1. Solution:

a) $\epsilon_x = \frac{\partial u}{\partial x} = c$, $\epsilon_y = \frac{\partial v}{\partial y} = -c$, $\epsilon_z = \frac{\partial w}{\partial z} = -c$

$$\gamma_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} = 2c - 2c = 0$$

$$\gamma_{xz} = \frac{\partial u}{\partial z} + \frac{\partial w}{\partial x} = 0$$

$$\gamma_{yz} = \frac{\partial w}{\partial y} + \frac{\partial v}{\partial z} = 0$$

Answer: $\epsilon_x = c$, $\gamma_{xy} = 0$

$\epsilon_y = -c$, $\gamma_{xz} = 0$

$\epsilon_z = -c$, $\gamma_{yz} = 0$

b) $\epsilon_x = \frac{\partial u}{\partial x} = \frac{2c}{1+\frac{x}{L}}$, $\epsilon_y = \frac{\partial v}{\partial y} = 0$

$$\epsilon_z = \frac{\partial w}{\partial z} = 0$$

$$\gamma_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} = -c\exp\left(-\frac{x}{L}\right)$$

$$\gamma_{xz} = \frac{\partial u}{\partial z} + \frac{\partial w}{\partial x} = 0$$

$$\gamma_{yz} = \frac{\partial v}{\partial z} + \frac{\partial w}{\partial y} = 0$$

Answer: $\epsilon_x = \frac{2c}{1+\frac{x}{L}}$, $\gamma_{xy} = -c\exp\left(-\frac{x}{L}\right)$

$\epsilon_y = 0$ $\gamma_{xz} = 0$

$\epsilon_z = 0$ $\gamma_{yz} = 0$

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2. SOLUTION:

a) Method 1, using compatibility eq'ns (suggested)

$$\frac{\partial^2 \epsilon_x}{\partial y^2} + \frac{\partial^2 \epsilon_y}{\partial x^2} - \frac{\partial^2 \gamma_{xy}}{\partial x \partial y} = 0$$

$$\Rightarrow \frac{\partial}{\partial y}\left(2\frac{Ay}{L^2} + \frac{Cx}{L^2}\right) + \frac{\partial}{\partial x}\left(2A\frac{x}{L^2}\right) - \frac{\partial}{\partial y}\left(\frac{Cy}{L^2}\right) = 0$$

$$\Rightarrow \frac{2A}{L^2} + \frac{2A}{L^2} - \frac{C}{L^2} = 0 \Rightarrow \underline{4A = C}$$

Method 2:

$$\epsilon_x = \frac{\partial u}{\partial x} \Rightarrow u = \frac{Ay^2}{L^2}x + \frac{Cx^2y}{2L^2} + f_1(y)$$

$$\epsilon_y = \frac{\partial v}{\partial y} \Rightarrow v = \frac{A}{L^2}\left(x^2y + \frac{y^3}{3}\right) + f_2(x)$$

$$\gamma_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} = \frac{2Axy}{L^2} + \frac{Cx^2}{2L^2} + \frac{df_1}{dy} + \frac{2Axy}{L^2} + \frac{df_2}{dx}$$

$$= \frac{Cxy}{L^2}$$

$$\Rightarrow \frac{\partial^2}{\partial x \partial y}\left(\frac{4Axy}{L^2} + \frac{Cx^2}{2L^2} + \frac{df_1}{dy} + \frac{df_2}{dx}\right) = \frac{\partial^2}{\partial x \partial y}\left(\frac{Cxy}{L^2}\right)$$

$$\Rightarrow \underline{4A = C}$$

b) Method 1, using compatibility equations (suggested).

$$\frac{\partial^2 \epsilon_x}{\partial y^2} + \frac{\partial^2 \epsilon_y}{\partial x^2} - \frac{\partial^2 \gamma_{xy}}{\partial x \partial y} = 0$$

$$\frac{\partial}{\partial y}\left[3A\cos\left(\frac{\pi x}{L}\right)\left(\frac{2\pi}{L}\right)\cos\left(\frac{2\pi y}{L}\right)\right] + \frac{\partial}{\partial x}\left[-C\frac{\pi}{L}\sin\left(\frac{\pi x}{L}\right)\sin\left(\frac{2\pi y}{L}\right)\right]$$

