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Geometric Motion Planning and Formation Optimization for a Fleet of Nonholonomic Wheeled Mobile Robots

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Abstract – **In this paper, we present a geometric method for motion planning for a fleet of differentially-driven wheeled mobile robots moving in formation, that explicitly takes in to account their nonholonomic constraints. The relative position within the formation induces different motion plans each individual wheeled mobile robot (WMR). We can quantitatively evaluate the *performance* of such induced motion plans using suitable metrics defined for the motions of each WMR. These performance metrics in cumulative form or individual form are used to optimize the overall formation (i.e. their relative positions) for performing a given task. The approach is well suited for online implementation and is demonstrated using case studies.**

Keywords- Team-based multi-agent motion planning, nonholonomy, Performance evaluation, Formation optimization.

I. INTRODUCTION

In recent years, there has been considerable interest in teaming and cooperation of multiple mobile robotic agents, targeting applications in surveillance, exploration and cooperative manipulation. Interest in such cooperating systems arises when the tasks may be inherently too complex for a single system to accomplish; or when building and using several simple systems can be more flexible, fault-tolerant or cheaper than using a single large system.

Several approaches to coordination of such fleet of robotic-agents are reported in the literature – see [1]-[3] for surveys. "Behavior"-based control adherents address the complexity of multi-agent coordination by decomposing the high-level goals into primitive tasks and implementing simple controllers for them, with the notion of an emergent intelligence as the sets of interacting behaviors grow. The benefits of such behavior based control include ease of decentralization, limited communication needs and surprisingly good performance using relatively simple behaviors [4],[5]. However, a systematic method for synthesizing desired emergent behavior and analyzing its performance/robustness has proven difficult.

In parallel, the "formation" paradigm has also emerged as a convenient mechanism for abstraction and coordination with approaches ranging from leader-following [6],[7], virtual structures [8],[9] and virtual leaders [10],[11]. The group control problem now reduces to a well-known single-agent control problem from which the other agents derive their control laws. However, this approach requires communication of some coordination information. The formation paradigm has evolved to allow prescription of parameterized formation maneuvers [11],[12] and group feedback [11]-[14]. From these seemingly disparate approaches, a dynamic system-theoretic perspective has emerged for examining the decentralized multi-agent "behavioral control" of "formations" [10]-[16] and our work is set in this context.

In this paper, we develop the formation and individual agent motion planning algorithms for a system of differentially-driven mobile robots moving in the plane. Many formation planning approaches treat the mobile agents simply as point objects with unconstrained motions in $\mathbb{R}^2$. Nonholonomic constraints, if considered, are tackled by defining a suitable end-effector point for the wheeled base and applying nonlinear input-output feedback linearization techniques. The planning problem is converted back into one of planning the unconstrained Cartesian motions of this end-effector point in $\mathbb{R}^2$, with guaranteed-stable zero dynamics [17]. Motion plans are first created for the end effector positions of the multiple robots within the formation and in a second stage translated into motion plans for each individual robots with respect to its end effector.

In contrast, we focus on developing a kinematic motion planning problem for rigid formations of differentially driven mobile robots, while retaining their full nonholonomic form. While [18] presents motion planning approach for formations of mobile robots with nonholonomic constraints, we also optimize the formations for specific motion paths. In particular, we consider the differentially driven mobile robots to be attached to the virtual structure at their center of axle (a point that traditionally has been avoided because of the singularity induced in the input-output feedback linearization). For any given (planar) path, we first determine the instantaneous motion of the Serret-Frenet frame and the corresponding pole of the motion. Aligning a "team-fixed frame" with this Serret-Frenet frame now induces a helicoidal velocity vector field for the vertices of the virtual structure. Our planning algorithm for the individual mobile robots then aligns the forward direction of travel with this helicoidal velocity vector field.

The relative position of each mobile robot within the virtual structure induces different motion plans for the individual mobile robots. By using a suitable metric, defined on the motions of each mobile robot, we can evaluate the performance of such induced motion plans. These performance metrics in cumulative form or individual form are used to optimize the overall formation (i.e. their relative positions) for performing a given task. This optimization can be carried out for a given instant of time or for the entire motion. While a global formation optimization can be carried out for an entire prescribed path, we also discuss local optimization (at a given time-instant/position) that is well suited for online implementation.

