Kinematic Synthesis of Spatial R-R Dyads for Path Following With Applications to Coupled Serial Chain Mechanisms

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The dimensional synthesis of a spatial two revolute jointed dyad for path following tasks with applications to coupled serial chain mechanisms is presented. The precision point synthesis equations obtained using the rotation matrix approach form a rank-deficient linear system in the link-vector components. The nullspace of this rank-deficient linear system is derived analytically and interpreted geometrically. The nullspace vectors lead to the specification of additional constraints via the so-called auxiliary equations and to the solution of the linear system of equations. The geometry also allows the derivation of a closed form solution for the three design position problem. Finally, optimal path following by coupled R-R dyads is achieved by optimization over the free choice variables. [DOI: 10.1115/1.1370977]

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1. Introduction

Single Degree-of-freedom Coupled Serial Chain (SDCSC) mechanisms combine the benefits of serial chain manipulators and closed-loop linkages. They are constructed by coupling the rotations of the distal joints of a conventional serial chain to the base joint by way of cable and pulley drives or gear trains to reduce the degree of freedom to one. Like closed-loop linkages, they are inexpensive, easily controlled with a single actuator, and optimized for an application. The versatility is not compromised in SDCSCs because they can be designed to perform multiple tasks by changing the speed ratios of the pulleys and varying the effective link lengths. Like serial chains, they are suitable for manipulation tasks in environments with obstacles. The design of planar SDCSC mechanisms was described in Krovi et al. [1,2]. In this paper, the dimensional synthesis of two-link spatial SDCSC mechanisms is considered. Figure 1 shows the general R-R dyad and the coupled R-R dyad. Because the two mechanisms have a number of common design issues, the extensive literature on the R-R dyad (see Bodduluri et al. [3] for a thorough review) is briefly reviewed next.

1.1 Background. The methods for spatial dyadic synthesis can be broadly classified into *geometric* and *loop-closure* methods [4]. In geometric methods, the orientations and locations of the joints are used as the variables in the design equations and their loci are found on a moving body such that they can be easily mechanized [5–7]. On the other hand, the loop-closure method formulates the design equations using vectors, rotation matrices, quaternions, or dual quaternions with link dimensions and orientations as the variables [8–12].

Using the geometric approach, Suh [13] demonstrated that spatial R-R dyads which provide three finitely separated poses (positions and orientations) of a rigid body can exist only in pairs and that each pair forms a Bennett mechanism. Tsai and Roth [14] proved that such a pair is unique and presented an analytical solution to find it. It should be noted however that an analytical solution of the design equations obtained using the geometric approach is not possible in general, and may warrant numerical solutions [7].

The loop-closure approach was used in spatial dyadic synthesis by Sandor [8] and Sandor and Bisshopp [9] among many others, and more recently by Lee and Yoon [12]. As noted by Kaufman in the accompanying discussion of Sandor and Bisshopp [9], the design equations in the loop-closure approach are formulated so that they are linear in the unknown link vector components. It not only simplifies the solution but also helps in easily determining the maximum number of design positions and in appropriately choosing excess unknowns as *free-choice variables* when there are fewer equations than unknowns.

The focus of this paper is to synthesize an R-R dyad that passes through three finitely separated positions in the 3-D space disregarding orientations, with applications to spatial SDCSCs. It is important to note at the outset that there exists a trivial planar solution for this problem because three arbitrary points necessarily lie in a plane. Such a solution is not deemed appropriate here because the objective is synthesizing an R-R dyad which when used in a mechanism can approximately follow a specified path in the 3-D space while precisely passing through three points on this path. In this work, vectors and rotation matrices are used to formulate the design equations using the loop-closure method. In contrast to the planar case, the spatial loop-closure equations written in terms of linear link vector unknowns are not independent. In the previous research on the spatial R-R dyad [7,9,12,14] sufficient number of auxiliary equations of appropriate form were added to render the system of design equations independent and consistent in order to ensure a solution. Different types of auxiliary equations used by other researchers are summarized later in this paper. The goal of this paper is to systematically present the necessary theory and algorithms for the design of R-R dyads while making three new contributions to this subject. The first contribution is a clear explanation of the rationale behind the auxiliary equations and their analytical and geometric interpretation. The second contribution is the development of a systematic closed-form solution of the design equations that uses the additional insight gained about the R-R dyad. The third contribution is using the free-choice variables to optimize the coupled R-R dyad mechanism formed with the dyad solution so that specified re-

