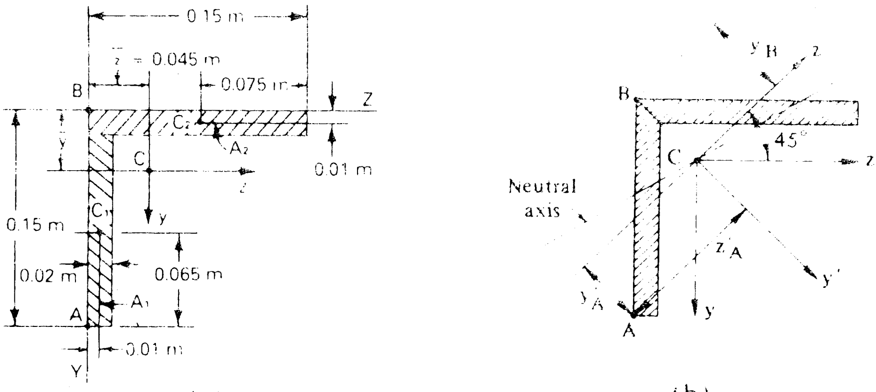
**Example 5.1.** A 0.15 by 0.15 m slender angle of 0.02 m thickness is subjected to oppositely directed end couples $M_z = 11{,}000\ \mathrm{N \cdot m}$, at the centroid of the cross section. What bending stresses exist at points $A$ and $B$ on a section away from the ends (Fig. 5.4a)? Determine the orientation of the neutral axis.

SOLUTION. Equations (5.13) and (5.15) will be applied to ascertain the normal stress. This requires first the determination of a number of section properties, through the use of familiar expressions of mechanics given in Appendix C. Note that the FORTRAN computer program presented in Table C.2 provides a check of the numerical values obtained here for the area characteristics, and may easily be extended to compute the stresses.

*Location of the centroid C*: Let $\bar{y}$ and $\bar{z}$ represent the distances from $C$ to arbitrary reference lines (denoted $Z$ and $Y$):

$$\bar{z} = \frac{\sum A_i \bar{z}_i}{\sum A_i} = \frac{A_1\bar{z}_1 + A_2\bar{z}_2}{A_1 + A_2} = \frac{0.13 \times 0.02 \times 0.01 + 0.15 \times 0.02 \times 0.075}{0.13 \times 0.02 + 0.15 \times 0.02} = 0.045\ \mathrm{m}$$

Here $\bar{z}_i$ represents the $z$ distance from the $Y$ reference line to the centroid of each




(a) (b)

**Figure 5.4**

subarea composing the total cross section. Since the section is symmetrical, $\bar{z} = \bar{y}$.

*Moments and products of inertia*: For a rectangular section of depth $h$ and width $b$, the moment of inertia about the neutral $z$ axis is $I_z = bh^3/12$ (Table C.1). Referring to Fig. 5.4a, and applying the parallel-axis theorem, we obtain

$$
\begin{aligned}
I_z = I_y &= \sum (I_{\bar{z}} + A\bar{y}^2) \\
&= \tfrac{1}{12} \times 0.02 \times (0.13)^3 + 0.13 \times 0.02 \times (0.04)^2 \\
&\quad + \tfrac{1}{12} \times 0.15 \times (0.02)^3 + 0.15 \times 0.02 \times (0.035)^2 \\
&= 11.596 \times 10^{-6}\ \mathrm{m}^4.
\end{aligned}
$$

The transfer formula (C.11) for a product of inertia yields

$$
\begin{aligned}
I_{yz} &= \sum (I_{\bar{y}\bar{z}} + A\bar{y}\bar{z}) \\
&= 0 + 0.13 \times 0.02 \times 0.04 \times (-0.035) + 0 + 0.15 \times 0.02 \times (-0.035) \times 0.03 \\
&= -6.79 \times 10^{-6}\ \mathrm{m}^4
\end{aligned}
$$

*Stresses using formula* (5.13): We have $y_A = 0.105$ m, $y_B = -0.045$ m, $z_A = -0.045$ m, $z_B = -0.045$ m, and $M_y = 0$. Thus

$$
\begin{aligned}
(\sigma_x)_A &= \frac{M_z(I_{yz}z_A - I_y y_A)}{I_y I_z - I_{yz}^2} \qquad \text{(h)} \\
&= \frac{11000[(-6.79)(-0.045) - (11.596)(0.105)]}{[(11.596)^2 - (-6.79)^2]10^{-6}} = -114\ \mathrm{MPa}
\end{aligned}
$$

Similarly,

$$
(\sigma_x)_B = \frac{11000[(-6.79)(-0.045) - (11.596)(-0.045)]}{[(11.596)^2 - (-6.79)^2]10^{-6}} = 103\ \mathrm{MPa}
$$

Alternatively, these stresses may be calculated by proceeding as follows.

