Problem 6.3  A coil consists of 100 turns of wire wrapped around a square frame of sides 0.25 m. The coil is centered at the origin with each of its sides parallel to the x- or y-axis. Find the induced emf across the open-circuited ends of the coil if the magnetic field is given by

(a) \( \mathbf{B} = \hat{z} 20e^{-3t} \) (T)

(b) \( \mathbf{B} = \hat{z} 20 \cos x \cos 10^3 t \) (T)

(c) \( \mathbf{B} = \hat{z} 20 \cos x \sin 2y \cos 10^3 t \) (T)

Solution: Since the coil is not moving or changing shape, \( V_{\text{emf}}^{\text{in}} = 0 \) V and \( V_{\text{emf}} = V_{\text{emf}}^{\text{out}} \). From Eq. (6.6),

\[
V_{\text{emf}} = -N \frac{d}{dt} \int_{S} \mathbf{B} \cdot d\mathbf{s} = -N \frac{d}{dt} \int_{-0.125}^{0.125} \int_{-0.125}^{0.125} \mathbf{B} \cdot (\hat{z} \, dx \, dy),
\]

where \( N = 100 \) and the surface normal was chosen to be in the +\( \hat{z} \) direction.

(a) For \( \mathbf{B} = \hat{z} 20e^{-3t} \) (T),

\[
V_{\text{emf}} = -100 \frac{d}{dt} \left( 20e^{-3t} \right)^2 = 375e^{-3t} \quad \text{(V)}.
\]

(b) For \( \mathbf{B} = \hat{z} 20 \cos x \cos 10^3 t \) (T),

\[
V_{\text{emf}} = -100 \frac{d}{dt} \left( 20 \cos 10^3 t \int_{x=-0.125}^{0.125} \int_{y=-0.125}^{0.125} \cos x \, dx \, dy \right) = 124.6 \sin 10^3 t \quad \text{(kV)}.
\]

(c) For \( \mathbf{B} = \hat{z} 20 \cos x \sin 2y \cos 10^3 t \) (T),

\[
V_{\text{emf}} = -100 \frac{d}{dt} \left( 20 \cos 10^3 t \int_{x=-0.125}^{0.125} \int_{y=-0.125}^{0.125} \cos x \sin 2y \, dx \, dy \right) = 0.
\]

Problem 6.5  A circular-loop TV antenna with 0.02-m\(^2\) area is in the presence of a uniform-amplitude 300-MHz signal. When oriented for maximum response, the loop develops an emf with a peak value of 30 (mV). What is the peak magnitude of \( \mathbf{B} \) of the incident wave?

Solution: TV loop antennas have one turn. At maximum orientation, Eq. (6.5) evaluates to \( \Phi = \int \mathbf{B} \cdot d\mathbf{s} = \pm BA \) for a loop of area \( A \) and a uniform magnetic field with magnitude \( B = |B| \). Since we know the frequency of the field is \( f = 300 \) MHz, we can express \( B \) as \( B = B_0 \cos (\omega t + \alpha_0) \) with \( \omega = 2\pi \times 300 \times 10^6 \text{ rad/s} \) and \( \alpha_0 \) an arbitrary reference phase. From Eq. (6.6),

\[
V_{\text{emf}} = -N \frac{d\Phi}{dt} = -A \frac{d}{dt} \left[ B_0 \cos(\omega t + \alpha_0) \right] = AB_0 \omega \sin(\omega t + \alpha_0).
\]

\( V_{\text{emf}} \) is maximum when \( \sin(\omega t + \alpha_0) = 1 \). Hence,

\[
30 \times 10^{-3} = AB_0 \omega = 0.02 \times B_0 \times 6\pi \times 10^8,
\]

which yields \( B_0 = 0.8 \) (nA/m).
Problem 6.6  The square loop shown in Fig. P6.6 is coplanar with a long, straight wire carrying a current

\[ I(t) = 5 \cos(2\pi \times 10^4 t) \quad \text{(A)}. \]

(a) Determine the emf induced across a small gap created in the loop.

(b) Determine the direction and magnitude of the current that would flow through a 4-\(\Omega\) resistor connected across the gap. The loop has an internal resistance of 1 \(\Omega\).

