HOMEWORK#10 SOLUTION

Problem 7.1 The magnetic field of a wave propagating through a certain nonmagnetic material is given by

$$\mathbf{H} = \hat{\mathbf{z}}\, 30 \cos(10^8 t - 0.5y) \quad \text{(mA/m)}$$

Find the following:

(a) The direction of wave propagation.
(b) The phase velocity.
(c) The wavelength in the material.
(d) The relative permittivity of the material.
(e) The electric field phasor.

Solution:
(a) Positive y-direction.
(b) $\omega = 10^8$ rad/s, $k = 0.5$ rad/m.

$$u_p = \frac{\omega}{k} = \frac{10^8}{0.5} = 2 \times 10^8 \text{ m/s}.$$

(c) $\lambda = 2\pi/k = 2\pi/0.5 = 12.6$ m.
(d) $\varepsilon_r = \left(\dfrac{c}{u_p}\right)^2 = \left(\dfrac{3\times10^8}{2\times10^8}\right)^2 = 2.25.$
(e) From Eq. (7.39b),

$$\widetilde{\mathbf{E}} = -\eta \hat{\mathbf{k}} \times \widetilde{\mathbf{H}},$$
$$\eta = \sqrt{\frac{\mu}{\varepsilon}} = \frac{120\pi}{\sqrt{\varepsilon_r}} = \frac{120\pi}{1.5} = 251.33 \quad (\Omega),$$
$$\hat{\mathbf{k}} = \hat{\mathbf{y}}, \quad \text{and} \quad \widetilde{\mathbf{H}} = \hat{\mathbf{z}} 30 e^{-j0.5y} \times 10^{-3} \quad \text{(A/m)}.$$

Hence,

$$\widetilde{\mathbf{E}} = -251.33 \hat{\mathbf{y}} \times \hat{\mathbf{z}} 30 e^{-j0.5y} \times 10^{-3} = -\hat{\mathbf{x}} 7.54 e^{-j0.5y} \quad \text{(V/m)},$$

and

$$\mathbf{E}(y,t) = \mathfrak{Re}(\widetilde{\mathbf{E}} e^{j\omega t}) = -\hat{\mathbf{x}} 7.54 \cos(10^8 t - 0.5y) \quad \text{(V/m)}.$$

Problem 7.2 Write general expressions for the electric and magnetic fields of a 1-GHz sinusoidal plane wave traveling in the $+y$-direction in a lossless nonmagnetic medium with relative permittivity $\varepsilon_r = 9$. The electric field is polarized along the x-direction, its peak value is 6 V/m, and its intensity is 4 V/m at $t = 0$ and $y = 2$ cm.

Solution: For $f = 1$ GHz, $\mu_r = 1$, and $\varepsilon_r = 9$,

$$\omega = 2\pi f = 2\pi \times 10^9 \text{ rad/s},$$
$$k = \frac{2\pi}{\lambda} = \frac{2\pi}{\lambda_0}\sqrt{\varepsilon_r} = \frac{2\pi f}{c}\sqrt{\varepsilon_r} = \frac{2\pi \times 10^9}{3\times 10^8}\sqrt{9} = 20\pi \text{ rad/m},$$
$$\mathbf{E}(y,t) = \hat{\mathbf{x}} 6 \cos(2\pi \times 10^9 t - 20\pi y + \phi_0) \quad \text{(V/m)}.$$

At $t = 0$ and $y = 2$ cm, $E = 4$ V/m:

$$4 = 6\cos(-20\pi \times 2 \times 10^{-2} + \phi_0) = 6\cos(-0.4\pi + \phi_0).$$

Hence,

$$\phi_0 - 0.4\pi = \cos^{-1}\left(\frac{4}{6}\right) = 0.84 \text{ rad},$$

which gives

$$\phi_0 = 2.1 \text{ rad} = 120.19^\circ$$

and

$$\mathbf{E}(y,t) = \hat{\mathbf{x}}\, 6\cos(2\pi \times 10^9 t - 20\pi y + 120.19^\circ) \quad \text{(V/m)}.$$

Problem 7.4 The electric field of a plane wave propagating in a nonmagnetic material is given by

$$\mathbf{E} = [\hat{\mathbf{y}}\, 3\sin(\pi \times 10^7 t - 0.2\pi x) + \hat{\mathbf{z}}\, 4\cos(\pi \times 10^7 t - 0.2\pi x)] \quad \text{(V/m)}$$

Determine

(a) The wavelength.

