**SOLUTION HW # 11**

**Problem 7.19** Ignoring reflection at the air–soil boundary, if the amplitude of a 3-GHz incident wave is 10 V/m at the surface of a wet soil medium, at what depth will it be down to 1 mV/m? Wet soil is characterized by $\mu_r = 1$, $\varepsilon_r = 9$, and $\sigma = 5 \times 10^{-4}$ S/m.

**Solution:**

$$E(z) = E_0 e^{-\alpha z} = 10 e^{-\alpha z},$$

$$\frac{\sigma}{\omega \varepsilon} = \frac{5 \times 10^{-4} \times 36\pi}{2\pi \times 3 \times 10^9 \times 10^{-9} \times 9} = 3.32 \times 10^{-4}.$$  

Hence, medium is a low-loss dielectric.

$$\alpha = \frac{\sigma}{2} \sqrt{\frac{\mu}{\varepsilon}} = \frac{\sigma}{2} \cdot \frac{120\pi}{\sqrt{\varepsilon_r}} = \frac{5 \times 10^{-4} \times 120\pi}{2 \times \sqrt{9}} = 0.032 \text{  (Np/m)},$$

$$10^{-3} = 10 e^{-0.032z}, \quad \ln 10^{-4} = -0.032z, \quad z = 287.82 \text{ m}.$$

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**Problem 7.21** Based on wave attenuation and reflection measurements conducted at 1 MHz, it was determined that the intrinsic impedance of a certain medium is $28.1 \angle 45^\circ$ (Ω) and the skin depth is 2 m. Determine the following:

(a) The conductivity of the material.

(b) The wavelength in the medium.

(c) The phase velocity.

**Solution:**

(a) Since the phase angle of $\eta_c$ is $45^\circ$, the material is a good conductor. Hence,

$$\eta_c = (1 + j)\frac{\alpha}{\sigma} = 28.1 e^{j45^\circ} = 28.1 \cos 45^\circ + j28.1 \sin 45^\circ,$$

or

$$\frac{\alpha}{\sigma} = 28.1 \cos 45^\circ = 19.87.$$

Since $\alpha = 1/\delta_s = 1/2 = 0.5$ Np/m,

$$\sigma = \frac{\alpha}{19.87} = \frac{0.5}{19.87} = 2.52 \times 10^{-2} \text{ S/m}.$$

(b) Since $\alpha = \beta$ for a good conductor, and $\alpha = 0.5$, it follows that $\beta = 0.5$. Therefore,

$$\lambda = \frac{2\pi}{\beta} = \frac{2\pi}{0.5} = 4\pi = 12.57 \text{ m}.$$

(c) $v_p = f\lambda = 10^6 \times 12.57 = 1.26 \times 10^7 \text{ m/s}$. 
Problem 7.22 The electric field of a plane wave propagating in a nonmagnetic medium is given by

\[ E = \hat{z} \, 25 e^{-30x} \cos(2\pi \times 10^9 t - 40x) \quad (V/m) \]

Obtain the corresponding expression for \( H \).

**Solution:** From the given expression for \( E \),

\[ \omega = 2\pi \times 10^9 \quad \text{(rad/s)}, \]
\[ \alpha = 30 \quad \text{(Np/m)}, \]
\[ \beta = 40 \quad \text{(rad/m)}. \]

From (7.65a) and (7.65b),

\[ \alpha^2 - \beta^2 = -\omega^2 \mu \epsilon' = -\omega^2 \mu_0 \epsilon_0 \epsilon'_r = -\frac{\omega^2}{c^2} \epsilon'_r, \]
\[ 2\alpha\beta = \omega^2 \mu \epsilon'' = \frac{\omega^2}{c^2} \epsilon''_r. \]

Using the above values for \( \omega, \alpha, \) and \( \beta \), we obtain the following:

\[ \epsilon'_r = 1.6, \]
\[ \epsilon''_r = 5.47, \]
\[ \eta_c = \sqrt{\frac{\mu}{\epsilon'}} \left( 1 - j \frac{\epsilon''}{\epsilon'} \right)^{-1/2} \]
\[ = \eta_0 \sqrt{\frac{\epsilon'_r}{\epsilon''_r}} \left( 1 - j \frac{\epsilon''}{\epsilon'} \right)^{-1/2} = \frac{377}{\sqrt{1.6}} \left( 1 - j \frac{5.47}{1.6} \right)^{-1/2} = 157.9 e^{j36.85^\circ} \quad (\Omega). \]
\[ \vec{E} = \hat{z} \, 25 e^{-30x} e^{-j40x}, \]
\[ \vec{H} = \frac{1}{\eta_c} \vec{k} \times \vec{E} = \frac{1}{157.9 e^{j36.85^\circ}} \hat{x} \times \hat{z} \, 25 e^{-30x} e^{-j40x} = -\hat{y} \, 0.16 e^{-30x} e^{-j36.85^\circ}, \]
\[ \vec{H} = \Re\{\vec{H} e^{j\delta_s}\} = -\hat{y} \, 0.16 e^{-30x} \cos(2\pi \times 10^9 t - 40x - 36.85^\circ) \quad (A/m). \]

