# Homework # 9 Solution

**Problem 6.3** A coil consists of 100 turns of wire wrapped around a square frame of sides 0.25 m. The coil is centered at the origin with each of its sides parallel to the $x$- or $y$-axis. Find the induced emf across the open-circuited ends of the coil if the magnetic field is given by

**(a)** $\mathbf{B} = \hat{\mathbf{z}}\, 20e^{-3t}$ (T)

**(b)** $\mathbf{B} = \hat{\mathbf{z}}\, 20\cos x \cos 10^3 t$ (T)

**(c)** $\mathbf{B} = \hat{\mathbf{z}}\, 20\cos x \sin 2y \cos 10^3 t$ (T)

**Solution:** Since the coil is not moving or changing shape, $V_{\text{emf}}^{\text{m}} = 0$ V and $V_{\text{emf}} = V_{\text{emf}}^{\text{tr}}$. From Eq. (6.6),

$$V_{\text{emf}} = -N\frac{d}{dt}\int_S \mathbf{B}\cdot d\mathbf{s} = -N\frac{d}{dt}\int_{-0.125}^{0.125}\int_{-0.125}^{0.125} \mathbf{B}\cdot(\hat{\mathbf{z}}\, dx\, dy),$$

where $N = 100$ and the surface normal was chosen to be in the $+\hat{\mathbf{z}}$ direction.

**(a)** For $\mathbf{B} = \hat{\mathbf{z}}20e^{-3t}$ (T),

$$V_{\text{emf}} = -100\frac{d}{dt}(20e^{-3t}(0.25)^2) = 375e^{-3t} \quad \text{(V)}.$$

**(b)** For $\mathbf{B} = \hat{\mathbf{z}}20\cos x \cos 10^3 t$ (T),

$$V_{\text{emf}} = -100\frac{d}{dt}\left(20\cos 10^3 t\int_{x=-0.125}^{0.125}\int_{y=-0.125}^{0.125}\cos x\, dx\, dy\right) = 124.6\sin 10^3 t \quad \text{(kV)}.$$

**(c)** For $\mathbf{B} = \hat{\mathbf{z}}20\cos x \sin 2y \cos 10^3 t$ (T),

$$V_{\text{emf}} = -100\frac{d}{dt}\left(20\cos 10^3 t\int_{x=-0.125}^{0.125}\int_{y=-0.125}^{0.125}\cos x \sin 2y\, dx\, dy\right) = 0.$$

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**Problem 6.5** A circular-loop TV antenna with 0.02-m$^2$ area is in the presence of a uniform-amplitude 300-MHz signal. When oriented for maximum response, the loop develops an emf with a peak value of 30 (mV). What is the peak magnitude of $\mathbf{B}$ of the incident wave?

**Solution:** TV loop antennas have one turn. At maximum orientation, Eq. (6.5) evaluates to $\Phi = \int \mathbf{B}\cdot d\mathbf{s} = \pm BA$ for a loop of area $A$ and a uniform magnetic field with magnitude $B = |\mathbf{B}|$. Since we know the frequency of the field is $f = 300$ MHz, we can express $B$ as $B = B_0\cos(\omega t + \alpha_0)$ with $\omega = 2\pi \times 300 \times 10^6$ rad/s and $\alpha_0$ an arbitrary reference phase. From Eq. (6.6),

$$V_{\text{emf}} = -N\frac{d\Phi}{dt} = -A\frac{d}{dt}[B_0\cos(\omega t + \alpha_0)] = AB_0\omega\sin(\omega t + \alpha_0).$$

$V_{\text{emf}}$ is maximum when $\sin(\omega t + \alpha_0) = 1$. Hence,

$$30 \times 10^{-3} = AB_0\omega = 0.02 \times B_0 \times 6\pi \times 10^8,$$

which yields $B_0 = 0.8$ (nA/m).

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**Problem 6.6** The square loop shown in Fig. P6.6 is coplanar with a long, straight wire carrying a current

$$I(t) = 5\cos(2\pi \times 10^4 t) \quad \text{(A)}.$$

**(a)** Determine the emf induced across a small gap created in the loop.

**(b)** Determine the direction and magnitude of the current that would flow through a 4-$\Omega$ resistor connected across the gap. The loop has an internal resistance of 1 $\Omega$.