(b) ε_r.

(c) $\mathbf{H}$.

Solution:

(a) Since $k = 0.2\pi$,

$$\lambda = \frac{2\pi}{k} = \frac{2\pi}{0.2\pi} = 10 \text{ m}.$$

(b)

$$u_p = \frac{\omega}{k} = \frac{\pi \times 10^7}{0.2\pi} = 5 \times 10^7 \text{ m/s}.$$

But

$$u_p = \frac{c}{\sqrt{\varepsilon_r}}.$$

Hence,

$$\varepsilon_r = \left(\frac{c}{u_p}\right)^2 = \left(\frac{3 \times 10^8}{5 \times 10^7}\right)^2 = 36.$$

(c)

$$\begin{aligned}\mathbf{H} = \frac{1}{\eta}\hat{\mathbf{k}} \times \mathbf{E} &= \frac{1}{\eta}\hat{\mathbf{x}} \times [\hat{\mathbf{y}}3\sin(\pi \times 10^7 t - 0.2\pi x) + \hat{\mathbf{z}}4\cos(\pi \times 10^7 t - 0.2\pi x)] \\ &= \hat{\mathbf{z}}\frac{3}{\eta}\sin(\pi \times 10^7 t - 0.2\pi x) - \hat{\mathbf{y}}\frac{4}{\eta}\cos(\pi \times 10^7 t - 0.2\pi x) \quad \text{(A/m)},\end{aligned}$$

with

$$\eta = \frac{\eta_0}{\sqrt{\varepsilon_r}} \simeq \frac{120\pi}{6} = 20\pi = 62.83 \quad (\Omega).$$

Problem 7.6 The electric field of a plane wave propagating in a lossless, nonmagnetic, dielectric material with $\varepsilon_r = 2.56$ is given by

$$\mathbf{E} = \hat{\mathbf{y}}\, 20\cos(6\pi \times 10^9 t - kz) \quad \text{(V/m)}$$

Determine:

(a) f, u_p, λ, k, and η.

(b) The magnetic field $\mathbf{H}$.

Solution:

(a)

$$\begin{aligned}\omega &= 2\pi f = 6\pi \times 10^9 \text{ rad/s}, \\ f &= 3 \times 10^9 \text{ Hz} = 3 \text{ GHz}, \\ u_p &= \frac{c}{\sqrt{\varepsilon_r}} = \frac{3 \times 10^8}{\sqrt{2.56}} = 1.875 \times 10^8 \text{ m/s}, \\ \lambda &= \frac{u_p}{f} = \frac{1.875 \times 10^8}{6 \times 10^9} = 3.12 \text{ cm}, \\ k &= \frac{2\pi}{\lambda} = \frac{2\pi}{3.12 \times 10^{-2}} = 201.4 \text{ rad/m}, \\ \eta &= \frac{\eta_0}{\sqrt{\varepsilon_r}} = \frac{377}{\sqrt{2.56}} = \frac{377}{1.6} = 235.62\ \Omega.\end{aligned}$$

(b)

$$\begin{aligned}\mathbf{H} &= -\hat{\mathbf{x}}\frac{20}{\eta}\cos(6\pi \times 10^9 t - kz) \\ &= -\hat{\mathbf{x}}\frac{20}{235.62}\cos(6\pi \times 10^9 t - 201.4z) \\ &= -\hat{\mathbf{x}}8.49 \times 10^{-2}\cos(6\pi \times 10^9 t - 201.4z) \quad \text{(A/m)}.\end{aligned}$$

Problem 7.8 An RHC-polarized wave with a modulus of 2 (V/m) is traveling in free space in the negative z-direction. Write the expression for the wave's electric field vector, given that the wavelength is 6 cm.