The organization of paper is as follows: In section II, we define our notation and discuss some of the metrics traditionally used for performance evaluation. In section III, we discuss the algorithm that is used to determine the motion plans for individual nonholonomic robots from motion of a team-fixed frame. In section IV, we present case studies of optimal formation planning for executing a screw motion and a general motion and discuss these results. Section V concludes the paper.

II. PROBLEM FORMULATION

A. Background

The set of rigid body displacements in the plane can be considered as elements of a Lie group, $A=(R,\vec{p})\in SE(2)$ and denoted by:

$$SE(2)=\left\{A \mid A=\begin{bmatrix} R & \vec{p} \\ 00 & 1 \end{bmatrix}, R\in\mathbb{R}^{2\times 2}, RR^T=I, \det R=+1, \vec{p}\in\mathbb{R}^2\right\} \quad (1)$$

The Lie algebra of $SE(2)$, denoted by $se(2)$, is given by

$$se(2)=\left\{T \mid T=\begin{bmatrix} \hat{\omega} & \vec{v} \\ 00 & 0 \end{bmatrix}, \hat{\omega}\in\mathbb{R}^{2\times 2}, \hat{\omega}^T=-\hat{\omega}, \vec{v}\in\mathbb{R}^2\right\} \quad (2)$$

where $\hat{\omega}$ is the 2×2 skew-symmetric matrix form of the vector $\vec{\omega}$. See [19] for a detailed description.

B. Motion Parameterization

Given a curve $A(s):[0,s_0]\rightarrow SE(2)$ and assuming a time-parameterization, $s(t):[t_1,t_2]\rightarrow[0,s_0]$, the configuration of the body can be represented by:

$$A(s(t))=\begin{bmatrix} R(s(t)) & \vec{p}(s(t)) \\ 00 & 1 \end{bmatrix}\in SE(2) \quad (3)$$

A curve on $SE(2)$ physically represents a motion of the rigid body. In the rest of this paper, we assume a constant speed of travel along the curve with $\dot{s}=1$. An element $T(s(t))$ of the Lie algebra $se(2)$ can be associated to the tangent vector $\dot{A}(s(t))$ along the curve at $s(t)$ by left translation as:

$$T(s(t))=\left[A(s(t))\right]^{-1}\dot{A}(s(t))=\begin{bmatrix} \hat{\omega} & R^T\dot{\vec{p}} \\ 0 & 0 \end{bmatrix} \quad (4)$$

where $\hat{\omega}=R^T\dot{R}$. The choice of associating the element of the Lie Algebra with an element from the tangent space by left translation results in a body-fixed representation of twists that is invariant with respect to location of the inertial frame (and hence the preferred description for locomotion systems). If $\vec{t}(s)=\left[\vec{\omega}^T \quad \vec{v}^T\right]^T$ is the vector pair representation of the matrix twist $T(s)$, then $\vec{\omega}$ corresponds to the angular velocity of the virtual structure while $\vec{v}$ is the linear velocity of the origin of $\{M_0\}$, both expressed in the team frame $\{M_0\}$.

An instantaneous screw axis can be associated with each twist. For a general motion along an arbitrary path, instantaneous screw axis changes with time. However, for a certain class of motions, the twist and thus corresponding screw axis remains fixed. Such motions are known as "screw motions" in kinematics. For a planar case, the screw axis reduces to "pole" and such motions reduce to rotations about a fixed pole with a constant angular velocity.

C. Virtual Structure

We consider a team of N differentially driven nonholonomic mobile robots in the plane. We affix a body-fixed frame $\{M_i\}$ to each mobile robot at the center of the axle with the x axis aligned in the direction of forward travel as shown in Figure 1.

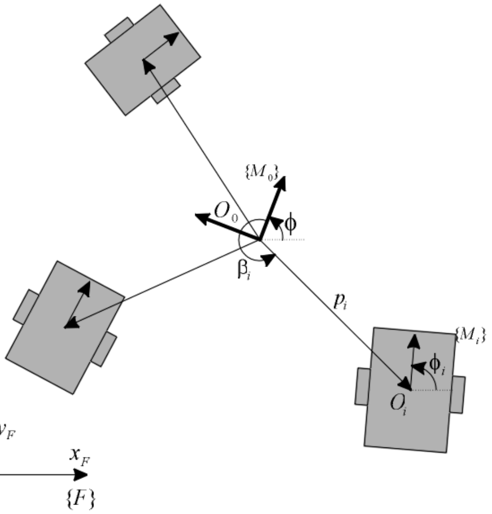

Figure 1 Virtual structure and formation of mobile robots

The configuration of each mobile robot is given by $g_i=({}^FR_{M_i},{}^F\vec{p}_i)\in SE(2)$ with respect to some inertial frame