![Figure P6.6: Loop coplanar with long wire (Problem 6.6).](image)

Solution:

(a) The magnetic field due to the wire is

\[ B = \frac{\mu_0 I}{2\pi r} = -\hat{x} \frac{\mu_0 I}{2\pi y}, \]

where in the plane of the loop, \(\hat{\phi} = -\hat{x}\) and \(r = y\). The flux passing through the loop is

\[
\Phi = \int_S B \cdot ds = \int_{5 \text{ cm}}^{15 \text{ cm}} \left(-\hat{x} \frac{\mu_0 I}{2\pi y}\right) \cdot [-\hat{x} \times 10 \text{ (cm)}] \ dy
\]

\[
= \frac{\mu_0 I \times 10^{-1}}{2\pi} \ln \frac{15}{5}
\]

\[
= \frac{4\pi \times 10^{-7}}{2\pi} \times 5 \cos(2\pi \times 10^4 t) \times 10^{-1} \times 1.1
\]

\[
= 1.1 \times 10^{-7} \cos(2\pi \times 10^4 t) \quad \text{(Wb)}.
\]

\[
V_{\text{emf}} = -\frac{d\Phi}{dt} = 1.1 \times 2\pi \times 10^4 \sin(2\pi \times 10^4 t) \times 10^{-7}
\]

\[
= 6.9 \times 10^{-3} \sin(2\pi \times 10^4 t) \quad \text{(V)}.
\]

(b)

\[
I_{\text{ind}} = \frac{V_{\text{emf}}}{4 + 1} = \frac{6.9 \times 10^{-3}}{5} \sin(2\pi \times 10^4 t) = 1.38 \sin(2\pi \times 10^4 t) \quad \text{(mA)}.
\]

At \(t = 0\), \(B\) is a maximum, it points in \(-\hat{x}\)-direction, and since it varies as \(\cos(2\pi \times 10^4 t)\), it is decreasing. Hence, the induced current has to be CCW when looking down on the loop, as shown in the figure.
Problem 6.7  The rectangular conducting loop shown in Fig. P6.7 rotates at 6,000 revolutions per minute in a uniform magnetic flux density given by

\[ \mathbf{B} = 50 \text{ mT}. \]

Determine the current induced in the loop if its internal resistance is 0.5 \( \Omega \).

![Figure P6.7: Rotating loop in a magnetic field (Problem 6.7).](image)

Solution:

\[
\Phi = \int_S \mathbf{B} \cdot d\mathbf{S} = 50 \times 10^{-3} \cdot (2 \times 3 \times 10^{-4}) \cos \phi(t) = 3 \times 10^{-5} \cos \phi(t),
\]

\[
\phi(t) = \omega t = \frac{2\pi \times 6 \times 10^3}{60} t = 200\pi t \quad \text{(rad/s)},
\]

\[
\Phi = 3 \times 10^{-5} \cos(200\pi t) \quad \text{(Wb)},
\]

\[
V_{\text{emf}} = -\frac{d\Phi}{dt} = 3 \times 10^{-5} \times 200\pi \sin(200\pi t) = 18.85 \times 10^{-3} \sin(200\pi t) \quad \text{(V)},
\]

\[
I_{\text{ind}} = \frac{V_{\text{emf}}}{0.5} = 37.7 \sin(200\pi t) \quad \text{(mA)}.
\]

The direction of the current is CW (if looking at it along \(-x\)-direction) when the loop is in the first quadrant \((0 \leq \phi \leq \pi/2)\). The current reverses direction in the second quadrant, and reverses again every quadrant.
Problem 6.8  The transformer shown in Fig. P6.8 consists of a long wire coincident with the z-axis carrying a current \( I = I_0 \cos \omega t \), coupling magnetic energy to a toroidal coil situated in the x–y plane and centered at the origin. The toroidal core uses iron material with relative permeability \( \mu_r \), around which 100 turns of a tightly wound coil serves to induce a voltage \( V_{\text{emf}} \), as shown in the figure.

![Figure P6.8: Problem 6.8.](image)

(a) Develop an expression for \( V_{\text{emf}} \).

(b) Calculate \( V_{\text{emf}} \) for \( f = 60 \text{ Hz} \), \( \mu_r = 4000 \), \( a = 5 \text{ cm} \), \( b = 6 \text{ cm} \), \( c = 2 \text{ cm} \), and \( I_0 = 50 \text{ A} \).