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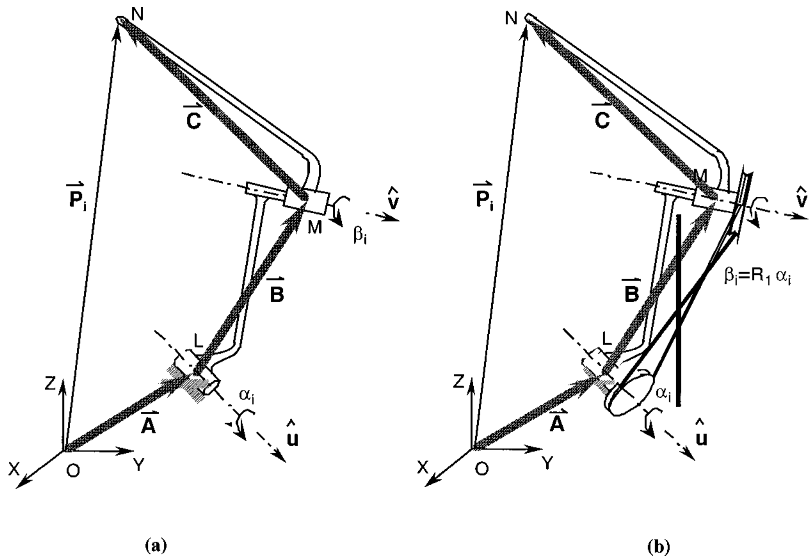

Fig. 1 (*a*) **Spatial R-R dyad with two degrees of freedom;** (*b*) **the coupled R-R dyad with only one degree of freedom**

quirements are met exactly at the design positions and in a least squares sense everywhere else on the specified 3-D path.

1.2 Organization of the Paper. The design equations are formulated in Section 2 using vectors and rotation matrices. The need for auxiliary equations is systematically presented and the result of an analytical derivation is interpreted using what is called a *nullspace dyad*. In Section 3, the rationale for why and how many auxiliary equations are necessary is explained, and a convenient form of these equations is presented in addition to briefly reviewing the auxiliary equations used by previous researchers. Arbitrarily chosen values for the free-choice variables might render the linear system of design equations inconsistent. This problem is discussed and solved in Section 4 using a fundamental theorem of linear algebra after analytically deriving the consistency conditions. Section 5 is an outline of the new closed-form solution procedure. In Section 6, the solution procedure is applied to the coupled R-R dyad which is optimized to trace a path in 3-D space. Numerical examples are presented in Section 7 and concluding remarks are made in Section 8.

2. Kinematic Design Equations and the Nullspace Dyad

2.1 Single Crank Passing Through Two Points in Space. In this subsection, the basic ideas will be presented using the simple example of a single crank passing through two arbitrarily specified points in space as shown in Fig. 2. There are nine scalar unknowns in this problem: three in the link vector $\vec{B}_1$ in the first design position, two in the unit vector representing the axis of rotation $\hat{u}$, three in the vector $\vec{A}$ that describes the crank, and the angle of rotation α_2 that takes first design position to the second. The two vector loop-closure equations shown below lead to six scalar equations giving rise to three *free-choice variables*.

