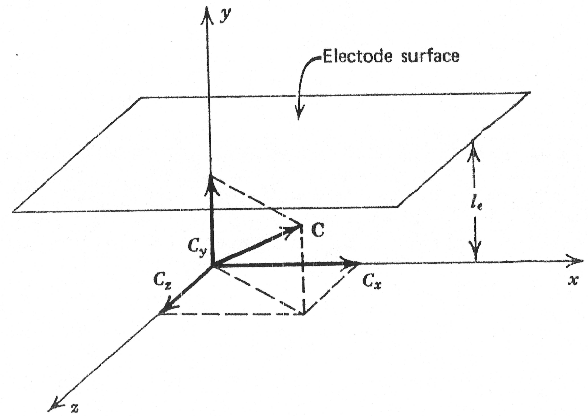


Figure 3. Particle flux density incident on a surface.

towards the surface and with sufficient energy to overcome the potential barrier will reach the electrode. The minimum y-component of velocity required C_{y0} is given by the equation

$$\frac{m_e C_{y0}^2}{2} \equiv -e\phi_0. \tag{3.1}$$

In accordance with equations (II 6.18a) and (II 6.20), the required one-way electron flux density is given by the expression

$$\Gamma_e^{\uparrow} = n\int_{C_y > C_{y0}} f_e(\mathbf{C}) C_y\, d^3C = n\left[\left(\frac{m_e}{2\pi kT}\right)^{1/2}\int_{-\infty}^{\infty} e^{-m_e C_x^2/2kT}\, dC_x\right]$$
$$\cdot\left[\left(\frac{m_e}{2\pi kT}\right)^{1/2}\int_{C_{y0}}^{\infty} e^{-m_e C_y^2/2kT} C_y\, dC_y\right]\cdot\left[\left(\frac{m_e}{2\pi kT}\right)^{1/2}\int_{-\infty}^{\infty} e^{-m_e C_z^2/2kT}\, dC_z\right].$$

Noting that the brackets containing the dC_x and dC_z integrals have the value unity, we obtain the result

$$\Gamma_e^{\uparrow} = \frac{n\overline{C}_e}{4} e^{e\phi_0/kT}. \tag{3.2}$$

Here $\overline{C}_e = (8kT/\pi m_e)^{1/2}$ is the electron mean thermal speed [cf. equation (II 6.34)]. The one-way ion flux density $\Gamma_i^{\uparrow}$ striking the electrode may be calculated in a similar manner, but since the ions are not impeded by the floating potential, one obtains the result

$$\Gamma_i^{\uparrow} = \frac{n\overline{C}_i}{4}. \tag{3.3}$$

The floating potential is obtained from the condition

$$\Gamma_i^{\uparrow} = \Gamma_e^{\uparrow}. \tag{3.4}$$

For the case of equal electron and ion temperatures, this condition yields the result

$$-\phi_0 = \frac{kT}{e} \ln \frac{\bar{C}_e}{\bar{C}_i} = \frac{kT}{e} \ln\left(\frac{m_i}{m_e}\right)^{1/2}. \tag{3.5}$$

We may note that ϕ_0 is proportional to the temperature in eV, as would be expected. However, the factor $\ln(m_i/m_e)^{1/2}$, which has a minimum value of 3.8 (corresponding to hydrogen ions), provides a significant amplification of the temperature effect.

A more detailed analysis substantiates the result (3.2), but for an infinite flat electrode, equation (3.3) should be changed (see Davison, 1957, p. 116) to read

$$\Gamma_i^{\uparrow} \simeq \frac{n\bar{C}_i}{2}. \tag{3.6}$$

A large fraction of the electrons moving toward the electrode from a distance one mean free path away are reflected by the potential barrier and return to this location. The electron distribution function is therefore nearly isotropic in velocity space and can be closely approximated by a Maxwellian, thereby justifying the assumption made in obtaining (3.2). On the other hand, all of the ions which leave this location are captured by the electrode so that the ion distribution function is highly nonisotropic in velocity space, and the assumption of a Maxwellian provides a rather crude approximation.

Turning now to the calculation of the potential distribution $\phi(y)$ in the sheath, we find that treating the ions accurately is made difficult for the aforementioned reason. We may assume that the electrons in the sheath are approximately in thermodynamic equilibrium and thus, in accord with statistical mechanical considerations (see, for example, Tolman, 1938, p. 89), have the Boltzmann distribution

$$n_e(y) = n e^{e\phi(y)/kT}. \tag{3.7}$$

We have used here the fact that the potential energy of an electron at the position y is $-e\phi(y)$. [This result may also be regarded from a dynamical point of view as a statement of conservation of momentum, the electric field force being balanced by a gradient in momentum flux density—see equations (7.4) and (7.6).] If the ions were also in thermodynamic equilibrium, we would have

$$n_i(y) = n e^{-e\phi(y)/kT}. \tag{3.8}$$

This relation would describe the ions well if the electrode reflected all the ions, but it is not a good description for a catalytic surface. However, for simplicity, we shall use the relation (3.8), and we may anticipate that the results will be approximately correct, particularly in view of the fact that equations (3.3) and (3.6) differ only by a factor of two. This statement is supported by more detailed calculations (see Thompson, 1962, p. 29; and Tanenbaum, 1967, p. 210).

The potential distribution for a steady-state problem is governed in general by Poisson's equation

$$\nabla^2\phi = -\frac{\rho^c}{\epsilon_0}. \tag{3.9}$$

In terms of ϕ, the electric field is given by

$$\mathbf{E} = -\nabla\phi. \tag{3.10}$$

The existence of such a potential is a consequence of Faraday's equation for a steady state, $\nabla \times \mathbf{E} = 0$ [cf. equation (VI 2.1c)]. Poisson's equation results upon substituting for $\mathbf{E}$ in Gauss' equation (2.1).

Employing equations (3.7) and (3.8) in the relation $\rho^c = e(n_i - n_e)$, the equation for $\phi(y)$ becomes

$$\frac{d^2\phi}{dy^2} = -\frac{ne}{\epsilon_0}\left(e^{-e\phi(y)/kT} - e^{e\phi(y)/kT}\right). \tag{3.11}$$

Towards the plasma edge of the sheath where $|e\phi/kT| \ll 1$, the exponentials may be expanded so that

$$\frac{d^2\phi}{dy^2} = -\frac{ne}{\epsilon_0}\left(1 - \frac{e\phi}{kT} - 1 - \frac{e\phi}{kT} + \cdots\right) \simeq \frac{2ne^2}{\epsilon_0\, kT}\phi = \frac{2}{\lambda_D^2}\phi,$$

and therefore

$$\phi(y) \propto e^{-\sqrt{2}y/\lambda_D}. \tag{3.12}$$

An exact integration of equation (3.11) subject to the boundary conditions that $\phi(0) = \phi_0$ and that as $y \to \infty$, ϕ and $d\phi/dy \to 0$, yields the result

$$\frac{e\phi}{kT} = \ln\left(\frac{1 - \tanh(e\phi_0/4kT)\, e^{-\sqrt{2}y/\lambda_D}}{1 + \tanh(e\phi_0/4kT)\, e^{-\sqrt{2}y/\lambda_D}}\right). \tag{3.13}$$

Numerical comparison of equations (3.13) and (3.12) shows that the solution for all y is well-represented by the relation

$$\phi(y) = \phi_0\, e^{-\sqrt{2}y/\lambda_D}. \tag{3.14}$$

Our major conclusion from this result is to observe that the spatial extent of the sheath is of the order of the Debye length, confirming what we had anticipated. The fact that the plasma attempts to adjust itself near the electrode surface so as to shield the main body of the plasma from the electric field is a general property of plasmas.