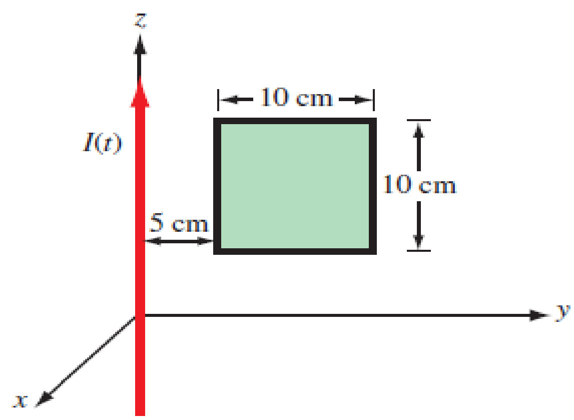


**Figure P6.6:** Loop coplanar with long wire (Problem 6.6).

**Solution:**

**(a)** The magnetic field due to the wire is

$$\mathbf{B} = \hat{\boldsymbol{\phi}} \frac{\mu_0 I}{2\pi r} = -\hat{\mathbf{x}} \frac{\mu_0 I}{2\pi y},$$

where in the plane of the loop, $\hat{\boldsymbol{\phi}} = -\hat{\mathbf{x}}$ and $r = y$. The flux passing through the loop is

$$\begin{aligned}
\Phi = \int_S \mathbf{B} \cdot d\mathbf{s} &= \int_{5\text{ cm}}^{15\text{ cm}} \left(-\hat{\mathbf{x}} \frac{\mu_0 I}{2\pi y}\right) \cdot [-\hat{\mathbf{x}}\, 10 \text{ (cm)}]\, dy \\
&= \frac{\mu_0 I \times 10^{-1}}{2\pi} \ln \frac{15}{5} \\
&= \frac{4\pi \times 10^{-7} \times 5\cos(2\pi \times 10^4 t) \times 10^{-1}}{2\pi} \times 1.1 \\
&= 1.1 \times 10^{-7} \cos(2\pi \times 10^4 t) \quad \text{(Wb)}.
\end{aligned}$$

$$\begin{aligned}
V_{\text{emf}} = -\frac{d\Phi}{dt} &= 1.1 \times 2\pi \times 10^4 \sin(2\pi \times 10^4 t) \times 10^{-7} \\
&= 6.9 \times 10^{-3} \sin(2\pi \times 10^4 t) \quad \text{(V)}.
\end{aligned}$$

**(b)**

$$I_{\text{ind}} = \frac{V_{\text{emf}}}{4+1} = \frac{6.9 \times 10^{-3}}{5} \sin(2\pi \times 10^4 t) = 1.38 \sin(2\pi \times 10^4 t) \quad \text{(mA)}.$$

At $t = 0$, $\mathbf{B}$ is a maximum, it points in $-\hat{\mathbf{x}}$-direction, and since it varies as $\cos(2\pi \times 10^4 t)$, it is decreasing. Hence, the induced current has to be CCW when looking down on the loop, as shown in the figure.

**Problem 6.7** The rectangular conducting loop shown in Fig. P6.7 rotates at 6,000 revolutions per minute in a uniform magnetic flux density given by

$$\mathbf{B} = \hat{\mathbf{y}}\, 50 \quad \text{(mT)}.$$

Determine the current induced in the loop if its internal resistance is 0.5 Ω.

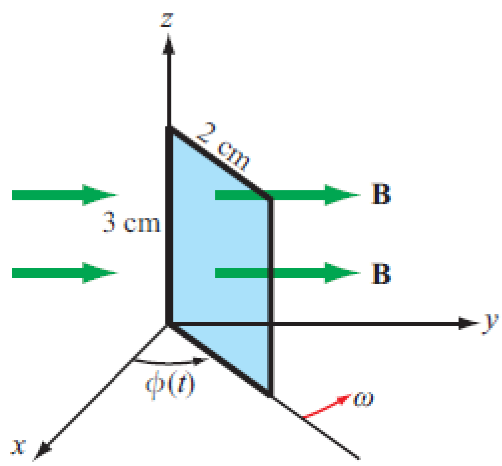


**Figure P6.7:** Rotating loop in a magnetic field (Problem 6.7).

**Solution:**

$$\Phi = \int_S \mathbf{B} \cdot d\mathbf{S} = \hat{\mathbf{y}}\, 50 \times 10^{-3} \cdot \hat{\mathbf{y}}(2 \times 3 \times 10^{-4}) \cos\phi(t) = 3 \times 10^{-5} \cos\phi(t),$$

$$\phi(t) = \omega t = \frac{2\pi \times 6 \times 10^3}{60} t = 200\pi t \quad \text{(rad/s)},$$

$$\Phi = 3 \times 10^{-5} \cos(200\pi t) \quad \text{(Wb)},$$

$$V_{\text{emf}} = -\frac{d\Phi}{dt} = 3 \times 10^{-5} \times 200\pi \sin(200\pi t) = 18.85 \times 10^{-3} \sin(200\pi t) \quad \text{(V)},$$

$$I_{\text{ind}} = \frac{V_{\text{emf}}}{0.5} = 37.7 \sin(200\pi t) \quad \text{(mA)}.$$

The direction of the current is CW (if looking at it along $-\hat{\mathbf{x}}$-direction) when the loop is in the first quadrant ($0 \le \phi \le \pi/2$). The current reverses direction in the second quadrant, and reverses again every quadrant.