$$-\frac{\partial}{\partial y}\left[2A\frac{\pi}{L}\cos\left(\frac{\pi x}{L}\right)\cos\left(\frac{2\pi y}{L}\right)\right] = 0$$

$$\Rightarrow -\frac{6A\pi}{L}\cos\left(\frac{\pi x}{L}\right)\left(\frac{2\pi}{L}\right)\sin\left(\frac{2\pi y}{L}\right) - \frac{C\pi}{L}\left(\frac{\pi}{L}\right)\cos\left(\frac{\pi x}{L}\right)\sin\left(\frac{2\pi y}{L}\right)$$

$$+\frac{2A\pi}{L}\left(\frac{2\pi}{L}\right)\cos\left(\frac{\pi x}{L}\right)\sin\left(\frac{2\pi y}{L}\right) = 0$$

$$\Rightarrow \frac{-12A\pi^2}{L^2}\cos\left(\frac{\pi x}{L}\right)\sin\left(\frac{2\pi y}{L}\right) - \frac{C\pi^2}{L^2}\cos\left(\frac{\pi x}{L}\right)\sin\left(\frac{2\pi y}{L}\right)$$

$$+\frac{4A\pi^2}{L^2}\cos\left(\frac{\pi x}{L}\right)\sin\left(\frac{2\pi y}{L}\right) = 0$$

$$\Rightarrow (-8A - C)\frac{\pi^2}{L^2}\cos\left(\frac{\pi x}{L}\right)\sin\left(\frac{2\pi y}{L}\right) = 0$$

$$\therefore \quad \underline{8A = -C}$$

Method 2:

$$\epsilon_x = \frac{\partial u}{\partial x} \Rightarrow u = \frac{3AL}{\pi}\sin\left(\frac{\pi x}{L}\right)\sin\left(\frac{2\pi y}{L}\right) + f_1(y)$$

$$\epsilon_y = \frac{\partial v}{\partial y} \Rightarrow v = -\frac{CL}{2\pi}\cos\left(\frac{\pi x}{L}\right)\cos\left(\frac{2\pi y}{L}\right) + f_2(x)$$

$$\therefore \gamma_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} = 6A\sin\left(\frac{\pi x}{L}\right)\cos\left(\frac{2\pi y}{L}\right) + \frac{df_1}{dy}$$

$$+\frac{C}{2}\cos\left(\frac{2\pi y}{L}\right)\sin\left(\frac{\pi x}{L}\right) + \frac{df_2}{dx}$$

$$= 2A\sin\left(\frac{\pi x}{L}\right)\cos\left(\frac{2\pi y}{L}\right)$$

$$\frac{\partial^2}{\partial x \partial y}\left[\left(6A+\frac{C}{2}\right)\sin\left(\frac{\pi x}{L}\right)\cos\left(\frac{2\pi y}{L}\right) + \frac{df_1}{dy} + \frac{df_2}{dx} \right]$$

$$= \frac{\partial^2}{\partial x \partial y}\left[2A\sin\left(\frac{\pi x}{L}\right)\cos\left(\frac{2\pi y}{L}\right) \right]$$

$$\Rightarrow -\frac{2\pi^2}{L^2}\left(6A+\frac{C}{2}\right)\cos\left(\frac{\pi x}{L}\right)\sin\left(\frac{2\pi y}{L}\right)$$

$$= -\frac{2\pi^2}{L^2}\cdot 2A \cdot \cos\left(\frac{\pi x}{L}\right)\sin\left(\frac{2\pi y}{L}\right)$$

$$\Rightarrow \quad 6A + \frac{C}{2} = 2A \quad \Rightarrow \quad \underline{8A = -C}$$

c) Method 1 (Suggested).