Problem 7.26 The inner and outer conductors of a coaxial cable have radii of 0.5 cm and 1 cm, respectively. The conductors are made of copper with \( \epsilon_r = 1 \), \( \mu_r = 1 \), and \( \sigma = 5.8 \times 10^7 \) S/m, and the outer conductor is 0.5 mm thick. At 10 MHz:

(a) Are the conductors thick enough to be considered infinitely thick as far as the flow of current through them is concerned?

(b) Determine the surface resistance \( R_s \).

(c) Determine the ac resistance per unit length of the cable.

**Solution:**

(a) From Eqs. (7.72) and (7.77b),

\[ \delta_s = [\pi f \mu \sigma]^{-1/2} = [\pi \times 10^7 \times 4\pi \times 10^{-7} \times 5.8 \times 10^7]^{-1/2} = 0.021 \text{ mm}. \]

Hence,

\[ \frac{d}{\delta_s} = \frac{0.5 \text{ mm}}{0.021 \text{ mm}} \approx 25. \]

Hence, conductor is plenty thick.

(b) From Eq. (7.82a),

\[ R_s = \frac{1}{\sigma \delta_s} = \frac{1}{5.8 \times 10^7 \times 2.1 \times 10^{-5}} = 8.2 \times 10^{-4} \ \Omega. \]

(c) From Eq. (7.96),

\[ R' = \frac{R_s}{2\pi} \left( \frac{1}{a} + \frac{1}{b} \right) = \frac{8.2 \times 10^{-4}}{2\pi} \left( \frac{1}{5 \times 10^{-3}} + \frac{1}{10^{-2}} \right) = 0.039 \quad (\Omega/m). \]
Problem 7.29  The electric-field phasor of a uniform plane wave traveling downward in water is given by

\[ \vec{E} = 5 e^{-0.2z} e^{-j0.2z} \text{ (V/m)} \]

where \( \hat{z} \) is the downward direction and \( z = 0 \) is the water surface. If \( \sigma = 4 \text{ S/m} \),

(a) Obtain an expression for the average power density.
(b) Determine the attenuation rate.
(c) Determine the depth at which the power density has been reduced by 40 dB.

Solution:

(a) Since \( \alpha = \beta = 0.2 \), the medium is a good conductor.

\[ \eta_c = (1 + j) \frac{\alpha}{\sigma} = (1 + j) \frac{0.2}{4} = (1 + j)0.05 = 0.0707 e^{j45^\circ} \text{ (\Omega)} \]

From Eq. (7.109),

\[ S_{av} = \frac{\hat{z} |E_0|^2}{2|\eta_c|} e^{-2\alpha z} \cos \theta_0 = \frac{\hat{z} 25}{2 \times 0.0707} e^{-0.4z} \cos 45^\circ = \hat{z} 125 e^{-0.4z} \text{ (W/m}^2) \]

(b) \( A = -8.68 \alpha z = -8.68 \times 0.2z = -1.74z \text{ (dB)} \).
(c) 40 dB is equivalent to \( 10^{-4} \). Hence,

\[ 10^{-4} = e^{-2\alpha z} = e^{-0.4z} \]

\[ \ln(10^{-4}) = -0.4z \]

or \( z = 23.03 \text{ m} \).
Problem 8.2  A plane wave traveling in medium 1 with \( \varepsilon_1 = 2.25 \) is normally incident upon medium 2 with \( \varepsilon_2 = 4 \). Both media are made of nonmagnetic, nonconducting materials. If the electric field of the incident wave is given by

\[
E^i = \hat{y} 8 \cos(6\pi \times 10^9 t - 30\pi x) \quad \text{(V/m)}.
\]

(a) Obtain time-domain expressions for the electric and magnetic fields in each of the two media.

(b) Determine the average power densities of the incident, reflected and transmitted waves.