Solution:

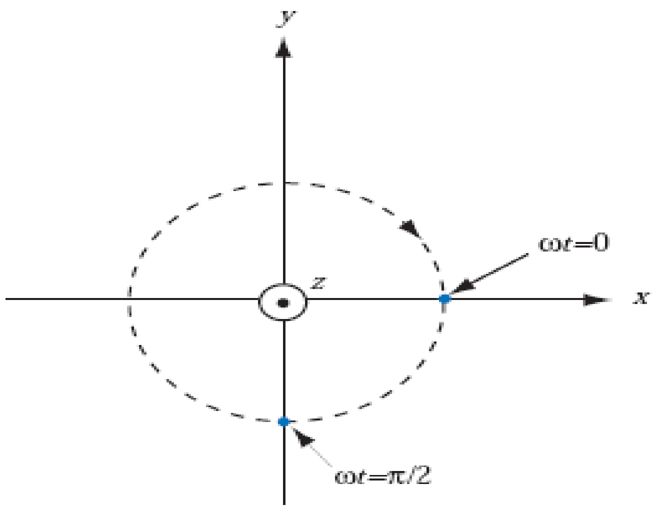

Figure P7.8: Locus of **E** versus time.

For an RHC wave traveling in $-\hat{\mathbf{z}}$, let us try the following:

$$\mathbf{E} = \hat{\mathbf{x}} a \cos(\omega t + kz) + \hat{\mathbf{y}} a \sin(\omega t + kz).$$

Modulus $|E| = \sqrt{a^2 + a^2} = a\sqrt{2} = 2$ (V/m). Hence,

$$a = \frac{2}{\sqrt{2}} = \sqrt{2}.$$

Next, we need to check the sign of the $\hat{\mathbf{y}}$-component relative to that of the $\hat{\mathbf{x}}$-component. We do this by examining the locus of $\mathbf{E}$ versus t at $z = 0$: Since the wave is traveling along $-\hat{\mathbf{z}}$, when the thumb of the right hand is along $-\hat{\mathbf{z}}$ (into the page), the other four fingers point in the direction shown (clockwise as seen from above). Hence, we should reverse the sign of the $\hat{\mathbf{y}}$-component:

$$\mathbf{E} = \hat{\mathbf{x}}\sqrt{2}\cos(\omega t + kz) - \hat{\mathbf{y}}\sqrt{2}\sin(\omega t + kz) \quad \text{(V/m)}$$

with

$$k = \frac{2\pi}{\lambda} = \frac{2\pi}{6 \times 10^{-2}} = 104.72 \quad \text{(rad/m)},$$

and

$$\omega = kc = \frac{2\pi}{\lambda} \times 3 \times 10^8 = \pi \times 10^{10} \quad \text{(rad/s)}.$$

Problem 7.9 For a wave characterized by the electric field

$$\mathbf{E}(z,t) = \hat{\mathbf{x}}\, a_x \cos(\omega t - kz) + \hat{\mathbf{y}}\, a_y \cos(\omega t - kz + \delta)$$

identify the polarization state, determine the polarization angles (γ, χ), and sketch the locus of $\mathbf{E}(0,t)$ for each of the following cases:

(a) $a_x = 3$ V/m, $a_y = 4$ V/m, and $\delta = 0$

(b) $a_x = 3$ V/m, $a_y = 4$ V/m, and $\delta = 180°$

(c) $a_x = 3$ V/m, $a_y = 3$ V/m, and $\delta = 45°$

(d) $a_x = 3$ V/m, $a_y = 4$ V/m, and $\delta = -135°$

Solution:

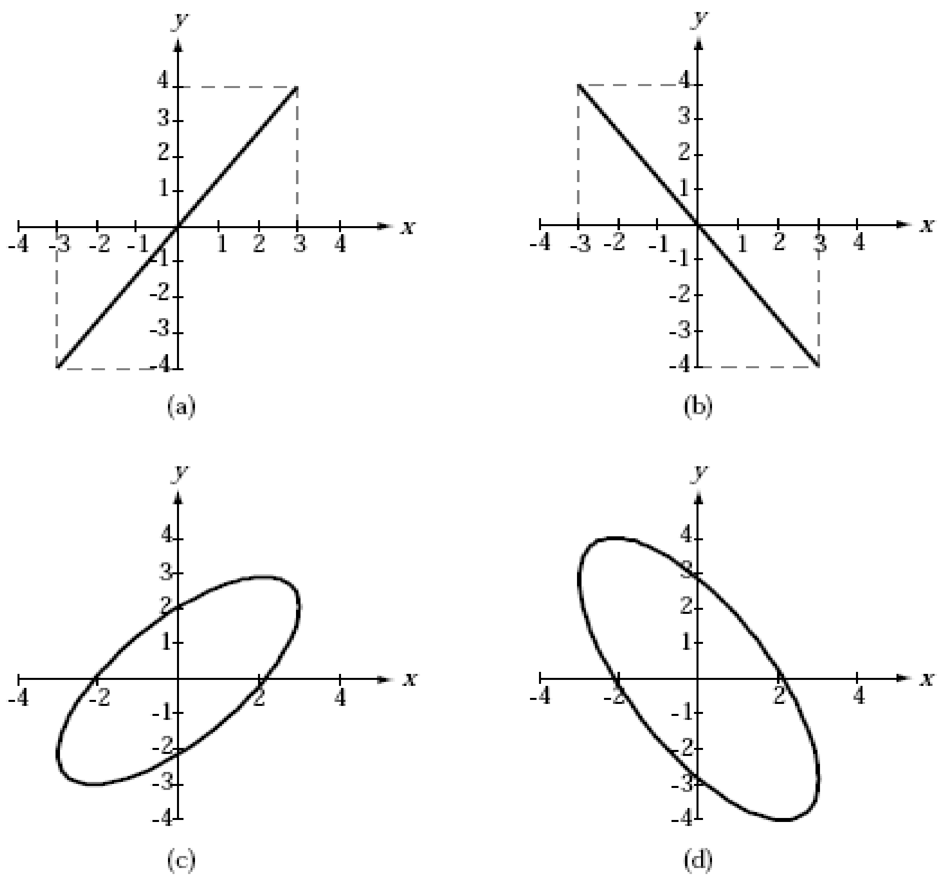

Figure P7.9: Plots of the locus of $\mathbf{E}(0,t)$.

$$\psi_0 = \tan^{-1}(a_y/a_x), \quad \text{[Eq. (7.60)]},$$
$$\tan 2\gamma = (\tan 2\psi_0)\cos\delta \quad \text{[Eq. (7.59a)]},$$
$$\sin 2\chi = (\sin 2\psi_0)\sin\delta \quad \text{[Eq. (7.59b)]}.$$

Case	a_x	a_y	δ	ψ_0	γ	χ	Polarization State
(a)	3	4	0	53.13°	53.13°	0	Linear
(b)	3	4	180°	53.13°	−53.13°	0	Linear
(c)	3	3	45°	45°	45°	22.5°	Left elliptical
(d)	3	4	−135°	53.13°	−56.2°	−21.37°	Right elliptical

(a) $\mathbf{E}(z,t) = \hat{\mathbf{x}} 3\cos(\omega t - kz) + \hat{\mathbf{y}} 4\cos(\omega t - kz)$.
(b) $\mathbf{E}(z,t) = \hat{\mathbf{x}} 3\cos(\omega t - kz) - \hat{\mathbf{y}} 4\cos(\omega t - kz)$.
(c) $\mathbf{E}(z,t) = \hat{\mathbf{x}} 3\cos(\omega t - kz) + \hat{\mathbf{y}} 3\cos(\omega t - kz + 45°)$.
(d) $\mathbf{E}(z,t) = \hat{\mathbf{x}} 3\cos(\omega t - kz) + \hat{\mathbf{y}} 4\cos(\omega t - kz - 135°)$.

Problem 7.12 The electric field of an elliptically polarized plane wave is given by

$$\mathbf{E}(z,t) = [-\hat{\mathbf{x}}\,10\sin(\omega t - kz - 60^\circ) \\ + \hat{\mathbf{y}}\,30\cos(\omega t - kz)] \quad \text{(V/m)}$$

Determine the following:

(a) The polarization angles (γ, χ).

(b) The direction of rotation.