$$\vec{A}+\vec{B}_1=\vec{P}_1$$

$$\vec{A}+R[\hat{u},\alpha_2]\vec{B}_1=\vec{P}_2 \tag{1}$$

where $\vec{P}_1$ and $\vec{P}_2$ are position vectors of the specified points, and $R[\hat{u},\alpha_2]$ is the rotation matrix that rotates vector $\vec{B}_1$ from its first design position to the second:

$$R[\hat{u},\alpha_2]=[\cos(\alpha_2)I+\sin(\alpha_2)(\hat{u}\times)+(1-\cos(\alpha_2))\hat{u}\hat{u}^T] \tag{2}$$

where I is the 3×3 identity matrix and $(\hat{u}\times)$ is the skew matrix of $\hat{u}$. Subtracting the first of Eq. (1) from the second, we eliminate three scalar unknowns of $\vec{A}$ leaving six scalar unknowns in three equations in the vector form as:

$$\{R[\hat{u},\alpha_2]-I\}\vec{B}_1=\vec{P}_2-\vec{P}_1 \tag{3}$$

Although it may appear that three unknowns ($\hat{u}$ and α_2) can be freely chosen to solve for the three unknowns in $\vec{B}_1$ in the linear system in Eq. (3), it is not possible because, for general values of $\hat{u}$, α_2, $\vec{P}_1$ and $\vec{P}_2$,

(i) the rank of $\{R[\hat{u},\alpha_2]-I\}$ is two, *i.e.*, it is rank-deficient by one, and

(ii) the rank of the 3×4 augmented matrix $\{R[\hat{u},\alpha_2]-I\,\vdots\,(P_2-P_1)\}$ is three (it should be two as explained later).

Since the matrix in Eq. (3) is rank-deficient by one, it has a non trivial homogeneous solution in addition to a particular solution. The homogeneous component of $\vec{B}_1$ can be determined if we know the nullspace of $\{R[\hat{u},\alpha_2]-I\}$. From the properties of the

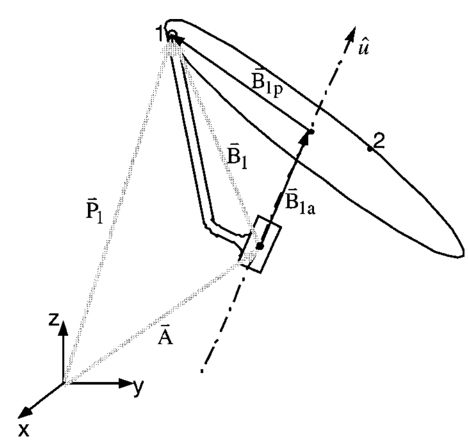

Fig. 2 Single crank passing through two points in space

rotation matrices, we know that the nullspace vector of $\{R[\hat{u},\alpha_2]-I\}$ is the axis of rotation, $\hat{u}$. Thus, the complete solution for $\vec{B}_1$ is given by:

$$\vec{B}_1=\vec{B}_{1,\text{particular}}+\vec{B}_{1,\text{homogeneous}}=\vec{B}_{1,\text{particular}}+s\hat{u} \tag{4}$$

where s is one of the three possible free choices.

The result of Eq. (4) can be interpreted geometrically by observing in Fig. 2 that $\vec{B}_1$ can be decomposed into $\vec{B}_{1a}$ and $\vec{B}_{1p}$ which are respectively parallel and perpendicular components to the unit vector $\hat{u}$. It can seen that the length of $\vec{B}_{1a}$ can be arbitrarily chosen without affecting the capability of the single crank to pass through the two points. This means that the fixed pivot of the crank can be arbitrarily positioned along the axis $\hat{u}$. Thus, s is one free choice variable. If it is intended that this free choice be not exercised freely and instead some other objective is to be fulfilled, an *auxiliary equation* can be added. For example, $\vec{B}_1$ in Eq. (3) may be required to satisfy the following auxiliary equation to minimize its length:

$$\vec{B}_1\cdot\hat{u}=0 \tag{5}$$

As per the fundamental theorem of linear algebra, a solution exists for a system of linear equations only if the right hand side vector lies in the column space of the matrix [15]. This implies that the rank of the matrix is equal to the rank of the augmented matrix that is obtained by adding the right hand side vector as an extra column to the matrix [16]. Therefore, unless $\hat{u}$ and α_2 are chosen appropriately to make the rank of the augmented matrix equal to two, the linear system in Eq. (3) will be inconsistent without a solution for specified $\vec{P}_1$ and $\vec{P}_2$. A *consistency condition* can be posed in a computationally convenient form using the indirect form of the fundamental theorem of linear algebra. The indirect form requires that the left nullspace of the matrix be orthogonal to the right hand side vector. In this case, the left nullspace of the matrix in Eq. (3) has a dimension of one and is spanned by the vector $\hat{u}$. Therefore, $\hat{u}$ must be perpendicular to the vector $(\vec{P}_2-\vec{P}_1)$ leading to the consistency condition:

$$\hat{u}\cdot(\vec{P}_2-\vec{P}_1)=0 \tag{6}$$

Using one free choice and the consistency condition of Eq. (6), $\hat{u}$ can be determined before solving the design equation of Eq. (3). The angle α_2 is the second free choice that is needed to solve Eq. (3). The three scalar equations of Eq. (3) are now guaranteed to be consistent. Any two independent equations in Eq. (3) and the auxiliary equation in Eq. (5) can now be solved for $\vec{B}_1$. And then, using first of Eq. (1), $\vec{A}$ can be obtained.

This simple example, as explained above, served to make the following observations that are generally true with spatial revolute jointed serial chains (as will be seen later in the paper):

1 The linear system of design equations resulting only from the loop-closure equations in the spatial case cannot be solved for arbitrary design specifications since the resulting matrix is rank-deficient.

2 The free choices must be chosen so as to render the linear system of design equations consistent and solvable.

3 The nullspace and the homogeneous solution must be interpreted correctly in order to add the auxiliary equations systematically and to choose the free-choice variables sensibly.

2.2 Spatial R-R Dyad. The above ideas will now be extended to the R-R dyad. As shown in Fig.1(a), first link vector $\vec{B}_1$, positioned away from the origin by $\vec{A}$, rotates about the fixed axis denoted by the unit vector $\hat{u}$ by an angle α_i to reach the i^{th} design position. The second link vector is $\vec{C}_1$ which is rotated along with $\vec{B}_1$ in addition to rotating about the moving axis $\hat{v}_i$ by an angle β_i. By Euler's theorem, the composite rotation of the second link vector can be obtained as a single rotation about an equivalent axis $\hat{w}_i$ by an angle γ_i. The loop-closure equation for the i^{th} design position can then be written as:

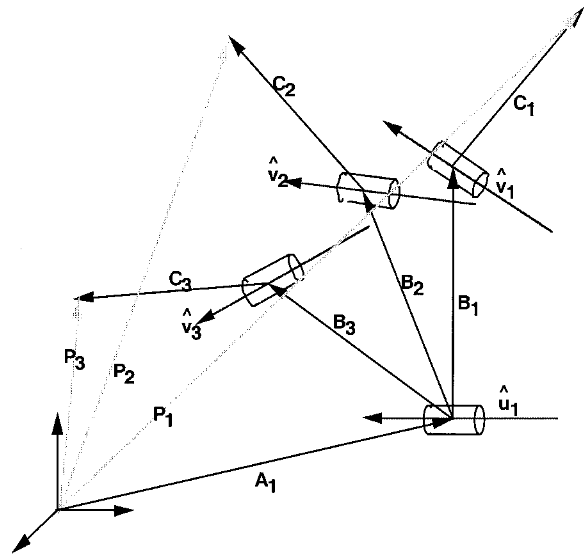

Fig. 3 Spatial R-R dyad passing through three design positions

$$\vec{A}+\vec{B}_1+\vec{C}_1=\vec{P}_1$$

$$\vec{A}+R[u,\alpha_i]\vec{B}_1+R[u,\alpha_i]R[v_i,\beta_i]\vec{C}_1=\vec{P}_i,\quad i=2,3 \tag{7a}$$

$$\text{or }\ \vec{A}+R[u,\alpha_i]\vec{B}_1+R[w_i,\gamma_i]\vec{C}_1=\vec{P}_i \tag{7b}$$