Exercise 3.1. Calculate the sheath thickness for a plasma adjacent to an isolated material surface when the electrons and ions are in thermal equilibrium at *different* temperatures T_e and T_i. Assume that the sheath thickness is much less than a mean free path.

4. SHIELDED COULOMB POTENTIAL

Suppose we place a point charge $q > 0$ in a plasma and inquire as to the steady-state distribution of charge that results in the neighborhood of q. As illustrated in Fig. 4, on the average there will be a surplus of negative

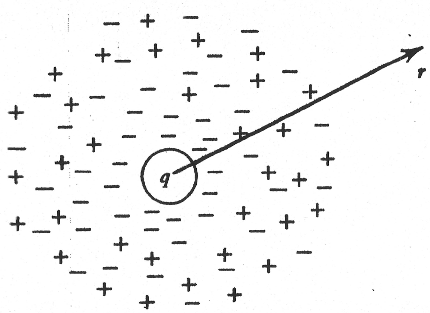


Figure 4. Shielding of a charge by a plasma.

charge in the immediate vicinity of q which gradually diminishes as the radial distance r from q is increased. This distribution results from the simultaneous tendency of q to attract electrons and repel positive ions. For sufficiently large values of r, the plasma will return to a condition of macroscopic electrical neutrality. The result of this redistribution of plasma charge is to produce an electric field at distant points in the plasma which completely cancels the electric field produced by q.

The shielded Coulomb potential $\phi(r)$ which describes the net effect of the charge q and the distributed plasma space charge is determined by Poisson's equation (3.9) in the form

$$\nabla^2\phi = -\frac{e}{\epsilon_0}[n_i(r) - n_e(r)]. \tag{4.1}$$

Since the plasma near q is in thermodynamic equilibrium, we may use relations (3.7) and (3.8), expressed in terms of the radial distance r, to

write equation (4.1) as

$$\frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{d\phi}{dr}\right) = -\frac{ne}{\epsilon_0}[e^{-e\phi(r)/kT} - e^{e\phi(r)/kT}]. \tag{4.2}$$

For large values of r where $|e\phi/kT| \ll 1$ we have, as in Sec. 3,

$$\frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{d\phi}{dr}\right) \simeq \frac{2}{\lambda_D^2}\phi. \tag{4.3}$$

To solve equation (4.3), one makes the substitution

$$\phi = \frac{g(r)}{r},$$

whereupon the differential equation becomes

$$\frac{d^2g}{dr^2} = \frac{2}{\lambda_D^2}g,$$

and therefore, for large r,

$$\phi(r) \sim \frac{e^{-\sqrt{2}r/\lambda_D}}{r}. \tag{4.4}$$

For sufficiently small r the effects of shielding are negligible, and the charge q must give rise to the usual Coulomb potential

$$\phi(r) = \frac{q}{4\pi\epsilon_0 r}. \tag{4.5}$$

For all values of r, we may take as an approximate solution of equation (4.2)

$$\boxed{\phi(r) = \frac{qe^{-\sqrt{2}r/\lambda_D}}{4\pi\epsilon_0 r}}, \tag{4.6}$$

since this expression certainly has the correct limiting behavior for both small and large values of r.

We may conclude from equation (4.6) that the plasma in effect confines the electric field of the charge q to a distance of the order of λ_D. For this reason, λ_D is sometimes called the *Debye shielding distance*. In discussing charged particle collisions in Sec. 8 of Chapter II, we noted that the momentum transfer cross section was infinite for interactions described by a pure Coulomb potential. The result (4.6) provides a more accurate representation of the effective interaction between charged particles in a plasma and leads to the finite value (II 8.4) for the charged particle cross section for momentum transfer.

The factor $\sqrt{2}$ appearing in equation (4.6) is often absorbed into the definition of the Debye length so that instead of equation (2.4a), one finds the shielding distance defined as

$$\lambda_D = \left[\frac{\epsilon_0 kT}{(n_e + n_i)e^2}\right]^{1/2} = \left(\frac{\epsilon_0 kT}{2ne^2}\right)^{1/2}. \tag{4.7}$$

For a *stationary* charge q, both ions and electrons participate equally in the shielding process and this definition of the shielding distance is quite appropriate. In any dynamical situation, however, the ions and the electrons will certainly contribute in different degrees to the shielding process. In particular, for rapidly fluctuating phenomena, the ions, being more massive than the electrons, will contribute only slightly to the shielding. To consider such effects with any precision requires detailed study. For simplicity we take the contribution only of the electrons to provide an approximate measure of the shielding distance.

Exercise 4.1. Calculate the potential distribution in a quiescent plasma in the vicinity of a stationary charge. The plasma consists of electrons at a temperature T_e and of two species of ions with charge numbers Z_1 and Z_2 at a temperature T. If $T_e = T$, is the shielding distance increased or decreased, compared to the case where $Z_1 = Z_2 = 1$?

5. RESPONSE TIME—THE PLASMA FREQUENCY

Since any slight distortion of the plasma from a condition of electrical neutrality gives rise to large restoring forces, the question may be raised as to how fast these restoring forces act. For simplicity, let us suppose that on the time scale of the response, the ions are sufficiently massive that they do not move. Accordingly, the ions may be considered as providing a background of uniform positive charge density ne. Consider a slab of plasma, and suppose that each electron initially on a plane surface located at y is displaced a distance $\xi(y)$. We wish to calculate what happens when the electrons are released.

Because of the displacement imposed on the electrons, the electron number density will be changed to the value

$$n_e(y) = n + \delta n(y). \tag{5.1}$$

The resulting space charge distribution $-e\,\delta n(y)$ will give rise to an electric field which in turn will interact with the electrons. To relate $\delta n(y)$ to $\xi(y)$, let us apply the condition of conservation of electrons to the slice of plasma contained originally between the planes y and $y_1 = y + \Delta y$, as shown in

Fig. 5. We then have

$$n\,\Delta y = (n + \delta n)\{[y_1 + \xi(y_1)] - [y + \xi(y)]\}$$
$$\simeq (n + \delta n)\left\{\left[y_1 + \xi(y) + \Delta y\frac{d\xi}{dy}\right] - [y + \xi(y)]\right\} = (n + \delta n)\left(1 + \frac{d\xi}{dy}\right)\Delta y,$$

and therefore, assuming $|d\xi/dy| \ll 1$

$$\delta n = \frac{-n\,d\xi/dy}{1 + d\xi/dy} \simeq -n\frac{d\xi}{dy}. \tag{5.2}$$

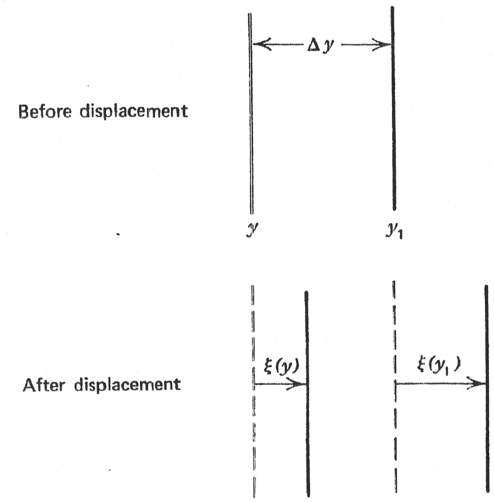


Figure 5. One-dimensional space charge distribution produced by variable electron displacement.