$\{F\}$. We assume that the origins of all these frames O_i form the vertices of a rigid virtual structure. A “team fixed coordinate frame” $\{M_0\}$ can now be affixed at some convenient location on this virtual structure. The configuration of this coordinate frame $\{M_0\}$ is then given by $g_0 = ({}^F R_{M_0}, {}^F \vec{p}_0) \in SE(2)$ with respect to $\{F\}$. The relative orientation of each individual frame $\{M_i\}$ with respect to $\{M_0\}$ can be written as $\left[{}^F R_{M_0} \right]^{-1} {}^F R_{M_i}$ and their relative positions by ${}^{M_0}\vec{p}_i = \left[{}^F R_{M_0} \right]^{-1} \left({}^F \vec{p}_0 - {}^F \vec{p}_i \right) \in \mathbb{R}^2$. The configuration of the overall formation can thus be written in terms of the configuration of the team-frame (determined by 3 parameters) and 2N parameters defining the relative locations of the origins of the individual robots as:

$$q = (g_0, \vec{p}_1, \vec{p}_2, ..., \vec{p}_n) \in Q = SE(2) \times \underbrace{\mathbb{R}^2 \times ... \times \mathbb{R}^2}_{n} \tag{5}$$

Note that we do not consider the relative orientations of the $\{M_i\}$ frame as part of the formation configuration for reasons discussed later.

We note that a number of approaches have been proposed for selecting the team coordinate frame $\{M\}$ with respect to various robot affixed frames $\{M_i\}$. For example, [20] choose the origin of $\{M\}$ to coincide with the origin of $\{M_1\}$ and orient the X axis in the direction of a nearest neighbor $\{M_2\}$. Others such as [21] assign $\{M\}$ at the center of mass of the system oriented along the principal inertial directions. Despite the added initial computation burden, such a selection makes the kinetic energy metric diagonal and allows ease of subsequent repeated evaluations of the system kinetic energy.

D. *Path Parameterization*

Given an arc-length parameterized planar path $X : [0, s_0] \to \mathbb{R}^2; X(s) = (x(s), y(s))$, we compute the Serret-Frenet frame at each point. The corresponding evolution equations may be written as

$$\begin{bmatrix} \dot{\hat{e}}_t(s) \\ \dot{\hat{e}}_n(s) \end{bmatrix} = \begin{bmatrix} 0 & \kappa(s) \\ -\kappa(s) & 0 \end{bmatrix} \begin{bmatrix} \hat{e}_t(s) \\ \hat{e}_n(s) \end{bmatrix} \tag{6}$$

where $\hat{e}_t$ is the unit tangent vector and $\hat{e}_n$ is is the unit normal vector. We now attach our team frame $\{M_0\}$ to this Serret-Frenet frame, thereby creating an arc-length-parameterized motion of the virtual structure. These twists at $\{M_0\}$, together with the rigid formation condition induce a *helicoidal velocity vector field* at the origins O_i of each of the robot fixed frames $\{M_i\}$.

E. *Riemannian metrics on* $SE(3)$ *(*$SE(2)$*)*

We adopt the notation of Zefran (and refer the reader to [22] for a detailed discussion). An inner product on $se(3)$ can be extended to a Riemannian metric on $SE(3)$. Let the inner product of two elements $T_1, T_2 \in se(3)$ be given by

$$\left\langle T_1, T_2 \right\rangle_I = \vec{t}_1^T W \vec{t}_2 \tag{7}$$

where $\vec{t}_1$ and $\vec{t}_2$ are 6×1 vectors of components of T_1 and T_2 with respect to some basis and W is a positive definite matrix representing the metric. If $\vec{V}_1$ and $\vec{V}_2$ are tangent vectors at an arbitrary point $A \in SE(3)$, the left-invariant inner product $\left\langle \vec{V}_1, \vec{V}_2 \right\rangle_A$ in the tangent space $T_A SE(3)$ can be defined by:

$$\left\langle \vec{V}_1, \vec{V}_2 \right\rangle_A = \left\langle A^{-1}\vec{V}_1, A^{-1}\vec{V}_2 \right\rangle_I \tag{8}$$