The kinematic design equations are obtained by taking the difference of the loop-closure equation in the i^{th} position and the first position. Thus, for three design positions shown in Fig. 3, the six scalar equations can be written in matrix form as follows:

$$\begin{bmatrix}(R[\hat{u},\alpha_2]-I) & (R[\hat{u},\alpha_2]R[\hat{v}_1,\beta_2]-I)\\ (R[\hat{u},\alpha_3]-I) & (R[\hat{u},\alpha_3]R[\hat{v}_1,\beta_3]-I)\end{bmatrix}\begin{Bmatrix}\vec{B}_1\\ \vec{C}_1\end{Bmatrix}$$

$$=\begin{Bmatrix}\vec{P}_2-\vec{P}_1\\ \vec{P}_3-\vec{P}_1\end{Bmatrix}\Rightarrow[M]_{6\times6}\begin{Bmatrix}\vec{B}_1\\ \vec{C}_1\end{Bmatrix}_{6\times1}=\begin{Bmatrix}\vec{P}_2-\vec{P}_1\\ \vec{P}_3-\vec{P}_1\end{Bmatrix}_{6\times1} \tag{7c}$$

There are a total of 14 scalar unknowns in 6 equations in Eq. (7c). the unknowns are: $\hat{u}$ (2), $\hat{v}_1$ (2), $\vec{B}_1$ (3), $\vec{C}_1$ (3), and four angles $\alpha_2,\alpha_3,\beta_2$, and β_3. As in the single crank case, it might appear that there are eight free-choice variables to assume freely and solve for $\vec{B}_1$ and $\vec{C}_1$ as a linear system. Once again, this is not possible as the matrix M is rank-deficient by two. Thus, two extra independent auxiliary equations or two additional specifications of free choices are necessary to find the homogeneous solution. Furthermore, the free-choice variables should be chosen such that the matrix equation is consistent and solvable. The homogeneous solution is analytically derived and geometrically interpreted next in this section and the consistency is discussed in Section 5.

Analytical Derivation of the Nullspace of $[M]_{6\times6}$. The homogeneous solution of Eq. (7c) is equivalent to finding the nullspace of the matrix M. Since the rank of this matrix is four, the dimension of its nullspace is two. Thus there exist two 6×1 vectors $\vec{X}_1$ and $\vec{X}_2$ for the solution of the homogeneous equation:

$$\begin{bmatrix}(R[\hat{u},\alpha_2]-I) & (R[\hat{u},\alpha_2]R[\hat{v}_1,\beta_2]-I)\\ (R[\hat{u},\alpha_3]-I) & (R[\hat{u},\alpha_3]R[\hat{v}_1,\beta_3]-I)\end{bmatrix}\begin{Bmatrix}\vec{B}_1\\ \vec{C}_1\end{Bmatrix}=\begin{Bmatrix}\vec{0}\\ \vec{0}\end{Bmatrix} \tag{8}$$

Let $\vec{X}_1$ and $\vec{X}_2$ be of the following form:

$$\vec{X}_{1,2}=\begin{Bmatrix} p_1\hat{u}+q_1\hat{v}_1 \\ p_2\hat{u}+q_2\hat{v}_1 \end{Bmatrix} \tag{9}$$

where p_1, q_1, p_2, and q_2 are unknown constants that assume two different sets of values to give $\vec{X}_1$ and $\vec{X}_2$.