The resulting electric field at y is determined by Gauss' equation

$$\frac{dE_y}{dy} = -\frac{e\,\delta n}{\epsilon_0} = \frac{ne}{\epsilon_0}\frac{d\xi}{dy},$$

which integrates directly to give

$$E_y(y) = \frac{ne}{\epsilon_0}\xi(y). \tag{5.3}$$

Finally, the equation of motion of the displaced electron at ξ is

$$m_e\frac{d^2\xi}{dt^2} = -eE_y(y + \xi) \simeq -eE_y(y) = -\frac{ne^2}{\epsilon_0}\xi. \tag{5.4}$$

According to equation (5.4), when the electrons are released they will execute simple harmonic motion[1] with the characteristic angular *plasma frequency*

$$\boxed{\omega_p = \left(\frac{ne^2}{\epsilon_0 m_e}\right)^{1/2}}. \tag{5.5a}$$

The value of the corresponding circular frequency in MKS units is

$$\nu_p = \frac{\omega_p}{2\pi} = 8.97 n_e^{1/2} \sec^{-1}. \tag{5.5b}$$

The response time of the plasma is then just the reciprocal of ν_p. As an example, for $n_e \simeq 10^{20}$ m^{-3} (which is typical for an MHD generator), the plasma frequency has the value $\nu_p \simeq 8.97 \times 10^{10}$ sec^{-1}. This value may be compared with a typical value for the average electron collision frequency $\bar{\nu}_{eH} \simeq 2 \times 10^{11}$ sec^{-1}.

The concept of the plasma frequency applies locally in a plasma and is not restricted to the planar geometry we have used to introduce this quantity. Suppose for some reason a fluctuation in electron charge density $-e\,\delta n$ develops in a plasma. Then this space charge sets up an electric field determined by the equation

$$\nabla \cdot \mathbf{E} = -\frac{e\,\delta n}{\epsilon_0}. \tag{5.6}$$

This electric field in turn will cause the electron fluid to be accelerated in accordance with the equation

$$\frac{\partial \mathbf{u}_e}{\partial t} = -\frac{e\mathbf{E}}{m_e}, \tag{5.7}$$

where $\mathbf{u}_e$, the mean velocity of the electrons, is assumed small. The resulting redistribution of electron density and mean velocity must be consistent with the requirement of conservation of electron number density

$$\frac{\partial n_e}{\partial t} + \nabla \cdot (n_e \mathbf{u}_e) = 0, \tag{5.8}$$

[1] If we write $\xi(y) = \xi_0 \sin \omega_p t$, where ξ_0 is the amplitude of the oscillation, the total energy of the electron is $m_e(\xi_0 \omega_p)^2/2$. If we suppose this energy derives from the average thermal energy $kT/2$ so that $m_e(\xi_0 \omega_p)^2 = kT$, we obtain for the value of oscillation amplitude $\xi_0 = \lambda_D$. This result is in accord with the interpretation of λ_D described in Sec. 2.

which, when linearized, becomes

$$\frac{\partial\,\delta n}{\partial t} + \nabla \cdot (n\mathbf{u}_e) = 0. \tag{5.9}$$

If we differentiate equation (5.9) with respect to time and employ equations (5.6) and (5.7), we obtain the following equation for δn,

$$\frac{\partial^2}{\partial t^2}\delta n + \nabla \cdot \left(n \frac{\partial \mathbf{u}_e}{\partial t}\right) = \frac{\partial^2}{\partial t^2}\delta n - \frac{ne}{m_e}\nabla \cdot \mathbf{E} = \frac{\partial^2}{\partial t^2}\delta n + \left(\frac{ne^2}{\epsilon_0 m_e}\right)\delta n = 0, \tag{5.10}$$

which again shows that the charge density fluctuation will oscillate with frequency ω_p.

If we multiply equations (2.4a) and (5.5a), we obtain the following relation between the Debye length and the plasma frequency,

$$\lambda_D \omega_p = \left(\frac{\epsilon_0 kT}{ne^2}\frac{ne^2}{\epsilon_0 m_e}\right)^{1/2} = \left(\frac{kT}{m_e}\right)^{1/2} \simeq \bar{C}_e, \tag{5.11}$$

where $\bar{C}_e$ is the mean thermal speed of the electrons. Suppose first we accept the interpretation of ω_p^{-1} as the local response time to charge fluctuations in the plasma. Then on the basis of equation (5.11) we may develop an interpretation of λ_D by the following reasoning: Let us imagine that the electrons are moving initially within some region of extent r in such a manner that if they continued to move freely, an excess of charge of one sign would soon result. The time interval required for this charge excess to build up is $r/\bar{C}_e$. However, this developing charge excess produces an electric field which acts so as to impede the charge excess from increasing. If the response time of the electrons is less than the time required for the charge excess to build up, the charge fluctuation will be prevented or reduced. We may therefore conclude that charge fluctuations are reduced in regions of extent r satisfying the inequality

$$\omega_p^{-1} < r/\bar{C}_e,$$

or, using equation (5.11), for

$$r > \frac{\bar{C}_e}{\omega_p} \simeq \lambda_D.$$

This argument shows that the interpretation of λ_D as a measure of the extent within which deviations from charge neutrality may occur is quite general and is really independent of the specialized model we used to introduce the concept in Sec. 2.

If we start with the premise that the interpretation of λ_D is in hand, then equation (5.11) may be used to obtain an interpretation of ω_p as follows:

According to equation (5.11), the electrons can move a distance λ_D in a time ω_p^{-1}. Therefore, for any disturbance of lower frequency, the plasma can respond sufficiently fast so as to maintain charge neutrality.

Exercise 5.1. Assuming that the ions and electrons can both move, calculate the oscillation frequency of the electron displacement relative to that of the ions and compare the result with equation (5.5a).

6. ELECTROSTATIC PROBES

One of the earliest methods for obtaining spatially resolved measurements of plasma properties was developed by Langmuir about 1924. A Langmuir probe consists basically of a small electrode, frequently just a partially exposed insulated wire, which is inserted into a plasma. A dc power supply is attached to the probe and is usually arranged so that the potential of the probe with respect to the plasma can be varied continuously over a range of both negative and positive values. The current collected by the probe is determined as a function of the biasing voltage, yielding a so-called current-voltage (or I–ϕ) characteristic. It is from the shape of this characteristic that one attempts to derive information concerning plasma properties.

Although it is relatively simple experimentally to obtain an I–ϕ characteristic, our present understanding of how probes behave is restricted to rather special plasma conditions. The utility of an electrostatic probe as a diagnostic device for other than these special kinds of plasmas is dubious and is still the subject of active investigation. Useful reviews concerning the theory and use of electrostatic probes have been written by Chen (1965), by de Leeuw (1963), and by Schott (1968).

In this section we shall discuss several aspects of probe theory applicable to plasma conditions similar to those prevailing in low-pressure discharges, as studied by Langmuir. Important features of such discharges, as far as probe theory is concerned, are that the plasma is nonmoving and the probe is nonemitting. Also, the probe dimensions are much smaller than the mean free paths of the plasma particles, so that the collection of charged particles occurs under essentially collision-free conditions. In addition, the sheath thickness is much less than the probe dimensions so that the sheath may be treated as having plane symmetry. Our reason for discussing this theory is that it is relatively simple and, at the same time, illustrates many of the features common to current collection from a plasma. We shall discuss some of the aspects involved in describing the effects of collisions in the following section.