**Solution:**

(a) We start by calculating the magnetic flux through the coil, noting that \( r \), the distance from the wire varies from \( a \) to \( b \)

\[
\Phi = \int_S \mathbf{B} \cdot d\mathbf{s} = \int_a^b \mathbf{x} \cdot \frac{\mu I}{2\pi r} \mathbf{c} \, dr = \frac{\mu c I}{2\pi} \ln \left( \frac{b}{a} \right)
\]

\[
V_{\text{emf}} = -N \frac{d\Phi}{dt} = -\frac{\mu c N}{2\pi} \ln \left( \frac{b}{a} \right) \frac{dI}{dt}
\]

\[
= \frac{\mu c N \omega I_0}{2\pi} \ln \left( \frac{b}{a} \right) \sin \omega t \quad \text{(V)}.
\]

(b)

\[
V_{\text{emf}} = \frac{4000 \times 4\pi \times 10^{-7} \times 2 \times 10^{-2} \times 100 \times 2\pi \times 60 \times 50 \ln(6/5)}{2\pi} \sin 377t
\]

\[= 5.5 \sin 377t \quad \text{(V)}.
\]
Problem 6.11  The loop shown in P6.11 moves away from a wire carrying a current \( I_1 = 10 \text{ A} \) at a constant velocity \( \mathbf{u} = \hat{y}7.5 \text{ (m/s)} \). If \( R = 10 \Omega \) and the direction of \( I_1 \) is as defined in the figure, find \( I_2 \) as a function of \( y_0 \), the distance between the wire and the loop. Ignore the internal resistance of the loop.

![Diagram of moving loop](image)

**Figure P6.11**: Moving loop of Problem 6.11.

**Solution**: Assume that the wire carrying current \( I_1 \) is in the same plane as the loop. The two identical resistors are in series, so \( I_2 = V_{\text{emf}}/2R \), where the induced voltage is due to motion of the loop and is given by Eq. (6.26):

\[
V_{\text{emf}} = V_{\text{emf}}^m = \int_C (\mathbf{u} \times \mathbf{B}) \cdot d\mathbf{l}.
\]

The magnetic field \( \mathbf{B} \) is created by the wire carrying \( I_1 \). Choosing \( \hat{z} \) to coincide with the direction of \( I_1 \), Eq. (5.30) gives the external magnetic field of a long wire to be

\[
\mathbf{B} = \hat{z} \frac{\mu_0 I_1}{2\pi r}.
\]

For positive values of \( y_0 \) in the y-z plane, \( \hat{y} = \hat{r} \), so

\[
\mathbf{u} \times \mathbf{B} = \hat{y} |\mathbf{u}| \times \mathbf{B} = \hat{r} |\mathbf{u}| \times \hat{z} \frac{\mu_0 I_1}{2\pi r}.
\]

Integrating around the four sides of the loop with \( d\mathbf{l} = \hat{z} dz \) and the limits of integration chosen in accordance with the assumed direction of \( I_2 \), and recognizing that only the two sides without the resistors contribute to \( V_{\text{emf}}^m \), we have

\[
V_{\text{emf}}^m = \int_0^{0.2} \left( \frac{\hat{z} \mu_0 I_1 u}{2\pi r} \right) \bigg|_{r=y_0}^{r=y_0+0.1} \cdot (\hat{z} \, dz) + \int_0^0 \left( \frac{\hat{z} \mu_0 I_1 u}{2\pi r} \right) \bigg|_{r=y_0}^{r=y_0+0.1} \cdot (\hat{z} \, dz)
\]

\[
= 4\pi \times 10^{-7} \times 10 \times 7.5 \times 0.2 \left( \frac{1}{y_0} - \frac{1}{y_0 + 0.1} \right)
\]

\[
= 3 \times 10^{-6} \left( \frac{1}{y_0} - \frac{1}{y_0 + 0.1} \right) \text{ (V)},
\]

and therefore

\[
I_2 = \frac{V_{\text{emf}}}{2R} = 150 \left( \frac{1}{y_0} - \frac{1}{y_0 + 0.1} \right) \text{ (nA)}.
\]
Problem 6.14  The plates of a parallel-plate capacitor have areas of 10 cm$^2$ each and are separated by 2 cm. The capacitor is filled with a dielectric material with $\varepsilon = 4\varepsilon_0$, and the voltage across it is given by $V(t) = 30\cos 2\pi \times 10^6 t$ (V). Find the displacement current.

Solution: Since the voltage is of the form given by Eq. (6.46) with $V_0 = 30$ V and $\omega = 2\pi \times 10^6$ rad/s, the displacement current is given by Eq. (6.49):

$$I_d = -\frac{\varepsilon A}{d} V_0 \omega \sin \omega t$$
$$= -\frac{4 \times 8.854 \times 10^{-12} \times 10 \times 10^{-4}}{2 \times 10^{-2}} \times 30 \times 2\pi \times 10^6 \sin(2\pi \times 10^6 t)$$
$$= -0.33 \sin (2\pi \times 10^6 t) \text{ (mA)}.$$

Problem 6.19  At $t = 0$, charge density $\rho_{v0}$ was introduced into the interior of a material with a relative permittivity $\varepsilon_r = 9$. If at $t = 1$ $\mu$s the charge density has dissipated down to $10^{-3}\rho_{v0}$, what is the conductivity of the material?