Substitution of Eq. (9) into (8) and the block multiplication of the top row, after simplification (note: $(R[\hat{u},\alpha_2]-I)\hat{u}=\vec{0}$ and $R[\hat{v}_1,\beta_2]\hat{v}_1=\hat{v}_1$) yields:

$$(q_1+q_2)(R[\hat{u},\alpha_2]-I)\hat{v}_1+p_2(R[\hat{u},\alpha_2]R[\hat{v}_1,\beta_2]-I)\hat{u}=\vec{0} \tag{10}$$

Pre-multiplication of Eq. (10) by $\hat{u}^T$ makes the first term vanish yielding

$$p_2\hat{u}^T(R[\hat{u},\alpha_2]R[\hat{v}_1,\beta_2]-I)\hat{u}=\vec{0} \tag{11}$$

from which we can infer that p_2 must be zero unless $\hat{u}$ is parallel to $\hat{v}_1$ which is the trivial case. By substituting $p_2=0$ in Eq. (10), we can infer that

$$q_1+q_2=0 \tag{12}$$

There is no restriction on the value of p_1. Block multiplication of the second row gives the same results. By choosing values for p_1, q_1, p_2 and q_2 in accordance with the above relationships, two sets of values can be obtained as follows:

$$\begin{aligned} p_1=1;\quad p_2=0;\quad q_1=0;\quad q_2=0 &\Rightarrow \vec{X}_1=\begin{Bmatrix} \hat{u} \\ \vec{0} \end{Bmatrix} \\ p_1=0;\quad p_2=0;\quad q_1=-1;\quad q_2=1 &\Rightarrow \vec{X}_2=\begin{Bmatrix} -\hat{v}_1 \\ \hat{v}_1 \end{Bmatrix} \end{aligned} \tag{13}$$

These nullspace vectors can be interpreted geometrically as described next.

The Nullspace Dyad. In the R-R dyad formed with the homogeneous solution of $\vec{B}_1$ and $\vec{C}_1$ (*i.e.*, the nullspace vectors), the tip of the end-effector experiences zero displacement for any α and β. Such a dyad, called the *nullspace dyad*, is shown geometrically in Fig. 4. It can be seen in the figure that the tip of the vector $\vec{C}'$, the end-effector, remains fixed for any rotation α of $\vec{B}'$ about $\hat{u}$ and any rotation β of $\vec{C}'$ about the rotated $\hat{v}$. By denoting the distance from the end-effector point to the fixed revolute joint of $\hat{u}$ by λ_1, and the distance from the end-effector point to the moving revolute joint of $\hat{v}$ by λ_2, from Fig. 4 we obtain,

$$\vec{B}'=\lambda_1\hat{u}-\lambda_2\hat{v}$$

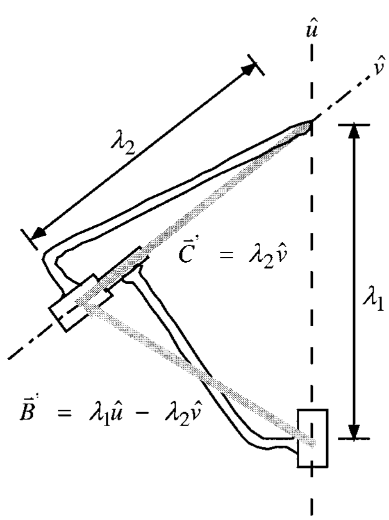

Fig. 4 The nullspace dyad, a geometric interpretation of *M*

$$\vec{C}'=\lambda_2\hat{v} \tag{14}$$

It should be noted that the nullspace dyad holds good for any values of λ_1 and λ_2. Therefore, the homogeneous solution of Eq. (7*c*) and therefore the nullspace of the matrix *M* can be written as:

$$\begin{Bmatrix} \vec{B}' \\ \vec{C}' \end{Bmatrix}=\lambda_1\begin{Bmatrix} \hat{u} \\ \vec{0} \end{Bmatrix}+\lambda_2\begin{Bmatrix} -\hat{v} \\ \hat{v} \end{Bmatrix} \tag{15}$$

which is same as Eq. (13).