The general appearance of an I–ϕ characteristic is shown in Fig. 6. The

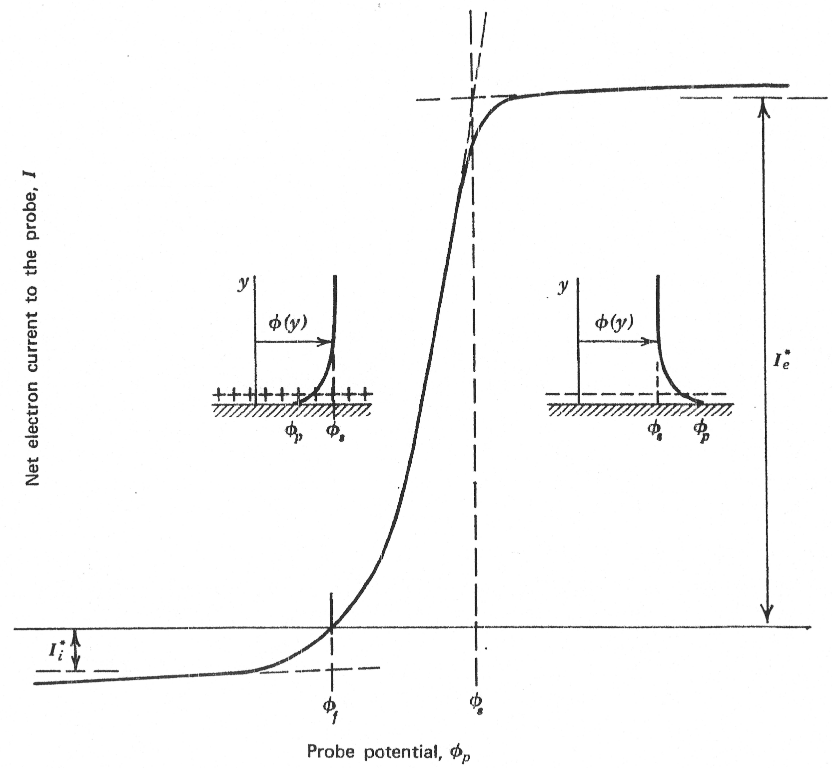


Figure 6. Schematic of a typical probe current-voltage characteristic. The inserts indicate the potential variation in the sheath for the two cases $\phi_p < \phi_s$ and $\phi_p > \phi_s$.

conventional current I is defined as positive for flow from the electrode into the plasma, and is thus equivalent to regarding the net electron current as positive for flow from the plasma towards the electrode. The potential ϕ_p refers to the probe voltage relative to an arbitrary reference, often the anode or cathode for a discharge-produced plasma or a part of the plasma container. The net current collected by the probe is zero when the probe voltage is equal to the floating potential ϕ_f. As discussed in Sec. 3, most of the electrons leaving the plasma are repelled by the probe at this condition in order that the separate electron and ion currents reaching the probe just balance. If ϕ_p is made negative with respect to ϕ_f, more ions will reach the probe than electrons, resulting in a negative net electron current to the probe. If ϕ_p is sufficiently negative, all the electrons will be repelled, and the current collected by the probe will attain a nearly constant magnitude called the *ion saturation* current.

When the probe potential somewhat exceeds the floating potential, more electrons will be collected than ions, resulting in a positive value of I. As ϕ_p is increased, the electron current will continue to increase, resulting in an increasing value of I, until the probe voltage equals the plasma or space potential ϕ_s. At this condition both the random electron and ion currents from the plasma reach the probe unimpeded. For all $\phi_p < \phi_s$, the probe is surrounded by a positive sheath as indicated in Fig. 6. When ϕ_p exceeds ϕ_s some of the ions are prevented from reaching the probe, and thus the net electron current to the probe is again increased, but the probe is now surrounded by a negative sheath. If ϕ_p is sufficiently positive, all the ions will be repelled and the probe will collect the *electron saturation* current. In practice, the probe current at either large negative or positive voltages does not saturate at a constant value, but continues to increase slowly in magnitude. This behavior is often attributable to an increasing sheath thickness, which results in an increasing effective current collection area for the probe.

In addition to the conditions previously discussed we shall assume, for simplicity, a single temperature. For $\phi_p \leq \phi_s$, equations (3.2) and (3.3) may be employed to obtain the expression

$$I = I_e^* \exp\left[\frac{-e(\phi_s - \phi_p)}{kT}\right] - I_i^*. \tag{6.1a}$$

For $\phi_p \geq \phi_s$,

$$I = I_e^* - I_i^* \exp\left[\frac{-e(\phi_p - \phi_s)}{kT}\right]. \tag{6.1b}$$

Here

$$I_e^* = AenC_e/4, \tag{6.2a}$$

$$I_i^* = AenC_i/4, \tag{6.2b}$$

and A denotes the current-collecting area of the probe. The theoretical current-voltage characteristic given by equations (6.1) is in accord with Fig. 6.

These results may be applied to obtain diagnostic information about plasma conditions in the following way. The ion saturation current I_i^* can be measured directly by applying large negative bias voltages to the probe. The electron temperature can then be determined by fitting a straight line on semi-log graph paper to a plot of $\ln(I + I_i^*)$ vs. ϕ_p, for values of ϕ_p in the transition regime (*i.e.*, for ϕ_p near ϕ_f). If the temperature is known, the electron number density may be calculated from the measured value of I_i^*.

In principle, the relation

$$\frac{kT}{e} = I_i^* \left[\frac{d\phi_p}{dI}\right]_{I=0} \tag{6.3}$$

provides an alternative method for determining the temperature.

For charge neutrality to be maintained, an equal ion current must leave the plasma to balance the electron current drawn by the probe. In effect, the wall of the plasma container, or whatever other voltage reference is being used, acts as a second electrode to complete the circuit carrying the probe current. If the area of this effective electrode is too small, the value of the saturation current of the probe at large positive bias voltages will not be equal to the value I_e^* predicted by equation (6.1b). To see in detail how this comes about, and at the same time to discuss an alternative probe method proposed by Johnson and Malter (1950), let us consider next the two-electrode probe shown in Fig. 7.

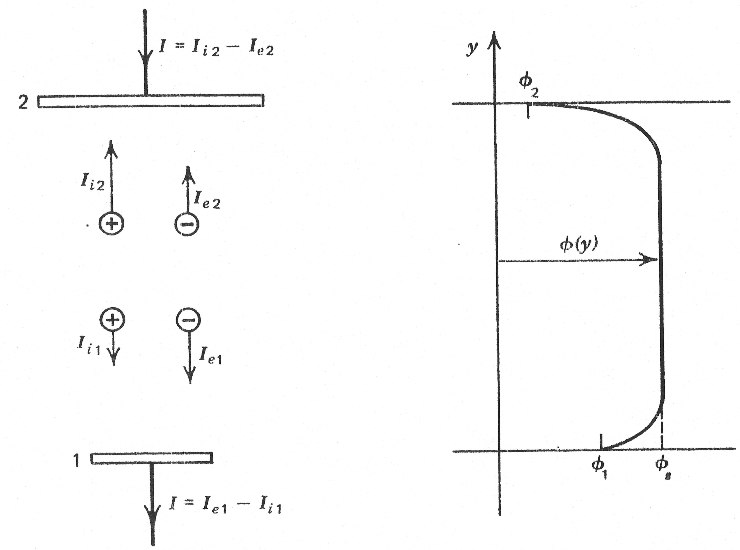


Figure 7. Currents and potential distribution for a double probe.

Let us denote properties associated with the two electrodes by the subscripts 1 and 2, respectively, and let us suppose that $A_2 \geq A_1$. We may, if we wish, identify electrode number 1 with the single-electrode probe discussed previously. If A_2 is not too much greater than A_1, then as the applied voltage $(\phi_1 - \phi_2)$ is increased, ϕ_1 will approach a constant value while ϕ_2 becomes more and more negative. The electron current into electrode 1 will be *limited* by the maximum ion current that electrode 2 will accept. Provided the ion saturation current for electrode 2 satisfies the condition

$$I_{i2}^* < I_{e1}^* - I_{i1}^*, \tag{6.4}$$

the potential ϕ_1 will always remain less than the space potential ϕ_s. We shall limit our discussion to this case. The current-voltage characteristic for a double probe is shown schematically in Fig. 8.