*Direction of the principal axes and the principal moments of inertia.* Employing Eq. (5.17) we have

$$\tan 2\theta_p = \frac{-2(-6.79)}{11.596 - 11.596} = \infty, \qquad 2\theta_p = 90^\circ \text{ or } 270^\circ$$

Therefore the two values of $\theta_p$ are 45° and 135°. Substituting the first of these values into Eq. (5.16), we obtain $I_{y'} = [11.596 + 6.79 \sin 90^\circ]10^{-6} = 18.386 \times 10^{-6}\ \mathrm{m}^4$. Since the principal moments of inertia are, by application of Eq. (5.18),

$$I_{1,2} = [11.596 \pm \sqrt{0 + 6.79^2}]10^{-6} = [11.596 \pm 6.79]10^{-6}$$

it is observed that $I_1 = I_{y'} = 18.386 \times 10^{-6}\ \mathrm{m}^4$ and $I_2 = I_{z'} = 4.806 \times 10^{-6}\ \mathrm{m}^4$. The principal axes are indicated in Fig. 5.4b as the $y'z'$ axes.

*Stresses using formula* (5.15): The components of bending moment about the principal axes are

$$M_{y'} = 11{,}000 \sin 45^\circ = 7778\ \mathrm{N \cdot m}$$

$$M_{z'} = 11{,}000 \cos 45^\circ = 7778\ \mathrm{N \cdot m}$$

Equation (5.15) is now applied, referring to Fig. 5.4b, with $y'_A = 0.043\ \mathrm{m}$, $z'_A = -0.106\ \mathrm{m}$, $y'_B = -0.0636\ \mathrm{m}$, and $z'_B = 0$, determined from geometrical considerations:

$$(\sigma_x)_A = \frac{7778(-0.106)}{18.386 \times 10^{-6}} - \frac{7778(0.043)}{4.806 \times 10^{-6}} = -114\ \mathrm{MPa}$$

$$(\sigma_x)_B = 0 - \frac{7778(-0.0636)}{4.806 \times 10^{-6}} = 103\ \mathrm{MPa}$$

as before.

*Direction of the neutral axis*: From Eq. (5.14), with $M_y = 0$,

$$(M_z I_y)y - (M_z I_{yz})z = 0, \qquad \text{from which} \qquad 11.596y + 6.79z = 0$$

or

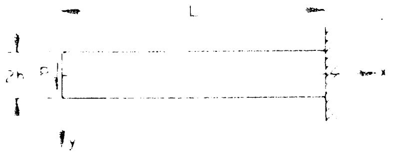
$$z = -1.71y$$

This result is plotted in Fig. 5.4b.

## 5.4 Bending of a Cantilever of Narrow Section

Consider a narrow cantilever beam of rectangular cross section, loaded at its free end by a concentrated force of such magnitude that the beam weight may be neglected (Fig. 5.5). The situation described may be

Figure 5.5



However, we can show for the small deformation that this is an *exact solution* according to the theory of elasticity. For the case of more general bending including loads *normal* to the center line we can use the generalized flexure formula *locally* for long slender beams to get very good approximate results. We shall now illustrate the use of the generalized flexure formula.

**Example 11.15**

What is the maximum tensile stress $\tau_{xx}$ in the cantilever beam shown in Fig. 11.39? The line of action of the 100-lb force goes through the centroid of the cross section.