Solution: We start by using Eq. (6.61) to find $\tau_r$:

$$\rho_v(t) = \rho_{v0} e^{-t/\tau_r},$$

or

$$10^{-3}\rho_{v0} = \rho_{v0} e^{-10^{-6}/\tau_r},$$

which gives

$$\ln 10^{-3} = -\frac{10^{-6}}{\tau_r},$$

or

$$\tau_r = -\frac{10^{-6}}{\ln 10^{-3}} = 1.45 \times 10^{-7} \text{ (s)}.$$

But $\tau_r = \varepsilon / \sigma = 9\varepsilon_0 / \sigma$. Hence

$$\sigma = \frac{9\varepsilon_0}{\tau_r} = \frac{9 \times 8.854 \times 10^{-12}}{1.45 \times 10^{-7}} = 5.5 \times 10^{-4} \text{ (S/m)}.$$
**Problem 6.24**  The magnetic field in a dielectric material with \( \varepsilon = 4\varepsilon_0, \mu = \mu_0, \) and \( \sigma = 0 \) is given by

\[
\mathbf{H}(y, t) = \hat{x}5\cos(2\pi \times 10^7 t + ky) \quad \text{(A/m)}.
\]

Find \( k \) and the associated electric field \( \mathbf{E} \).

**Solution:** In phasor form, the magnetic field is given by \( \mathbf{H} = \hat{x}5e^{jky} \) (A/m). From Eq. (6.86),

\[
\hat{\mathbf{E}} = \frac{1}{j\omega\varepsilon} \nabla \times \mathbf{H} = \frac{-jk}{j\omega\varepsilon} \hat{z}5e^{jky}
\]

and, from Eq. (6.87),

\[
\mathbf{H} = \frac{1}{-j\omega\mu} \nabla \times \hat{\mathbf{E}} = \frac{-jk^2}{-j\omega^2\varepsilon\mu} \hat{x}5e^{jky},
\]

which, together with the original phasor expression for \( \mathbf{H} \), implies that

\[
k = \omega\sqrt{\varepsilon\mu} = \frac{\omega}{c} = \frac{2\pi \times 10^7 \sqrt{4}}{3 \times 10^8} = \frac{4\pi}{30} \quad \text{(rad/m)}.
\]

Inserting this value in the expression for \( \hat{\mathbf{E}} \) above,

\[
\hat{\mathbf{E}} = -\frac{4\pi/30}{2\pi \times 10^7 \times 4 \times 8.854 \times 10^{-12}} 5e^{j4\pi y/30} = -2941e^{j4\pi y/30} \quad \text{(V/m)}.
\]

---

**Problem 6.25**  Given an electric field

\[
\mathbf{E} = \hat{x}E_0 \sin ay \cos(\omega t - kz),
\]

where \( E_0, a, \omega, \) and \( k \) are constants, find \( \mathbf{H} \).

**Solution:**

\[
\begin{align*}
\mathbf{E} &= \hat{x}E_0 \sin ay \cos(\omega t - kz), \\
\mathbf{E} &= \hat{x}E_0 \sin ay e^{-jkz}, \\
\mathbf{H} &= -\frac{1}{j\omega\mu} \nabla \times \mathbf{E} \\
&= -\frac{1}{j\omega\mu} \left[ \hat{y} \frac{\partial}{\partial z} (E_0 \sin ay e^{-jkz}) - \hat{z} \frac{\partial}{\partial y} (E_0 \sin ay e^{-jkz}) \right] \\
&= \frac{E_0}{\omega\mu} \left[ \hat{y}k \sin ay - \hat{z} \alpha \cos ay \right] e^{-jkz}, \\
\mathbf{H} &= \Re[\mathbf{H}e^{j\omega t}] \\
&= \Re \left\{ \frac{E_0}{\omega\mu} \left[ \hat{y}k \sin ay + \hat{z} \alpha \cos ay e^{-j\pi/2} \right] e^{-jkz} e^{j\omega t} \right\} \\
&= \frac{E_0}{\omega\mu} \left[ \hat{y}k \sin ay \cos(\omega t - kz) + \hat{z} \alpha \cos ay \cos(\omega t - k\frac{\pi}{2}) \right] \\
&= \frac{E_0}{\omega\mu} \left[ \hat{y}k \sin ay \cos(\omega t - kz) + \hat{z} \alpha ay \sin(\omega t - kz) \right].
\end{align*}
\]