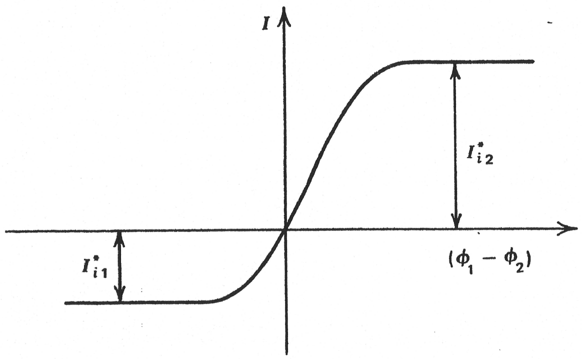


Figure 8. Schematic of the current-voltage characteristic for an idealized double probe with $A_2 > A_1$.

To obtain a theoretical expression for the current-voltage characteristic, we note first that for a specified net electron current flow I to electrode 1, the potential ϕ_1 of this electrode is determined by the relation [cf. equation (6.1a)]

$$I = I_{e1}^* \exp\left[\frac{-e(\phi_s - \phi_1)}{kT}\right] - I_{i1}^*. \tag{6.5}$$

The potential ϕ_2 of electrode 2 is then determined in terms of ϕ_1 from the requirement that the current be continuous, which leads to the equation

$$I_{i2}^* - I_{e2}^* \exp\left[\frac{-e(\phi_s - \phi_2)}{kT}\right] = I_{e1}^* \exp\left[\frac{-e(\phi_s - \phi_1)}{kT}\right] - I_{i1}^*. \tag{6.6}$$

Solving equation (6.6) for $(\phi_2 - \phi_s)$, the applied potential difference $(\phi_1 - \phi_2)$ is given in terms of $(\phi_1 - \phi_s)$ by the relation

$$\exp\left[\frac{e(\phi_1 - \phi_2)}{kT}\right] = \exp\left[\frac{e(\phi_1 - \phi_s)}{kT}\right]\exp\left[\frac{e(\phi_s - \phi_2)}{kT}\right]$$

$$= \frac{\exp[e(\phi_1 - \phi_s)/kT]}{\dfrac{I_{i1}^* + I_{i2}^*}{I_{e2}^*} - \dfrac{I_{e1}^*}{I_{e2}^*}\exp\left[\dfrac{e(\phi_1 - \phi_s)}{kT}\right]}.$$

From this equation $(\phi_1 - \phi_s)$ may be obtained in terms of $(\phi_1 - \phi_2)$, and the result substituted into equations (6.5). One then obtains for the current-voltage characteristic, the relation

$$I = \frac{I_{i2}^*(I_{e1}^*/I_{e2}^*)\exp[e(\phi_1 - \phi_2)/kT] - I_{i1}^*}{(I_{e1}^*/I_{e2}^*)\exp[e(\phi_1 - \phi_2)/kT] + 1}. \tag{6.7}$$

For $(\phi_1 - \phi_2) \to -\infty$ we recover from equation (6.7) the same result as for a single probe, that $I \to -I_{i1}^*$. However, for $(\phi_1 - \phi_2) \to \infty$ equation (6.7) shows that $I \to I_{i2}^*$, in contrast with the result $I \to I_{e1}^*$ for a single probe. This behavior illustrates the need for caution in interpreting electron saturation current data obtained with single probes.

In the limit $A_2 \gg A_1$, and for currents I in the ion saturation and transition regimes for electrode 1, one can show that the current-voltage characteristic for a double probe is identical with that for a single probe. For the conditions stated, the current collected by electrode 2 will be negligible in comparison with I_{i2}^*, so that $\phi_2 \simeq \phi_f$ and

$$I_{i2}^* \simeq I_{e2}^* \exp[-e(\phi_s - \phi_f)/kT].$$

Making these substitutions in equation (6.7) and neglecting the first term in the denominator, which is justified since for the conditions stipulated $e(\phi_1 - \phi_2)/kT \lesssim 1$, we obtain

$$I \simeq I_{e1}^* \exp[-e(\phi_s - \phi_1)/kT] - I_{i1}^*.$$

Comparison of this result with equation (6.1a) for a single probe shows that the characteristics are indeed identical under the conditions stated.

For diagnostic applications, corresponding to equation (6.3) for a single probe, one may show from equation (6.7) that for a double probe

$$\frac{kT}{e} = \frac{I_{i1}^*}{1 + (A_1/A_2)}\left[\frac{d(\phi_1 - \phi_2)}{dI}\right]_{I=0}. \tag{6.8}$$

Double probes are usually used with $A_2 = A_1$, in which case equation (6.7) can be written in the simplified form

$$I = I_{i1}^* \tanh[e(\phi_1 - \phi_2)/2kT].$$

To conclude this section, it should be pointed out that the measured potential of an electrode differs from the effective potential at the surface by a so-called *contact potential*, or *work function*. As long as the contact potential is uniform over the probe surface and does not change with time, the shapes of the probe characteristics discussed above remain unaltered. Under some experimental conditions, variations in contact potential can be important, and steps must be taken to minimize their effects.

Exercise 6.1. Derive the relations (6.3) and (6.8).

Exercise 6.2. Discuss the theory of double probe characteristics when the condition (6.4) does not apply.

7. AMBIPOLAR DIFFUSION

For collision-dominated partially ionized gases, the diffusion of charged particles is determined not only by the electric field, as discussed in Sec. II 13, but also by gradients in plasma properties. Thus, in place of equation (II 13.6a), the electron diffusion velocity may be written

$$\mathbf{U}_e = -\mu_e\left(\mathbf{E} + \frac{\nabla p_e}{en_e}\right). \tag{7.1a}$$

Here $p_e = n_e kT_e$ is the electron pressure, and μ_e is the electron mobility. As shown by equation (II 13.6b), $\mu_e \simeq e/m_e\bar{\nu}_{eH}$. (In this section we shall take the magnetic field as zero, and thus $\mathbf{E}' = \mathbf{E}$. The effects of a magnetic field are discussed in Sec. IV 8.) Equation (7.1a) is obtained in Sec. IV 8 using an approach based on fluid conservation equations, and it is derived more rigorously in Chapter VIII on the basis of kinetic theory. The conditions for the validity of this expression are discussed in detail in Chapter VIII. Briefly, the use of equation (7.1a) requires that the effects of thermal diffusion be small and that the electron velocity distribution function be approximately Maxwellian. The expression for $\mathbf{U}_e$ is often written in the more general form

$$\mathbf{U}_e = -\mu_e\mathbf{E} - D_e\frac{\nabla p_e}{p_e}, \tag{7.1b}$$

where D_e is the electron diffusion coefficient. For the conditions of validity of equation (7.1a),

$$D_e/\mu_e = kT_e/e. \tag{7.2}$$

This result is frequently referred to as the Einstein relation.

The ion diffusion velocity for a weakly ionized gas, and for approximately uniform total pressure, may be written

$$\mathbf{U}_i = \mu_i\left(\mathbf{E} - \frac{\nabla p_i}{en_i}\right). \tag{7.3a}$$

Here $p_i = n_i kT$ is the ion pressure, T is the heavy particle temperature, and μ_i is the ion mobility. For a three-species gas $\mu_i \simeq e/m_{in}\bar{\nu}_{in}$ in accordance with equation (II 13.13). Equation (7.3a) may be derived on the basis of fluid conservation equations, as described in Exercise IV 8.3, and it is discussed in greater detail in Chapter VIII in the context of kinetic theory. In analogy with equation (7.1b), one may also express $\mathbf{U}_i$ in the more general form

$$\mathbf{U}_i = \mu_i\mathbf{E} - D_i\frac{\nabla p_i}{p_i}, \tag{7.3b}$$

where D_i is the ion diffusion coefficient. For our present purposes, we shall assume that D_i and μ_i satisfy the Einstein relation $D_i/\mu_i = kT/e$.