Using compatibility eq'ns : $\frac{\partial^2 \epsilon_x}{\partial y^2} + \frac{\partial^2 \epsilon_y}{\partial x^2} - \frac{\partial^2 \gamma_{xy}}{\partial x \partial y} = 0$

$$\frac{\partial \epsilon_x}{\partial y} = \frac{-\frac{\pi A}{L}\sin\left(\frac{\pi y}{L}\right)}{\left(1+\frac{\pi x}{L}\right)^2}, \quad \boxed{\frac{\partial^2 \epsilon_x}{\partial y} = \frac{-\frac{\pi^2 A}{L^2}\cos\left(\frac{\pi y}{L}\right)}{\left(1+\frac{\pi x}{L}\right)^2}}$$

$$\frac{\partial \epsilon_y}{\partial x} = -B\cos\left(\frac{\pi y}{L}\right)\frac{\pi}{L}e^{-\frac{\pi x}{L}}, \quad \boxed{\frac{\partial^2 \epsilon_y}{\partial x} = B\frac{\pi^2}{L^2}\cos\left(\frac{\pi y}{L}\right)e^{-\frac{\pi x}{L}}}$$

$$\frac{\partial \gamma_{xy}}{\partial x} = -\frac{A\pi}{L}e^{-\frac{\pi x}{L}}\sin\left(\frac{\pi y}{L}\right) - \frac{\sin\left(\frac{\pi y}{L}\right)\cdot C\frac{\pi}{L}}{\left(1+\frac{\pi x}{L}\right)^2}$$

$$\boxed{\frac{\partial^2 \gamma_{xy}}{\partial x \partial y} = \frac{-A\pi^2}{L^2}e^{-\frac{\pi x}{L}}\cos\left(\frac{\pi y}{L}\right) - \frac{C\frac{\pi^2}{L^2}\cos\left(\frac{\pi y}{L}\right)}{\left(1+\frac{\pi x}{L}\right)^2}}$$

$$\therefore \frac{-\frac{\pi^2 A}{L^2}\cos\left(\frac{\pi y}{L}\right)}{\left(1+\frac{\pi x}{L}\right)^2} + \frac{B\pi^2}{L^2}\cos\left(\frac{\pi y}{L}\right)e^{-\frac{\pi x}{L}} + \frac{A\pi^2}{L^2}e^{-\frac{\pi x}{L}}\cos\left(\frac{\pi y}{L}\right)$$

$$+ \frac{\frac{C\pi^2}{L^2}\cos(\pi y/L)}{(1+\pi x/L)^2} = 0$$

Upon inspection, we have $\underline{A = C = -B}$

Method 2:

$$\epsilon_x = \frac{\partial u}{\partial x} \Rightarrow u = -\frac{AL \cdot \cos(\frac{\pi y}{L})}{\pi\ (1+\frac{\pi x}{L})} + f_1(y)$$

$$\epsilon_y = \frac{\partial v}{\partial y} \Rightarrow v = \frac{BL}{\pi} \cdot \sin(\frac{\pi y}{L}) \cdot \exp(-\frac{\pi x}{L}) + f_2(x)$$

$$\gamma_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} = \frac{A \cdot \sin(\frac{\pi y}{L})}{1+\frac{\pi x}{L}} - B\sin(\frac{\pi y}{L})\exp(-\frac{\pi x}{L})$$

$$+ f_1'(y) + f_2'(x)$$

$$= A\exp(-\frac{\pi x}{L})\sin(\frac{\pi y}{L}) + \frac{C}{1+\frac{\pi x}{L}}\sin(\frac{\pi y}{L})$$

$$\Rightarrow \frac{\partial^2}{\partial x \partial y}\left[\frac{A \cdot \sin(\frac{\pi y}{L})}{1+\frac{\pi x}{L}} - B\sin(\frac{\pi y}{L})\exp(-\frac{\pi x}{L}) + f_1'(y) + f_2'(x)\right]$$

$$= \frac{\partial^2}{\partial x \partial y}\left[A\exp(-\frac{\pi x}{L})\sin(\frac{\pi y}{L}) + \frac{C}{1+\frac{\pi x}{L}}\sin(\frac{\pi y}{L})\right]$$

$$\Rightarrow \frac{-\frac{\pi^2 A}{L^2}\cos(\frac{\pi y}{L})}{(1+\frac{\pi x}{L})^2} + \frac{B\pi^2}{L^2}\cos(\frac{\pi y}{L})e^{-\frac{\pi x}{L}}$$

$$= -\frac{A\pi^2}{L^2}e^{-\frac{\pi x}{L}}\cos(\frac{\pi y}{L}) - \frac{\frac{C\pi^2}{L^2}\cos(\frac{\pi y}{L})}{(1+\frac{\pi x}{L})^2}$$

$$\Rightarrow \underline{A = -B = C}$$

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