**Problem 6.8** The transformer shown in Fig. P6.8 consists of a long wire coincident with the $z$-axis carrying a current $I = I_0 \cos \omega t$, coupling magnetic energy to a toroidal coil situated in the $x$–$y$ plane and centered at the origin. The toroidal core uses iron material with relative permeability $\mu_r$, around which 100 turns of a tightly wound coil serves to induce a voltage $V_{emf}$, as shown in the figure.

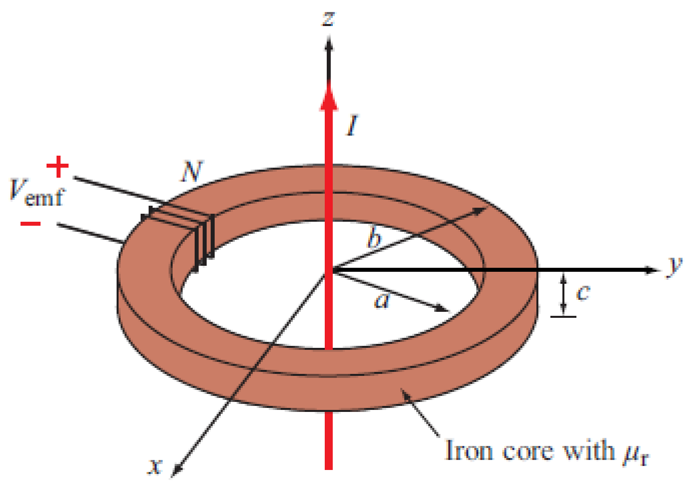


**Figure P6.8:** Problem 6.8.

**(a)** Develop an expression for $V_{emf}$.

**(b)** Calculate $V_{emf}$ for $f = 60$ Hz, $\mu_r = 4000$, $a = 5$ cm, $b = 6$ cm, $c = 2$ cm, and $I_0 = 50$ A.

**Solution:**

**(a)** We start by calculating the magnetic flux through the coil, noting that $r$, the distance from the wire varies from $a$ to $b$

$$\Phi = \int_S \mathbf{B} \cdot d\mathbf{s} = \int_a^b \hat{\mathbf{x}} \frac{\mu I}{2\pi r} \cdot \hat{\mathbf{x}} c \, dr = \frac{\mu c I}{2\pi} \ln\left(\frac{b}{a}\right)$$

$$V_{emf} = -N \frac{d\Phi}{dt} = -\frac{\mu c N}{2\pi} \ln\left(\frac{b}{a}\right) \frac{dI}{dt}$$

$$= \frac{\mu c N \omega I_0}{2\pi} \ln\left(\frac{b}{a}\right) \sin \omega t \quad \text{(V)}.$$

**(b)**

$$V_{emf} = \frac{4000 \times 4\pi \times 10^{-7} \times 2 \times 10^{-2} \times 100 \times 2\pi \times 60 \times 50 \ln(6/5)}{2\pi} \sin 377t$$

$$= 5.5 \sin 377t \quad \text{(V)}.$$

**Problem 6.11** The loop shown in P6.11 moves away from a wire carrying a current $I_1 = 10$ A at a constant velocity $\mathbf{u} = \hat{\mathbf{y}}7.5$ (m/s). If $R = 10\ \Omega$ and the direction of $I_2$ is as defined in the figure, find $I_2$ as a function of $y_0$, the distance between the wire and the loop. Ignore the internal resistance of the loop.

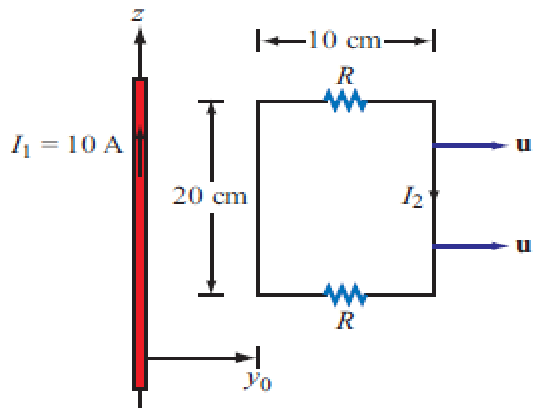


**Figure P6.11:** Moving loop of Problem 6.11.

**Solution:** Assume that the wire carrying current $I_1$ is in the same plane as the loop. The two identical resistors are in series, so $I_2 = V_{\text{emf}}/2R$, where the induced voltage is due to motion of the loop and is given by Eq. (6.26):

$$V_{\text{emf}} = V_{\text{emf}}^{\text{m}} = \oint_C (\mathbf{u} \times \mathbf{B}) \cdot d\mathbf{l}.$$