Let us consider first the case of gas in a steady state adjacent to a solid surface, where the surface draws no net current, and let us suppose there is no applied electric field. If the surface is nonemitting and catalytic, as discussed in Sec. 3, it will act as a sink for charged particles, and thus gradients in n_e and n_i will be established in the gas adjacent to the surface. In accordance with equations (7.1a) and (7.3a) these gradients will cause electrons and ions to diffuse toward the wall. If no net current is drawn from the gas, the electron and ion diffusion velocities normal to the wall must be equal (assuming, for simplicity, a single species of singly ionized ions). However, $\mu_e \gg \mu_i$, and so a space charge electric field must be established which will impede the diffusion of electrons. This field may be obtained approximately from equation (7.1a) as

$$\mathbf{E} \simeq -\frac{\nabla p_e}{en_e}. \tag{7.4}$$

The corresponding common diffusion velocity for the charged particles is obtained from equation (7.3a) as

$$\mathbf{U}_e = \mathbf{U}_i \simeq -\mu_i\left(\frac{\nabla p_e}{en_e} + \frac{\nabla p_i}{en_i}\right). \tag{7.5}$$

If we write $\mathbf{E} = -\nabla\phi$ and assume that T_e is uniform, equation (7.4) may be integrated to obtain the relation [cf. equation (3.7)]

$$n_e \propto \exp\left(\frac{e\phi}{kT_e}\right). \tag{7.6}$$

The preceding results may be derived more precisely as follows. Eliminating $\mathbf{E}$ between equations (7.1a) and (7.3a), we obtain

$$\mu_i\mathbf{U}_e + \mu_e\mathbf{U}_i = -\mu_e\mu_i\left(\frac{\nabla p_e}{en_e} + \frac{\nabla p_i}{en_i}\right).$$

For the condition $\mathbf{U}_e = \mathbf{U}_i$, which is commonly referred to as *ambipolar diffusion*, it follows that

$$\mathbf{U}_e = \mathbf{U}_i = -\frac{\mu_e\mu_i}{(\mu_e + \mu_i)}\left(\frac{\nabla p_e}{en_e} + \frac{\nabla p_i}{en_i}\right). \tag{7.7}$$

Substituting equation (7.7) into either equation (7.1a) or (7.3a), we obtain

$$\mathbf{E} = \frac{-\mu_e}{\mu_e + \mu_i} \frac{\nabla p_e}{en_e} + \frac{\mu_i}{\mu_e + \mu_i} \frac{\nabla p_i}{en_i}. \tag{7.8}$$

With $\mu_e \gg \mu_i$, equation (7.7) reduces to equation (7.5), and equation (7.8) reduces to equation (7.4). In the region of gas beyond the sheath, $n_e \simeq n_i$, and it is customary to write equation (7.7) in the form

$$\mathbf{U}_e = \mathbf{U}_i = -D_a \frac{\nabla(p_e + p_i)}{2p_i}, \tag{7.9}$$

where

$$D_a = \frac{2kT}{e} \frac{\mu_e \mu_i}{\mu_e + \mu_i} \simeq 2D_i \tag{7.10}$$

is the *ambipolar diffusion coefficient*. The form of equation (7.9) suggests that the diffusion of electrons and ions in a plasma to a noncurrent-collecting surface may be viewed as analogous to the diffusion of neutral particles (where the electric field **E** plays no role) provided one selects for the diffusion coefficient the value D_a.

In general, the distributions of n_e and n_i in a nonmoving collision-dominated gas are governed by the species conservation equations

$$\nabla \cdot (n_e \mathbf{U}_e) = \dot{n}_e, \tag{7.11a}$$

and

$$\nabla \cdot (n_i \mathbf{U}_i) = \dot{n}_i, \tag{7.11b}$$

where $\dot{n}_e$ and $\dot{n}_i = \dot{n}_e$ are the net rates of electron and ion production per unit volume. For a weakly ionized gas, $\mathbf{U}_e$ and $\mathbf{U}_i$ are given by equations (7.1a) and (7.3a). The potential distribution is governed by Poisson's equation

$$\nabla^2 \phi = -\frac{\rho^c}{\epsilon_0}. \tag{7.12}$$

Equations (7.11) and (7.12) provide three nonlinear coupled equations for n_e, n_i, and ϕ. The electric field is determined by the relation $\mathbf{E} = -\nabla\phi$.

The full formulation of the problem, as described above, is actually needed only in the sheath region and is applicable there only if the sheath is collision dominated. Outside the sheath region we may write $n_e \simeq n_i$, and equations (7.11) then suffice to provide a closed formulation for n_e and ϕ, viz.,

$$\nabla \cdot \left(-\mu_e n_e \mathbf{E} - \frac{\mu_e}{e} \nabla p_e\right) = \dot{n}_e, \tag{7.13a}$$

$$\nabla \cdot \left(\mu_i n_e \mathbf{E} - \frac{\mu_i}{e} \nabla p_i\right) = \dot{n}_e. \tag{7.13b}$$

For simplicity, let us take μ_e and μ_i as constant. Then we may eliminate **E** between equations (7.13) and arrive directly at the result

$$\nabla \cdot \left[-n_e D_a \frac{\nabla(p_e + p_i)}{2p_i}\right] = \dot{n}_e. \tag{7.14}$$

Equation (7.14) governs the electron distribution in what is frequently called the *ambipolar region* (although there is no restriction here on the relation between $\mathbf{U}_e$ and $\mathbf{U}_i$). Once n_e is determined, we may return to either of equations (7.13), or a combination thereof, to obtain the potential distribution. A useful form of these equations for this purpose is obtained by subtracting equation (7.13a) from (7.13b), thereby constructing the equation $\nabla \cdot \mathbf{J} = 0$, or equivalently,

$$\nabla \cdot (\sigma \mathbf{E} + \mu_e \nabla p_e - \mu_i \nabla p_i) = 0. \tag{7.15}$$

Here $\sigma = en_e(\mu_e + \mu_i)$ is the electrical conductivity, as defined by equation (II 13.15b).

To illustrate an application of these equations, let us assume that $T_e = T$ and that the temperature is uniform. Let us suppose also that the ionization-recombination process is the three-body reaction $e + A \rightarrow e + A^+ + e$, for which the electron production rate is given by equation (II 11.15). With these assumptions, equation (7.14) may be written in the form

$$l_R^2 \nabla^2 \left(\frac{n_e}{n_e^*}\right) = -\left(\frac{n_e}{n_e^*}\right)\left[1 - \left(\frac{n_e}{n_e^*}\right)^2\right]. \tag{7.16}$$

Here n_e^* is the equilibrium electron number density that would be attained in the body of the plasma at sufficiently large distances away from the influence of boundaries. The length l_R is defined by the relation

$$l_R^2 \equiv \frac{D_a}{n_e^* \alpha}. \tag{7.17}$$

Here $\alpha = n_e^* \beta(T)$ is the recombination rate coefficient, as discussed in conjunction with equation (II 11.13a), evaluated at equilibrium conditions. In accordance with equation (II 5.3), the characteristic time for an ion to recombine is given in terms of α by the relation

$$\tau^{(\text{recomb})} = (n_e^* \alpha)^{-1}.$$

To obtain a physical interpretation of l_R, we may rewrite the right-hand side of equation (7.17) as follows:

$$l_R^2 \simeq \frac{2\mu_i kT}{en_e^* \alpha} = \frac{2kT}{m_{in}} \frac{\tau^{(\text{recomb})}}{\bar{\nu}_{in}} = \frac{\pi}{4} \frac{\bar{g}_{in}^2}{\bar{\nu}_{in}^2} \frac{\tau^{(\text{recomb})}}{\bar{\nu}_{in}^{-1}} \simeq \left(\frac{\tau^{(\text{recomb})}}{\bar{\nu}_{in}^{-1}}\right) l_i^2. \tag{7.18}$$

Here $l_i \simeq (n_n \bar{Q}_{in})^{-1}$ is the mean free path of an ion [cf. equation (II 5.8)].