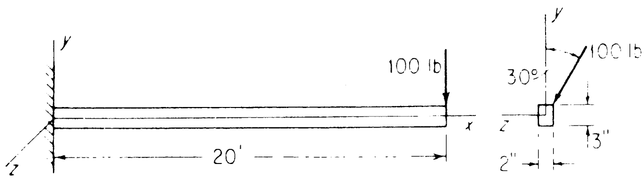



**Figure 11.39** Cantilever with unsymmetric loading.

Clearly, the maximum bending moment occurs at the base of the cantilever beam, and we show a free-body diagram in Fig. 11.40 exposing this section. Note that *positive* bending moments *conventionwise* for $M_z$ and $M_y$ have been shown in the diagram. Clearly, the algebraic sign from equilibrium equations using the right-hand rule will then correspond to the convention sign for the bending moments. Hence, from equilibrium we have

$\underline{\sum M_y = 0:}$

$$-M_y - (100)(\sin 30^\circ)(20) = 0$$

$$\therefore M_y = -1000 \text{ ft-lb}$$

$\underline{\sum M_z = 0:}$

$$-M_z - (100)(\cos 30^\circ)(20) = 0$$

$$\therefore M_z = -1732 \text{ ft-lb}$$

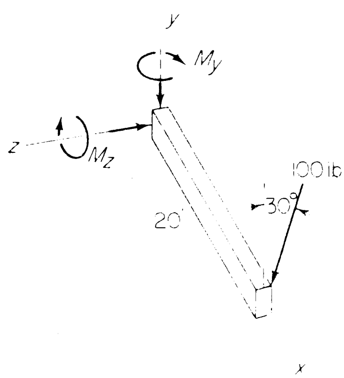



**Figure 11.40** Free-body diagram of cantilever.

We can employ Eq. (11.41) here as follows:

$$\tau_{xx} = \frac{[(-1000)(12)(\frac{1}{12})(2)(3^3) + (-1732)(12)(0)]z - [(-1732)(12)(\frac{1}{12})(3)(2^3) + (-1000)(12)(0)]y}{(\frac{1}{12})(2)(3^3)(\frac{1}{12})(3)(2^3) - (0)^2} \tag{a}$$

(Since the axes $yz$ are principal axes, we could also have used Eq. (11.37a) directly.) Carrying out the numerical work, we get

$$\tau_{xx} = -(6000z - 4619y) \text{ psi} \tag{b}$$

The maximum tensile stress $\tau_{xx}$ occurs when $z = -1$ and $y = 1.5$, giving

$$(\tau_{xx})_{max} = 6000 + 6928 = 12{,}928 \text{ psi} \tag{c}$$

We have solved this problem in a very formal manner being very careful of the signs for the bending moments. In complex problems you will have to proceed in this careful deliberate manner and that is why we solved this problem this way. However, when the problem is as easy as this, we encourage you to proceed in a simple direct manner using common sense and "physical feel."

Thus for the vertical component of the force, namely (100) cos 30°, you should be able to visualize that we will get a maximum tensile stress at the *top* fibers $AB$ of the base cross section (see Fig. 11.41). Calling this stress $(\tau_{xx})_1$, we can say

$$|(\tau_{xx})_1| = \frac{[(100)(\cos 30°)(20)(12)](1.5)}{(\frac{1}{12})(2)(3^3)} = 6928 \text{ psi}$$

As for the horizontal component of the 100-lb force, namely (100) sin 30°, we will get a maximum tensile stress along side $BC$ (see Fig. 11.41), as you should again be able to visualize. Calling this stress $(\tau_{xx})_2$, we have

$$|(\tau_{xx})_2| = \frac{[(100)(\sin 30°)(20)(12)](1)}{(\frac{1}{12})(3)(2^3)} = 6000 \text{ psi}$$

The total maximum tensile stress is then 12,928 psi at corner $B$.

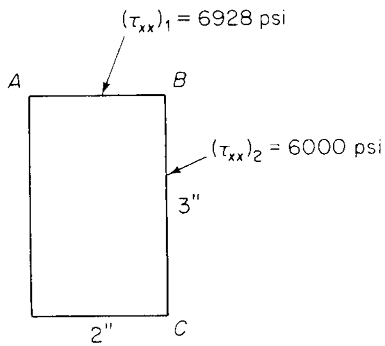



**Figure 11.41** Maximum tensile stress $\tau_{xx}$ is at $B$.

Now that we have formulated the generalized flexure formula for computing the normal stress $\tau_{xx}$, the next logical step would be to present a similar generalization for