Solution:

(a)

$$\begin{aligned}\mathbf{E}(z,t) &= [-\hat{\mathbf{x}}10\sin(\omega t - kz - 60^\circ) + \hat{\mathbf{y}}30\cos(\omega t - kz)] \\ &= [\hat{\mathbf{x}}10\cos(\omega t - kz + 30^\circ) + \hat{\mathbf{y}}30\cos(\omega t - kz)] \quad \text{(V/m)}.\end{aligned}$$

Phasor form:

$$\widetilde{\mathbf{E}} = (\hat{\mathbf{x}}10e^{j30^\circ} + \hat{\mathbf{y}}30)e^{-jkz}.$$

Since δ is defined as the phase of E_y relative to that of E_x,

$$\begin{aligned}\delta &= -30^\circ, \\ \psi_0 &= \tan^{-1}\left(\frac{30}{10}\right) = 71.56^\circ, \\ \tan 2\gamma &= (\tan 2\psi_0)\cos\delta = -0.65 \quad \text{or} \quad \gamma = 73.5^\circ, \\ \sin 2\chi &= (\sin 2\psi_0)\sin\delta = -0.40 \quad \text{or} \quad \chi = -8.73^\circ.\end{aligned}$$

(b) Since $\chi < 0$, the wave is right-hand elliptically polarized.

Problem # 5.0

Find the expression for the plane wave that propagates parallel to the xy-plane in the direction indicated in Figure 12-3. Assume that $\mathbf{E}$ has only a z-component.

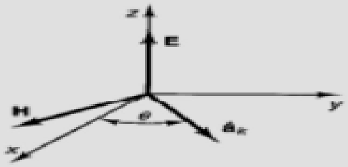

Figure 12-3 A plane wave propagating parallel to the xy-plane at an angle θ with respect to the x-axis.

Solution:

Since this wave propagates at an angle θ with respect to the x-axis, we can express the wave-number vector $\mathbf{k}$ as

$$\mathbf{k} = k(\cos\theta\,\hat{\mathbf{a}}_x + \sin\theta\,\hat{\mathbf{a}}_y).$$

As a result,

$$\mathbf{k}\cdot\mathbf{r} = k(x\cos\theta + y\sin\theta).$$

Since $\mathbf{E}$ has only a z-component, we also can write

$$\mathbf{E}_o = E_o\hat{\mathbf{a}}_z.$$

Substituting these expressions into Equation (12.35), we obtain

$$\mathbf{E} = E_o\hat{\mathbf{a}}_z e^{-j\beta(x\cos\theta + y\sin\theta)}.$$

To find $\mathbf{H}$, we first evaluate $\mathbf{k}\times\mathbf{E}_o$:

$$\mathbf{k}\times\mathbf{E}_o = k(\cos\theta\,\hat{\mathbf{a}}_x + \sin\theta\,\hat{\mathbf{a}}_y)\times E_o\hat{\mathbf{a}}_z$$

$$= kE_o(\sin\theta\,\hat{\mathbf{a}}_x - \cos\theta\,\hat{\mathbf{a}}_y).$$

Finally, using $(\omega\mu)/k = \eta$, we have, from Equation (12.36),

$$\mathbf{H} = \frac{E_o}{\eta}(\sin\theta\,\hat{\mathbf{a}}_x - \cos\theta\,\hat{\mathbf{a}}_y)e^{-j\beta(x\cos\theta + y\sin\theta)}.$$

Problem # 5.1

Find the polarization ellipse for a plane wave described by

$$\mathbf{E} = 4\cos(\omega t - \beta z)\hat{\mathbf{a}}_x + 2\cos(\omega t + 30° - \beta z)\hat{\mathbf{a}}_y.$$

Solution:

For this wave, we have $E_{xo} = 4.0$, $E_{yo} = 2.0$, and $\Delta\theta = 30°$. Using Equations (12.51) and (12.52), we find that

$$OA = \left[\frac{1}{2}\{4^2 + 2^2 + [4^4 + 2^4 + 2\times 4^2\times 2^2\cos(60°)]^{1/2}\}\right]^{1/2} = 4.38$$

$$OB = \left[\frac{1}{2}\{4^2 + 2^2 - [4^4 + 2^4 + 2\times 4^2\times 2^2\cos(60°)]^{1/2}\}\right]^{1/2} = 0.914.$$