The matrix form for a family of left-invariant metrics on se(3), parameterized by 3 scalars α, β and γ, is expressed as:

$$W = \begin{bmatrix} \alpha I_{3\times3} & \beta I_{3\times3} \\ \beta I_{3\times3} & \gamma I_{3\times3} \end{bmatrix} \tag{9}$$

Several left invariant metrics can be found in literature by using different values for α, β and γ. For example, $\beta = \gamma = 0$ is called Killing form which essentially is a measure of the angular velocities $(\alpha \vec{\omega}^T \vec{\omega})$ within the space of twists. $\alpha = \gamma = 0$ results in the Klein form and serves as a measure of $(2\vec{\omega}^T \vec{v})$. (However, when specialized to $SE(2)$, this measure is identically zero and we avoid its use). $\beta = 0$ yields the decoupled Park metric which serves as a weighted quadratic combination of the linear and angular velocities $\alpha(\vec{\omega}^T \vec{\omega}) + \gamma(\vec{v}^T \vec{v})$. Similarly, in the centroid fixed frame of reference aligned along the principal inertial directions, the kinetic energy metric can be written as:

$$W = \begin{bmatrix} H_{3\times3} & 0_{3\times3} \\ 0_{3\times3} & mI_{3\times3} \end{bmatrix} \tag{10}$$

The metrics shown in Equation. 9 can be specialized for $SE(2)$, and written as:

$$W = \begin{bmatrix} \alpha_{1\times1} & 0_{1\times2} \\ 0_{2\times1} & \gamma I_{2\times2} \end{bmatrix} \tag{11}$$

And similarly the metric shown in Equation. 11 may be specialized as:

$$W = \begin{bmatrix} I_{zz} & 0_{1\times 2} \\ 0_{2\times 1} & mI_{2\times 2} \end{bmatrix} \tag{12}$$

where m and I_{zz} are the mass and planar inertia of the rigid body.

III. FORMATION MOTION PLANNING

Given a desired path determines both, the location of ICR as well as the curvature. Let the tangent and normal directions of the curve at any point be $\hat{e}_t$ and $\hat{e}_n$ which uniquely defines the Serret-Frenet frame. Let $\dot{s}$ be the magnitude of velocity at which the frame travels along the path. The forward velocity of the origin of the Serret-Frenet frame may be written as ${}^{M_0}v = \hat{e}_t \dot{s}$ and the angular as ${}^{M_0}\omega = \dot{\hat{e}}_t = \hat{e}_k \kappa \dot{s}$. In this paper we assume that we travel along the path with constant $\dot{s} = 1$ leaving the case of path- and time-varying $\dot{s}(s,t)$ for future work. The motion of the team frame $\{M_0\}$ as it traces the desired path induces a helicoidal velocity field at the vertices of the virtual structure. The velocity of the ith robot with respect to frame $\{M_0\}$ is computed as:

$${}^{M_0}\begin{bmatrix} \vec{v}_i \\ 0 \end{bmatrix} = {}^{M_0}\begin{bmatrix} {}^{F}T_{M_0} \end{bmatrix} \begin{bmatrix} \vec{p}_i \\ 1 \end{bmatrix} = \begin{bmatrix} 0 & -\kappa(s) & 1 \\ \kappa(s) & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} p_i \cos\beta_i \\ p_i \sin\beta_i \\ 1 \end{bmatrix} \tag{13}$$

As the formation maneuvers, maintaining the rigidity of this virtual structure, our motion planning strategy aligns the direction of forward travel (the X axis of each robot) with the induced helicoidal velocity vector field.