Solution:

(a)

\[
E^i = \hat{y} 8 \cos(6\pi \times 10^9 t - 30\pi x) \quad \text{(V/m)},
\]

\[
\eta_1 = \frac{\eta_0}{\sqrt{\varepsilon_1}} = \frac{\eta_0}{\sqrt{2.25}} = \frac{\eta_0}{1.5} = \frac{377}{1.5} = 251.33 \Omega,
\]

\[
\eta_2 = \frac{\eta_0}{\sqrt{\varepsilon_2}} = \frac{\eta_0}{\sqrt{4}} = \frac{377}{2} = 188.5 \Omega,
\]

\[
\Gamma = \frac{\eta_2 - \eta_1}{\eta_2 + \eta_1} = \frac{1/2 - 1/1.5}{1/2 + 1/1.5} = -0.143,
\]

\[
\tau = 1 + \Gamma = 1 - 0.143 = 0.857,
\]

\[
E' = \Gamma E^i = -1.14 \hat{y} \cos(6\pi \times 10^9 t + 30\pi x) \quad \text{(V/m)}.
\]

Note that the coefficient of \( x \) is positive, denoting the fact that \( E' \) belongs to a wave traveling in \(-x\)-direction.

\[
E_1 = E^i + E' = \hat{y} [8 \cos(6\pi \times 10^9 t - 30\pi x) - 1.14 \cos(6\pi \times 10^9 t + 30\pi x)] \quad \text{(A/m)},
\]

\[
H^i = \frac{2}{\eta_1} \cos(6\pi \times 10^9 t - 30\pi x) = 31.83 \cos(6\pi \times 10^9 t - 30\pi x) \quad \text{(mA/m)},
\]

\[
H' = \frac{1.14}{\eta_1} \cos(6\pi \times 10^9 t + 30\pi x) = 4.54 \cos(6\pi \times 10^9 t + 30\pi x) \quad \text{(mA/m)},
\]

\[
H_1 = H^i + H' = \frac{2}{\eta_1} [31.83 \cos(6\pi \times 10^9 t - 30\pi x) + 4.54 \cos(6\pi \times 10^9 t + 30\pi x)] \quad \text{(mA/m)}.
\]

Since \( k_1 = \omega \sqrt{\mu \varepsilon_1} \) and \( k_2 = \omega \sqrt{\mu \varepsilon_2} \),

\[
k_2 = \sqrt{\frac{\varepsilon_2}{\varepsilon_1}} k_1 = \sqrt{\frac{4}{2.25}} 30\pi = 40\pi \quad \text{(rad/m)},
\]

\[
E_2 = E' = \hat{y} 8 \tau \cos(6\pi \times 10^9 t - 40\pi x) = \hat{y} 6.86 \cos(6\pi \times 10^9 t - 40\pi x) \quad \text{(V/m)},
\]

\[
H_2 = H' = \frac{8\tau}{\eta_2} \cos(6\pi \times 10^9 t - 40\pi x) = \frac{36.38}{188.5} \cos(6\pi \times 10^9 t - 40\pi x) \quad \text{(mA/m)}.
\]

(b)

\[
S_{\text{iv}} = x \frac{\varepsilon_1^2}{2 \eta_1} = 64 \frac{2}{2 \times 251.33} = 127.3 \quad \text{(mW/m}^2\text{)},
\]

\[
S_{\text{iv}} = -|\Gamma|^2 S_{\text{av}} = -\hat{x} (0.143)^2 \times 0.127 = -\hat{x} 2.6 \quad \text{(mW/m}^2\text{)},
\]

\[
S_{\text{iv}} = \frac{|E_0|^2}{2 \eta_2} = \frac{\hat{x} \tau^2 (8 \tau)^2}{2 \eta_2} = \frac{\hat{x} (0.86)^2 (8 \tau)^2}{2 \times 251.33} = 124.7 \quad \text{(mW/m}^2\text{)}.
\]

Within calculation error, \( S_{\text{iv}} + S_{\text{rv}} = S_{\text{sv}} \).
Problem 8.4 A 200-MHz, left-hand circularly polarized plane wave with an electric field modulus of 5 V/m is normally incident in air upon a dielectric medium with $\varepsilon_r = 4$, and occupies the region defined by $z \geq 0$.

(a) Write an expression for the electric field phasor of the incident wave, given that the field is a positive maximum at $z = 0$ and $t = 0$.

(b) Calculate the reflection and transmission coefficients.

(c) Write expressions for the electric field phasors of the reflected wave, the transmitted wave, and the total field in the region $z \leq 0$.

(d) Determine the percentages of the incident average power reflected by the boundary and transmitted into the second medium.