From Equation (12.50), the axial ratio is

$$AR = \frac{OA}{OB} = \frac{4.38}{0.914} = 4.79.$$

Finally, using Equation (12.53), we see that the tilt angle is

$$\tau = \frac{1}{2}\tan^{-1}\left[\frac{2\times 4\times 2}{4^2 - 2^2}\cos(30°)\right] = 24.55°.$$

Problem # 5.2

A linearly polarized plane wave propagates through free space at an angle θ with respect to the $z = 0$ plane, as shown in Figure 12-11. If the peak amplitude of $\mathbf{E}$ is 10 [mV/m], calculate the average power that passes through the 1 [m²] surface shown in the figure.

Solution:

Using Equation (12.107), we can represent the average Poynting vector as

$$\mathcal{P}_{ave} = \frac{1}{2}\frac{|10\times 10^{-3}|^2}{377}\hat{\mathbf{a}}_k = 132.6\,\hat{\mathbf{a}}_k \qquad [\text{nW/m}^2].$$

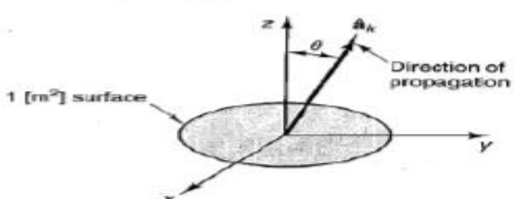

Figure 12-11 A plane wave propagating at an angle θ through a 1 [m²] surface.

where $\hat{\mathbf{a}}_k$ points in the direction of propagation.

The average power that passes through the surface is

$$P_{ave} = \int_s \mathcal{P}_{ave}\cdot d\mathbf{s} = \int_S \mathcal{P}_{ave}\cdot d\mathbf{s} = 132.6\times\int_S \hat{\mathbf{a}}_k\cdot\hat{\mathbf{a}}_z\,dz.$$

From Figure 12-11, we see that $\hat{\mathbf{a}}_k\cdot\hat{\mathbf{a}}_z = \cos\theta$. Since the total surface area is 1 [m²], we finally obtain

$$P_{ave} = 132.6\cos\theta \qquad [\text{nW}].$$

Thus, the power that passes through the surface is maximized when the direction of propagation is parallel to the surface normal.

Equations List:-

$$AR = \frac{\text{major axis}}{\text{minor axis}} = \frac{OA}{OB} \qquad 1 \leq AR \leq \infty, \tag{12.50}$$

$$OA = \left[\frac{1}{2}\left\{|E_{xo}|^2 + |E_{yo}|^2 + [|E_{xo}|^4 + |E_{yo}|^4 + 2|E_{xo}|^2|E_{yo}|^2\cos(2\Delta\theta)]^{1/2}\right\}\right]^{1/2}, \tag{12.51}$$

$$OB = \left[\frac{1}{2}\left\{|E_{xo}|^2 + |E_{yo}|^2 - [|E_{xo}|^4 + |E_{yo}|^4 + 2|E_{xo}|^2|E_{yo}|^2\cos(2\Delta\theta)]^{1/2}\right\}\right]^{1/2}. \tag{12.52}$$

$$\tau = \frac{1}{2}\tan^{-1}\left[\frac{2|E_{xo}||E_{yo}|}{|E_{xo}|^2 - |E_{yo}|^2}\cos(\Delta\theta)\right]. \tag{12.53}$$

$$\mathscr{S}_{\text{ave}} = \frac{1}{2}\frac{|E_o|^2}{\eta}\hat{\mathbf{a}}_k \quad [\text{W/m}^2] \quad \text{(Linearly polarized plane waves in lossless medium)}, \tag{12.107}$$

$$\mathbf{E} = \mathbf{E}_o e^{-j\mathbf{k}\cdot\mathbf{r}}, \tag{12.35}$$

$$\mathbf{H} = \frac{1}{\omega\mu}(\mathbf{k} \times \mathbf{E}_o)e^{-j\mathbf{k}\cdot\mathbf{r}}, \tag{12.36}$$

$$\mathbf{E}_o \cdot \mathbf{k} = \mathbf{E}_o \cdot k\hat{\mathbf{a}}_k = 0. \tag{12.37}$$