The magnetic field $\mathbf{B}$ is created by the wire carrying $I_1$. Choosing $\hat{\mathbf{z}}$ to coincide with the direction of $I_1$, Eq. (5.30) gives the external magnetic field of a long wire to be

$$\mathbf{B} = \hat{\boldsymbol{\phi}} \frac{\mu_0 I_1}{2\pi r} .$$

For positive values of $y_0$ in the $y$-$z$ plane, $\hat{\mathbf{y}} = \hat{\mathbf{r}}$, so

$$\mathbf{u} \times \mathbf{B} = \hat{\mathbf{y}}|\mathbf{u}| \times \mathbf{B} = \hat{\mathbf{r}}|\mathbf{u}| \times \hat{\boldsymbol{\phi}} \frac{\mu_0 I_1}{2\pi r} = \hat{\mathbf{z}} \frac{\mu_0 I_1 u}{2\pi r} .$$

Integrating around the four sides of the loop with $d\mathbf{l} = \hat{\mathbf{z}}\, dz$ and the limits of integration chosen in accordance with the assumed direction of $I_2$, and recognizing that only the two sides without the resistors contribute to $V_{\text{emf}}^{\text{m}}$, we have

$$\begin{aligned} V_{\text{emf}}^{\text{m}} &= \int_0^{0.2} \left( \hat{\mathbf{z}} \frac{\mu_0 I_1 u}{2\pi r} \right)\bigg|_{r=y_0} \cdot (\hat{\mathbf{z}}\, dz) + \int_{0.2}^{0} \left( \hat{\mathbf{z}} \frac{\mu_0 I_1 u}{2\pi r} \right)\bigg|_{r=y_0+0.1} \cdot (\hat{\mathbf{z}}\, dz) \\ &= \frac{4\pi \times 10^{-7} \times 10 \times 7.5 \times 0.2}{2\pi} \left( \frac{1}{y_0} - \frac{1}{y_0 + 0.1} \right) \\ &= 3 \times 10^{-6} \left( \frac{1}{y_0} - \frac{1}{y_0 + 0.1} \right) \quad \text{(V)}, \end{aligned}$$

and therefore

$$I_2 = \frac{V_{\text{emf}}^{\text{m}}}{2R} = 150 \left( \frac{1}{y_0} - \frac{1}{y_0 + 0.1} \right) \quad \text{(nA)}.$$

**Problem 6.14** The plates of a parallel-plate capacitor have areas of 10 cm$^2$ each and are separated by 2 cm. The capacitor is filled with a dielectric material with $\varepsilon = 4\varepsilon_0$, and the voltage across it is given by $V(t) = 30\cos 2\pi \times 10^6 t$ (V). Find the displacement current.

**Solution:** Since the voltage is of the form given by Eq. (6.46) with $V_0 = 30$ V and $\omega = 2\pi \times 10^6$ rad/s, the displacement current is given by Eq. (6.49):

$$\begin{aligned} I_{\text{d}} &= -\frac{\varepsilon A}{d} V_0 \omega \sin \omega t \\ &= -\frac{4 \times 8.854 \times 10^{-12} \times 10 \times 10^{-4}}{2 \times 10^{-2}} \times 30 \times 2\pi \times 10^6 \sin(2\pi \times 10^6 t) \\ &= -0.33 \sin(2\pi \times 10^6 t) \quad \text{(mA)}. \end{aligned}$$

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**Problem 6.19** At $t = 0$, charge density $\rho_{\text{v0}}$ was introduced into the interior of a material with a relative permittivity $\varepsilon_{\text{r}} = 9$. If at $t = 1$ $\mu$s the charge density has dissipated down to $10^{-3}\rho_{\text{v0}}$, what is the conductivity of the material?