Solution:

(a)

$$k_1 = \frac{\omega}{c} = \frac{2\pi \times 2 \times 10^8}{3 \times 10^8} = \frac{4\pi}{3} \text{ rad/m},$$

$$k_2 = \frac{\omega}{\mu_r c} = \frac{\omega}{\sqrt{\varepsilon_r}} = \frac{4\pi}{3} \sqrt{\frac{1}{4}} = \frac{8\pi}{3} \text{ rad/m}.$$

LHC wave:

$$\vec{E}^i = a_0(\hat{x} + j\hat{y}) e^{jkt} = a_0(\hat{x} + j\hat{y}) e^{-jkt},$$

$$\vec{E}^i(z,t) = \hat{x} a_0 \cos(\omega t - k z) - j\hat{y} a_0 \sin(\omega t - k z),$$

$$|\vec{E}^i| = [a_0^2 \cos^2(\omega t - k z) + a_0^2 \sin^2(\omega t - k z)]^{1/2} = a_0 = 5 \text{ (V/m)}.$$

Hence,

$$\vec{E}^i = 5(\hat{x} + j\hat{y}) e^{-j\omega t/3} \text{ (V/m)}.$$

(b)

$$\eta_1 = \eta_0 = 120\pi \text{ (}\Omega), \quad \eta_2 = \frac{\eta_0}{\sqrt{\varepsilon_r}} = \frac{\eta_0}{2} = 60\pi \text{ (}\Omega).$$

Equations (8.8a) and (8.9) give

$$\Gamma = \frac{\eta_2 - \eta_1}{\eta_2 + \eta_1} = \frac{60\pi - 120\pi}{60\pi + 120\pi} = \frac{-60}{180} = -\frac{1}{3}, \quad \tau = 1 + \Gamma = \frac{2}{3}.$$

(c)

$$\tilde{\vec{E}}^t = 5\Gamma(\hat{x} + j\hat{y}) e^{jk_2 z} = -\frac{5}{3}(\hat{x} + j\hat{y}) e^{j4\pi x/3} \text{ (V/m)},$$

$$\tilde{\vec{E}}^t = 5\tau(\hat{x} + j\hat{y}) e^{-jk_2 z} = \frac{10}{3}(\hat{x} + j\hat{y}) e^{-j8\pi x/3} \text{ (V/m)},$$

$$\tilde{\vec{E}}_1 = \tilde{\vec{E}}^i + \tilde{\vec{E}}^t = 5(\hat{x} + j\hat{y}) \left[ e^{-j4\pi x/3} - \frac{1}{3} e^{j4\pi x/3} \right] \text{ (V/m)}.$$

(d)

% of reflected power $= 100 \times |\Gamma|^2 = \frac{100}{9} = 11.11\%$,

% of transmitted power $= 100 \times |\tau|^2 \frac{n_1}{n_2} = 100 \times \left( \frac{2}{3} \right)^2 \times \frac{120\pi}{60\pi} = 88.89\%$. 

Problem 8.6  A 50-MHz plane wave with electric field amplitude of 50 V/m is normally incident in air onto a semi-infinite, perfect dielectric medium with $\varepsilon_r = 36$. Determine the following:

(a) $\Gamma$

(b) The average power densities of the incident and reflected waves.

(c) The distance in the air medium from the boundary to the nearest minimum of the electric field intensity, $|E|$.

Solution:

(a)

$$\eta_1 = \eta_0 = 120\pi \quad (\Omega), \quad \eta_2 = \sqrt{\frac{\mu_2}{\varepsilon_2}} = \frac{120\pi}{\sqrt{\varepsilon_r}} = \frac{120\pi}{6} = 20\pi \quad (\Omega),$$

$$\Gamma = \frac{\eta_2 - \eta_1}{\eta_2 + \eta_1} = \frac{20\pi - 120\pi}{20\pi + 120\pi} = -0.71.$$

Hence, $|\Gamma| = 0.71$ and $\theta_\eta = 180^\circ$.

(b)\[ S_{av} = \frac{|E_0|^2}{2\eta_1} = \frac{(50)^2}{2 \times 120\pi} = 3.32 \quad (\text{W/m}^2), \]

$$S_{av} = |\Gamma|^2 S_{av} = (0.71)^2 \times 3.32 = 1.67 \quad (\text{W/m}^2).$$

(c) In medium 1 (air),

$$\lambda_1 = \frac{c}{f} = \frac{3 \times 10^8}{5 \times 10^7} = 6 \text{ m}.$$

From Eqs. (8.16) and (8.17),

$$I_{\text{max}} = \frac{\theta_\lambda}{\lambda_1} = \frac{\pi \times 6}{4\pi} = 1.5 \text{ m},$$

$$I_{\text{min}} = I_{\text{max}} - \frac{\lambda_1}{4} = 1.5 - 1.5 = 0 \text{ m (at the boundary)}. $$
Problem # 6.1