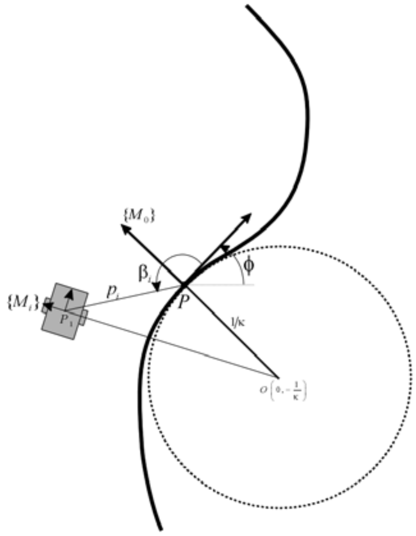

Figure 2 Visualization of instantaneous center of rotation constraint

This planning strategy for developing motion plans for each mobile robot can be visualized using the notion of the Instantaneous Center of Rotation (ICR). In frame $\{M_0\}$, the location of instant center of the osculating circle is given by $(0, -(1/\kappa(s)))$. Further, graphically, we see that the ICR of each robot is constrained to lie on the line passing through the axle of each robot (Figure 2). Thus, when multiple robots form part of a virtual structure moving with its helicoidal field, the ICR of each robot must now correspond to the instant center of the virtual structure (and thus the motions along the underlying path). This is used to uniquely determine the orientation of each mobile base.

The corresponding twists of the frames $\{M_i\}$, as they move to align themselves with this motion plan have a simplified representation in robot fixed frame $\{M_i\}$ as:

$${}^{M_i}\begin{bmatrix} {}^{F}t_{M_i} \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \omega_i + \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} v_i \tag{14}$$

where $v_i = \left\| {}^{M}\vec{v}_i \right\|$ and ω_i are the magnitudes of the linear and angular velocities. However, for any given path, the relative configurations $(\vec{p}_i)$ induce different motion plans (with different performance characteristics) for the individual mobile robots as they move in formation towards the final goal. Hence, using the left invariant metrics discussed in Section II.E we can evaluate the performance at each of these different configurations and use this to optimize the relative location as will be discussed in the next section. We also note that while the discussion above was with respect to a prescribed *path*, an online implementation is also possible given only the instantaneous motion representation (twist) of a suitable task frame.

IV. FORMATION OPTIMIZATION CASE STUDIES

We examine the optimization of the overall formation using case studies of a screw motion and a general motion. In particular, for both cases, we show the results of the optimization using the Killing form and the decoupled kinetic energy metric. The Killing form metric evaluates $\alpha\omega^2 = \alpha(\kappa(s))^2$ while the kinetic energy metric evaluates $\frac{1}{2}(I_{zz}\omega^2 + mv^2)$ for each mobile robot (in its own frame $\{M_i\}$).

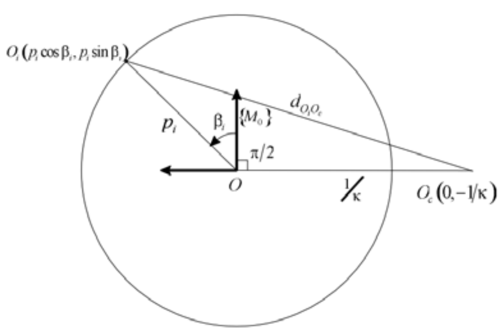

Figure 3 Formation moving with a constant v and ω

We note that an analytical solution for the kinetic energy can be computed in terms of the arc-length parameterization and the geometry of the problem as shown in Figure 3. Let the mobile robot be located at point O_i and the team frame located at O as shown in Figure 3. Then, the kinetic energy can be written as:

$$E = v_i^2\left(\frac{\kappa^2}{1+\kappa^2 p_i^2 - 2p_i\kappa\cos\left(\pi/2+\beta_i\right)} + m\right) \qquad (15)$$

By considering optimization variables to be p_i and β_i individually, the conditions for minimizing the kinetic energy can be derived analytically. For a constant β_i, the optimal p_i^* is:

$$p_i^* = \frac{\cos(\beta_i + \pi/2)}{\kappa} \qquad (16)$$

For a constant p_i, the optimal β_i^* is:

$$\beta_i^* = -\pi/2 \qquad (17)$$

In each of the subsequent figures, the first subplot depicts the desired trajectory in the XY plane, the second subplot depicts $\kappa(s)$ vs s, the third subplot depicts $p_i^*(s)$ vs s and the fourth subplot depicts $\beta_i^*(s)$ vs s.