**Solution:** We start by using Eq. (6.61) to find $\tau_{\text{r}}$:

$$\rho_{\text{v}}(t) = \rho_{\text{v0}} e^{-t/\tau_{\text{r}}},$$

or

$$10^{-3}\rho_{\text{v0}} = \rho_{\text{v0}} e^{-10^{-6}/\tau_{\text{r}}},$$

which gives

$$\ln 10^{-3} = -\frac{10^{-6}}{\tau_{\text{r}}},$$

or

$$\tau_{\text{r}} = -\frac{10^{-6}}{\ln 10^{-3}} = 1.45 \times 10^{-7} \quad \text{(s)}.$$

But $\tau_{\text{r}} = \varepsilon/\sigma = 9\varepsilon_0/\sigma$. Hence

$$\sigma = \frac{9\varepsilon_0}{\tau_{\text{r}}} = \frac{9 \times 8.854 \times 10^{-12}}{1.45 \times 10^{-7}} = 5.5 \times 10^{-4} \quad \text{(S/m)}.$$

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**Problem 6.24** The magnetic field in a dielectric material with $\varepsilon = 4\varepsilon_0$, $\mu = \mu_0$, and $\sigma = 0$ is given by

$$\mathbf{H}(y,t) = \hat{\mathbf{x}}5\cos(2\pi \times 10^7 t + ky) \quad \text{(A/m)}.$$

Find $k$ and the associated electric field $\mathbf{E}$.

**Solution:** In phasor form, the magnetic field is given by $\widetilde{\mathbf{H}} = \hat{\mathbf{x}}5e^{jky}$ (A/m). From Eq. (6.86),

$$\widetilde{\mathbf{E}} = \frac{1}{j\omega\varepsilon}\nabla\times\widetilde{\mathbf{H}} = \frac{-jk}{j\omega\varepsilon}\hat{\mathbf{z}}5e^{jky}$$

and, from Eq. (6.87),

$$\widetilde{\mathbf{H}} = \frac{1}{-j\omega\mu}\nabla\times\widetilde{\mathbf{E}} = \frac{-jk^2}{-j\omega^2\varepsilon\mu}\hat{\mathbf{x}}5e^{jky},$$

which, together with the original phasor expression for $\widetilde{\mathbf{H}}$, implies that

$$k = \omega\sqrt{\varepsilon\mu} = \frac{\omega\sqrt{\varepsilon_r}}{c} = \frac{2\pi\times 10^7\sqrt{4}}{3\times 10^8} = \frac{4\pi}{30} \quad \text{(rad/m)}.$$

Inserting this value in the expression for $\widetilde{\mathbf{E}}$ above,

$$\widetilde{\mathbf{E}} = -\hat{\mathbf{z}}\frac{4\pi/30}{2\pi\times 10^7\times 4\times 8.854\times 10^{-12}}5e^{j4\pi y/30} = -\hat{\mathbf{z}}941e^{j4\pi y/30} \quad \text{(V/m)}.$$

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**Problem 6.25** Given an electric field

$$\mathbf{E} = \hat{\mathbf{x}}E_0\sin ay\cos(\omega t - kz),$$

where $E_0$, $a$, $\omega$, and $k$ are constants, find $\mathbf{H}$.

**Solution:**

$$\mathbf{E} = \hat{\mathbf{x}}E_0\sin ay\cos(\omega t - kz),$$

$$\widetilde{\mathbf{E}} = \hat{\mathbf{x}}E_0\sin ay\, e^{-jkz},$$

$$\widetilde{\mathbf{H}} = -\frac{1}{j\omega\mu}\nabla\times\widetilde{\mathbf{E}}$$

$$= -\frac{1}{j\omega\mu}\left[\hat{\mathbf{y}}\frac{\partial}{\partial z}(E_0\sin ay\, e^{-jkz}) - \hat{\mathbf{z}}\frac{\partial}{\partial y}(E_0\sin ay\, e^{-jkz})\right]$$

$$= \frac{E_0}{\omega\mu}[\hat{\mathbf{y}}k\sin ay - \hat{\mathbf{z}}ja\cos ay]e^{-jkz},$$

$$\mathbf{H} = \mathfrak{Re}[\widetilde{\mathbf{H}}e^{j\omega t}]$$

$$= \mathfrak{Re}\left\{\frac{E_0}{\omega\mu}[\hat{\mathbf{y}}k\sin ay + \hat{\mathbf{z}}a\cos ay\, e^{-j\pi/2}]e^{-jkz}e^{j\omega t}\right\}$$

$$= \frac{E_0}{\omega\mu}\left[\hat{\mathbf{y}}k\sin ay\cos(\omega t - kz) + \hat{\mathbf{z}}a\cos ay\cos\left(\omega t - kz - \frac{\pi}{2}\right)\right]$$

$$= \frac{E_0}{\omega\mu}[\hat{\mathbf{y}}k\sin ay\cos(\omega t - kz) + \hat{\mathbf{z}}a\cos ay\sin(\omega t - kz)].$$

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