Calculate the net reflection coefficient $\Gamma$ for the plane wave incident upon the stratified media shown in Figure 12-25. Assume that the dielectric constants of the media are $\varepsilon_1 = 1$, $\varepsilon_2 = 6.25$, $\varepsilon_3 = 2.25$, and $\varepsilon_4 = L$. Also, assume that $\lambda_3 = \lambda_2/8$ and $\ell = \lambda_3/5$.

\[ \begin{array}{cccc}
\varepsilon_1 & \varepsilon_2 & \varepsilon_3 & \varepsilon_4 \\
\ell_1 & \ell_2 & \ell_3 & \ell_4 \\
\eta_1 & \eta_2 & \eta_3 & \eta_4 \\
\eta_5 & \eta_6 & \eta_7 & \eta_8 \\
\end{array} \]

Figure 12-25 Incident and reflected waves at a stack of three interfaces of dissimilar materials.

**Solution:**

Using the specified dielectric constants, we first calculate the following parameters:

\[ \eta_1 = \eta_4 - \frac{377}{\sqrt{1}} = 377 \quad [\Omega] \]

\[ \eta_2 = \frac{377}{\sqrt{6.25}} = 150.8 \quad [\Omega] \]

\[ \eta_3 = \frac{377}{\sqrt{2.25}} = 251.33 \quad [\Omega] \]

\[ \tan (\beta_3 \ell_3) = \tan \left[ \frac{2\pi \times \lambda_3}{\lambda_3} \right] = \tan (2\pi / 3) = 3.08 \]

\[ \tan (\beta_2 \ell_2) = \tan (2\pi / 8) = 0.785. \]

The effective wave impedance $\eta_b$ just to the right of the interface between regions 2 and 3 can be found from Equation (12.160):

\[ \eta_b = \eta_3 + j\eta_4 \tan (\beta_3 \ell_3) \]

\[ = 251.33 + j (251.33 \times 3.08) = 176.94 - j 43.34. \]

Next, the effective wave impedance $\eta_c$ just to the right of the interface between regions 1 and 2 is

\[ \eta_c = \eta_2 + j\eta_3 \tan (\beta_2 \ell_2) \]

\[ = \frac{150.8 (176.94 - j 43.34) + j(150.8)(0.785)}{150.8 + j (176.94 - j 43.34)(0.785)} = 121.66 - j 30.2 \quad [\Omega]. \]

Finally, the effective reflection coefficient is

\[ \Gamma = \frac{\eta_c - \eta_1}{\eta_c + \eta_1} = \frac{121.66 - j 30.2 - 377}{121.66 - j 30.2 + 377} = 0.515 \angle -169.8^\circ. \]

This results in a power reflection coefficient of

\[ |\Gamma|^2 = 0.265 - 26.5\%. \]
Problem # 6.2

A perpendicularly polarized plane wave is incident from free space onto a lossless dielectric surface at an angle of 30° with respect to the surface normal. If the material parameters are \( \varepsilon = 4.0 \varepsilon_0 \) and \( \mu = \mu_0 \), find the angle of transmission and the reflection and transmission coefficients.

**Solution:**

From Equation (12.174), the angle of transmission

\[
\theta_r = \sin^{-1} \left( \frac{\eta_1 \sin \theta_i}{\eta_2} \right) - \sin^{-1} \left( \frac{1}{\sqrt{\frac{\mu_2}{\varepsilon_2}}} \sin (30°) \right) = 14.48°.
\]

The intrinsic impedances of the two media are

\[
\eta_1 = \sqrt{\frac{\mu_1}{\varepsilon_0}} = 377 \quad [\Omega] , \quad \eta_2 = \sqrt{\frac{\mu_2}{4\varepsilon_0}} = 188.5 \quad [\Omega].
\]

Substituting these values into Equations (12.177) and (12.178), we find that

\[
\Gamma_\perp = \frac{188.5 \cos (30°) - 377 \cos (14.48°)}{188.5 \cos (30°) + 377 \cos (14.48°)} = -0.382
\]

\[
T_\perp = \frac{2 \times 188.5 \cos (30°)}{188.5 \cos (30°) + 377 \cos (14.48°)} = 0.618.
\]

**Equation List**

\[
\Gamma_\perp = \frac{\eta_2 \cos \theta_i - \eta_1 \cos \theta_i}{\eta_2 \cos \theta_i + \eta_1 \cos \theta_i} \quad (12.177)
\]

\[
T_\perp = \frac{2 \eta_2 \cos \theta_i}{\eta_2 \cos \theta_i + \eta_1 \cos \theta_i} \quad (12.178)
\]