The ratio $(\tau^{(\text{recomb})}/\bar{\nu}_{in}^{-1})$ is the average *number* of collisions an ion experiences, before it recombines. Thus, in accordance with equation (II 5.9), we may interpret l_R as the average distance traversed by an ion before it recombines. (For atmospheric pressure potassium-seeded argon, $l_R \sim 1$ mm—see Exercise II 5.4.) For situations which have a characteristic geometric dimension L, nondimensionalization of the distance coordinate leads to a factor $(l_R/L)^2$ on the left-hand side of equation (7.16). For $(l_R/L) \ll 1$, the left-hand side of equation (7.16) may be replaced by zero, and we obtain the equilibrium limit solution; for $(l_R/L) \gg 1$, the right-hand side of equation (7.16) may be replaced by zero, and the resulting equation defines the solution in the "frozen" limit.

In the remainder of the section, we shall discuss the problem of a plasma adjacent to a plane infinite plate. If the coordinate normal to the plate is denoted by y, the nondimensional form of equation (7.16) describing the electron density in the ambipolar region may be written

$$2\frac{d^2\bar{n}}{d\bar{y}^2} = -\bar{n}(1-\bar{n}^2). \tag{7.19}$$

Here $\bar{n} = n_e(y)/n_e^*$, and $\bar{y} = \sqrt{2}\, y/l_R$. Since this problem has no characteristic geometric length, the presence of the plate will produce a disturbance in the plasma extending a distance of the order of l_R from the plate. The domain of applicability of equation (7.19) is

$$y_b \le y < \infty, \tag{7.20a}$$

where y_b denotes the distance from the plate of the lower boundary of the ambipolar region. The domain

$$0 \le y \le y_b \tag{7.20b}$$

defines the sheath region. There is of course no sharp separation between these regions in reality, and so the model we have developed is somewhat artificial in this respect. More refined mathematical treatments are possible, but the essential physical elements of the problem are brought out with sufficient accuracy by the present approach. (See for example, Lam, 1964, who discusses the case in which the sheath and ambipolar regions are both collision dominated and the fluid may be in motion.)

To integrate equation (7.19) we may write

$$2\frac{d^2\bar{n}}{d\bar{y}^2} = \frac{d}{d\bar{n}}\left(\frac{d\bar{n}}{d\bar{y}}\right)^2.$$

Employing the boundary condition that $d\bar{n}/d\bar{y} \to 0$ as $\bar{n} \to 1$, and the requirement that $d\bar{n}/d\bar{y} \ge 0$, we obtain

$$\frac{d\bar{n}}{d\bar{y}} = \frac{(1-\bar{n}^2)}{2}. \tag{7.21}$$

Using the formula $\int 2\, d\bar{n}/(1-\bar{n}^2) = \ln[(1+\bar{n})/(1-\bar{n})]$, we may then show that

$$\bar{n} = \frac{(1+\bar{n}_b) - (1-\bar{n}_b)\exp - (\bar{y}-\bar{y}_b)}{(1+\bar{n}_b) + (1-\bar{n}_b)\exp - (\bar{y}-\bar{y}_b)}. \tag{7.22}$$

Here $\bar{n}_b$ denotes the value of $\bar{n}$ at the boundary of the ambipolar region adjacent to the plate. The dependence of $\bar{n}$ on $\bar{y}$ is shown in Fig. 9 for the case $\bar{n}_b \simeq 10^{-4}$.

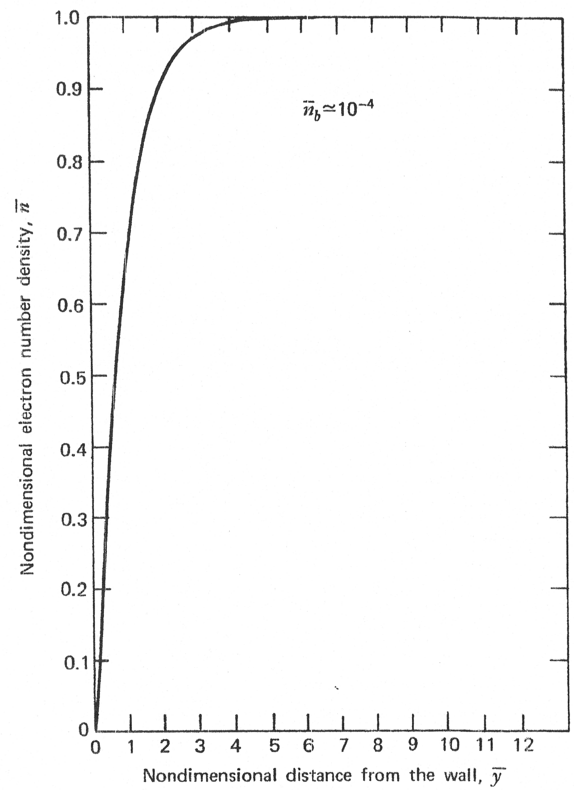


Figure 9. Electron number density profile resulting from ambipolar diffusion.

To obtain the potential distribution we may use equation (7.15). For the problem being considered, equation (7.15) states that the y-component of the current density $\mathbf{J}$ is a constant. The first integral of equation (7.15) may then be written

$$\frac{d\phi}{dy} = \frac{kT}{e}\left(\frac{\mu_e - \mu_i}{\mu_e + \mu_i}\right)\frac{d}{dy}\ln n_e - \frac{J}{\sigma}, \tag{7.23}$$

and thus the potential distribution is given by the relation

$$\phi(y) - \phi_b = \frac{kT}{e}\left(\frac{\mu_e - \mu_i}{\mu_e + \mu_i}\right)\ln\frac{n_e(y)}{n_{eb}} - J\int_{y_b}^{y}\frac{dy}{\sigma(y)}. \tag{7.24}$$

The second term on the right-hand side of equation (7.24) represents the usual Ohmic potential variation associated with current flow in a resistive medium. The first term describes the potential drop associated with ambipolar diffusion and is present even in the absence of current flow. The total potential drop associated with ambipolar diffusion alone has the value

$$[\phi_\infty - \phi_b]_{\text{amb}} \simeq \frac{kT}{e}\ln\frac{n_e^*}{n_{eb}} = -\frac{kT}{e}\ln\bar{n}_b. \tag{7.25}$$

Employing equation (7.21), the resistive part of the potential variation may be written

$$[\phi(y) - \phi_b]_{\text{res}} = -\sqrt{2}\,\frac{J}{\sigma^*}\,l_R\int_{\bar{n}_b}^{\bar{n}}\frac{d\bar{n}}{\bar{n}(1-\bar{n}^2)}$$

$$= -\sqrt{2}\,\frac{J}{\sigma^*}\,l_R\left[\ln\frac{\bar{n}}{\bar{n}_b} - \frac{1}{2}\ln\left(\frac{1-\bar{n}^2}{1-\bar{n}_b^2}\right)\right]. \tag{7.26}$$

Using the relations (7.21) and (7.23), the diffusion velocities may be written in the form

$$U_e = -\frac{\mu_e}{\mu_e + \mu_i}\frac{J}{en_e} - \frac{D_a}{n_e}\frac{dn_e}{dy}, \tag{7.27a}$$

$$U_i = \frac{\mu_i}{\mu_e + \mu_i}\frac{J}{en_e} - \frac{D_a}{n_e}\frac{dn_e}{dy}. \tag{7.27b}$$

These expressions exhibit explicitly the transition from the condition $|U_e| \gg |U_i|$ in the body of a plasma to a condition where $|U_e|$ and $|U_i|$ can be of the same order near a boundary.