A. *Screw Motion*

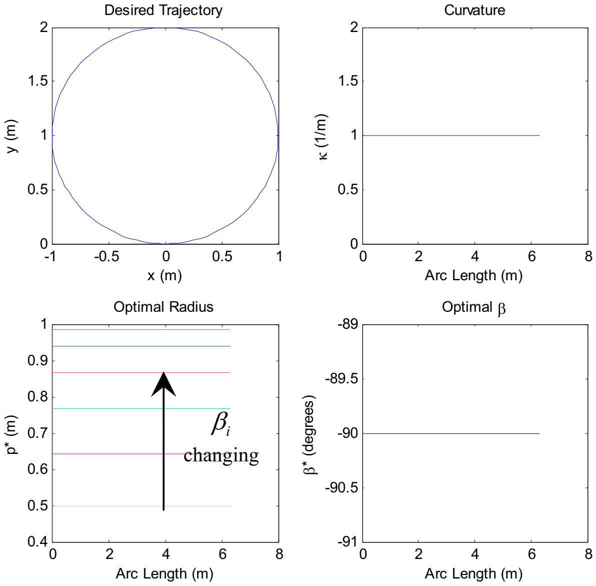

Figure 4 Desired trajectory and corresponding results for a screw motion

Figure 4 shows the results for the screw motion using the kinetic energy metric. For a screw motion, for which ω is constant, the performance measure also remains constant regardless of the relative configuration of the mobile base from the team-frame. Thus, any feasible relative configuration is optimal. The path will be a circle as shown in first subplot in Figure 4. For this case-study $v = 2\pi$ and $\omega = 1$ and so $\kappa = 1$. Moreover, the values of $p_i^*(s)$ are constant for given $\beta_i(s)$ for optimal configuration – see Equation 16. This is shown in third subplot where different constant lines corresponds to different values of $\beta_i(s)$. Also value $\beta_i^*(s)$ is constant for any $p_i(s)$.

B. *General motion*

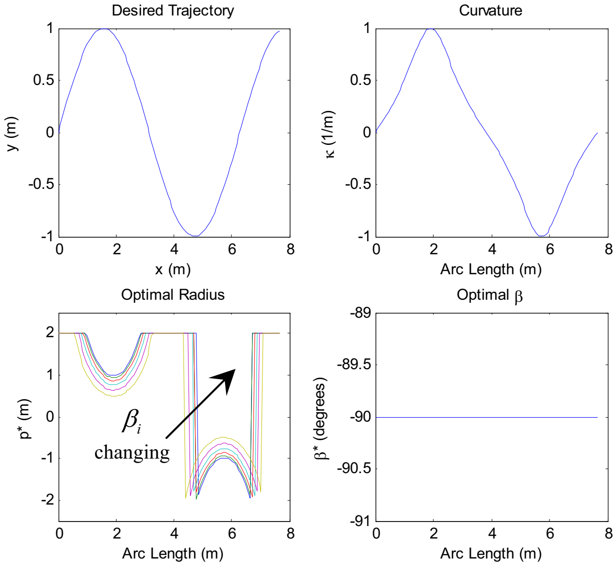

Figure 5 Desired trajectory and corresponding results for a non-screw (Sinusoidal) motion

Figure 5 depicts the results obtained for the case where the Serret-Frenet frame travels along a sinusoidal path along the X axis of the inertial frame with a constant linear speed $\dot{s} = 1$. For general motions κ (and therefore ω) are not constants but functions of parameter s. While analytical expressions for $\kappa(s)$ may be difficult to obtain, it is always possible to numerically compute these values for any given path. However, for the case of a sinusoidal desired path, an analytical equation for the curvature $\kappa(s)$ exists and can be written as:

$$\kappa(s) = \frac{v\omega^2\sin\left(\omega s\right)}{1+v^2\omega^2\cos^2\left(\omega s\right)} \qquad (18)$$

This expression is graphically shown in the second subplot of Figure 5. Further, from Equation 16, we note that as $\kappa \to 0$ $p_i^*(s) \to \infty$ which is reflected in the third subplot. However, please note that we capped the value of $p_i^*(s)$ to 2. The fourth subplot depicts the fact that optimal values for $\beta_i^*(s)$ for a constant radius p_i always remains unchanged at $-\pi/2$.

V. CONCLUSION

In this paper we presented method of motion planning that takes into consideration the nonholonomic constraints in the planning phase. We also specialized metrics defined on $SE(3)$ for use with $SE(2)$. Using these metrics, we optimized the performance of the formation of robots and derive analytical solutions for the optimal values of parameters. The analytical expressions developed for the optimal configuration in terms of $\kappa(s)$ make it very easy to compute optimal formations and are well suited for online implementation.

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