To complete the solution of the problem we have been considering, it is still necessary to specify the value of the electron number density n_{eb} at the boundary of the ambipolar region. This value will depend on the structure

of the sheath region. For simplicity we shall assume a free-fall sheath, as discussed in Secs. 3 and 6. In terms of our present discussion this means that we may identify the distance of the ambipolar region from the plate y_b with the particle mean free path l_e. (We shall assume $l_e = l_i$.)

In accordance with equation (3.2) the microscopic flux densities of electrons and ions striking the plate may be written

$$\Gamma_e^\dagger = -\frac{n_{eb}\bar{C}_e}{4}h_e, \tag{7.28a}$$

and

$$\Gamma_i^\dagger = -\frac{n_{eb}\bar{C}_i}{4}h_i. \tag{7.28b}$$

The negative sign is required here because our sign convention has been to regard flux from the plate to the gas as positive. The functions h_e and h_i [cf. equation (6.1)] are defined as follows: For $\phi_b - \phi_p \geq 0$,

$$h_e = \exp -\frac{e(\phi_b - \phi_p)}{kT}, \qquad h_i = 1, \tag{7.29a}$$

and for $\phi_b - \phi_p \leq 0$,

$$h_i = \exp -\frac{e(\phi_p - \phi_b)}{kT}, \qquad h_e = 1. \tag{7.29b}$$

By continuity, the macroscopic expressions for the flux densities (7.1a) and (7.3a) evaluated at $y = y_b$ must equal the corresponding microscopic values given by equations (7.28). Therefore

$$-\frac{\bar{C}_e h_e}{4} = -\mu_e E_b - \frac{\mu_e kT}{en_{eb}}\left(\frac{dn_e}{dy}\right)_b, \tag{7.30a}$$

and

$$-\frac{\bar{C}_i h_i}{4} = \mu_i E_b - \frac{\mu_i kT}{en_{eb}}\left(\frac{dn_e}{dy}\right)_b. \tag{7.30b}$$

Eliminating E_b between equations (7.30), we obtain the following boundary condition for n_e:

$$\left(\frac{dn_e}{dy}\right)_b = Qn_{eb}. \tag{7.31a}$$

Here

$$Q = \frac{e}{8kT}\left(\frac{\bar{C}_e}{\mu_e}h_e + \frac{\bar{C}_i}{\mu_i}h_i\right).$$

In nondimensional form equation (7.31a) becomes

$$\left(\frac{d\bar{n}}{dy}\right)_b = \bar{Q}\bar{n}_b, \tag{7.31b}$$

where

$$\bar{Q} = \frac{l_R Q}{\sqrt{2}} = \frac{l_R e}{\sqrt{2}\,8kT}\left(\frac{\bar{C}_e}{\mu_e}h_e + \frac{\bar{C}_i}{\mu_i}h_i\right). \tag{7.32}$$

Employing equation (7.21), we obtain for the value of $\bar{n}_b$,

$$\bar{n}_b = -\bar{Q} + \sqrt{\bar{Q}^2 + 1}. \tag{7.33}$$

To obtain an estimate for $\bar{Q}$, we may rewrite $\bar{Q}$ in the form

$$\bar{Q} = \frac{l_R}{\sqrt{2}\pi}\left(\frac{h_e}{l_e} + \frac{m_{in}}{m_i}\frac{h_i}{l_i}\right) \sim \frac{1}{\sqrt{2}\pi}\frac{l_R}{l_e}. \tag{7.34}$$

Thus, $\bar{Q}$ is a weak function of the potential difference $(\phi_b - \phi_p)$, and it is of order (l_R/l_e), which is usually a large number. For $\bar{Q} \gg 1$,

$$\bar{n}_b \simeq \frac{1}{2\bar{Q}} \sim \frac{\pi}{\sqrt{2}}\frac{l_e}{l_R}, \tag{7.35}$$

and thus $\bar{n}_b$ is usually small. Referring to equation (7.25), we see that the ambipolar potential drop can thus become quite significant relative to (kT/e).

The complete procedure for determining $n_e(y)$ and $\phi(y)$ may now be summarized as folllows: The potential drop across the sheath $(\phi_b - \phi_p)$ is first assumed. This drop fixes the value of n_{eb}, and the ambipolar equations then yield $n_e(y)$. The current density collected by the plate is calculated next from the free-fall sheath equations [cf. equations (6.1)]

$$J = \frac{en_{eb}}{4}(\bar{C}_e h_e - \bar{C}_i h_i). \tag{7.36}$$

With the value of J specified, the ambipolar equations serve to determine $\phi(y) - \phi_b$. The potential with respect to the plate is then obtained by adding $[\phi(y) - \phi_b]$ to $[\phi_b - \phi_p]$.

Exercise 7.1. Using the results of Exercise IV 8.3, show that the ambipolar diffusion velocity for a gas of arbitrary degree of ionization is given approximately by the expression

$$\mathbf{U}_e = \mathbf{U}_i \simeq -\frac{\rho_n}{\rho}\mu_i\left(\frac{\nabla p_e}{en_e} + \frac{\nabla p_i}{en_i}\right).$$

Exercise 7.2. Discuss the voltage-current characteristic for a collision-dominated gas contained between two parallel flat plates. Assume the plates are separated by a distance which is large compared with the ion recombination length l_R. Sketch the potential distribution between the plates. (The general case of arbitrary separation between the plates is discussed by McKee, 1967.)

8. PROPAGATION OF ELECTROMAGNETIC WAVES

The propagation of electromagnetic radiation is governed by Maxwell's equations. These equations are discussed in Sec. 2 of Chapter VI. As shown in Sec. VI 4, a fundamental solution of Maxwell's equations in a vacuum corresponds to the existence of plane, transverse traveling waves which can transport energy. In this section we wish to describe how the propagation of such waves is modified by the presence of a plasma and how one may derive diagnostic information about the plasma from this fact. We shall limit our discussion to the case of a uniform, unbounded, and nonmoving plasma, characterized by an electron number density n_e and average collision frequence $\bar{\nu}_{eH}$. We shall neglect compressibility effects. In this section, to simplify notation, we shall write $\bar{\nu} \equiv \bar{\nu}_{eH}$.

In a plasma, the magnetic induction **B** and the magnetic intensity **H** are almost always treated by the same constitutive relation $\mathbf{B} = \mu_0 \mathbf{H}$ [cf. equation (VI 2.1f)] as in free space. [See the discussion preceding equations (IV 2.12).] To account for the polarizability associated with the bound electrons of the neutral particles and of the ions, one may write in place of equation (VI 2.1e) the relation $\mathbf{D} = \epsilon\mathbf{E}$. Taking the time derivative of the Maxwell-Ampere equation (VI 2.1d) and employing Faraday's equation (VI 2.1c), we then obtain the following equation for the electric field:

$$\nabla^2\mathbf{E} - \nabla(\nabla \cdot \mathbf{E}) = \mu_0\frac{\partial \mathbf{J}}{\partial t} + \epsilon\mu_0\frac{\partial^2 \mathbf{E}}{\partial t^2}. \tag{8.1}$$

Since the plasma is assumed stationary, the total current density **j** is equal to the conduction current density **J**.

The generalized Ohm's law for a partially ionized gas given by equation (IV 8.17) serves to relate **J** to **E**. For our present purposes we will suppose that there is no applied B field. There is thus no possibility of ion slip, and equation (IV 8.17) can be shown to have the same form as equation (IV 8.9), which was derived for a fully ionized gas. For a plane transverse electromagnetic wave in a vacuum, the magnitude of **B** is given by equation (VI 4.5b) as $|\mathbf{B}| = |\mathbf{E}|/c$, where $c = (\epsilon_0\mu_0)^{-1/2}$ is the speed of light in a vacuum. The ratio of the force on an electron resulting from **B** as compared