It is also worth noting that the above geometric interpretation and derivation of the nullspace dyad extends to serial revolute chains of three or more links. The case of the *nullspace triad* is shown in Fig. 5 and is given by:

$$\begin{Bmatrix} \vec{B}' \\ \vec{C}' \\ \vec{D}' \end{Bmatrix}=\lambda_1\begin{Bmatrix} \hat{u} \\ 0 \\ 0 \end{Bmatrix}+\lambda_2\begin{Bmatrix} -\hat{v} \\ \hat{v} \\ 0 \end{Bmatrix}+\lambda_3\begin{Bmatrix} 0 \\ -\hat{w} \\ \hat{w} \end{Bmatrix} \tag{16}$$

Alternatively, the nullspace vectors shown in Eq. (13) can be written using a different basis, which has the attractive property of being orthogonal:

$$\begin{Bmatrix} \hat{u} \\ \vec{0} \end{Bmatrix} \quad \text{and} \quad \begin{Bmatrix} -\hat{v}_1+(\hat{u}\cdot\hat{v}_1)\hat{u} \\ \hat{v} \end{Bmatrix} \tag{17}$$

In a similar manner, an alternative representation of the nullspace vectors of Eq. (16) for the triad can be written using orthogonal basis vectors as:

$$\begin{Bmatrix} \hat{u} \\ 0 \\ 0 \end{Bmatrix}, \quad \begin{Bmatrix} -\hat{v}_1+(\hat{u}\cdot\hat{v}_1)\hat{u} \\ \hat{v} \\ 0 \end{Bmatrix}$$

and

$$\begin{Bmatrix} (\hat{v}_1\cdot\hat{w}_1)(-\hat{v}_1+(\hat{v}_1\,.\,\hat{u})\hat{u}) \\ -\hat{w}_1+(\hat{w}_1\cdot\hat{v}_1)\hat{v}_1 \\ \hat{w}_1 \end{Bmatrix} \tag{18}$$

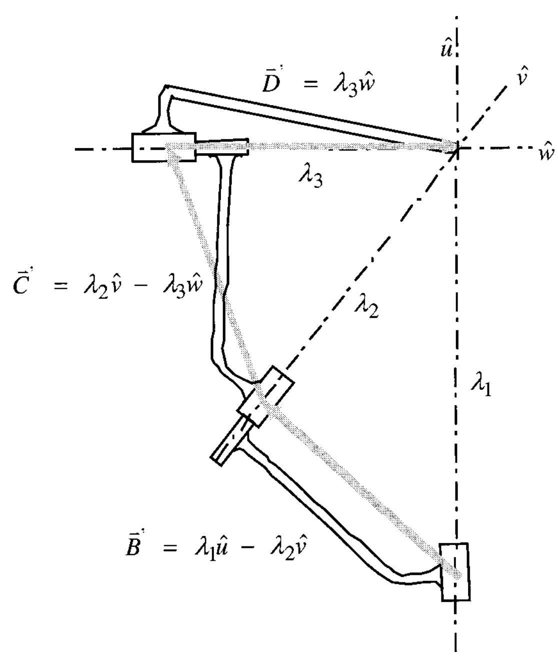

Fig. 5 Nullspace triad, a geometric interpretation of the nullspace of *M*

In the next section, the insight gained with the nullspace vectors is used in systematically arriving at the auxiliary equations that make the linear system of equations in Eq. (7*c*) consistent and solvable.

3 Auxiliary Equations

In the single crank case, instead of using the free-choice variable s in the homogeneous solution in Eq. (3), an auxiliary equation (Eq. (5)) was added to minimize the length of the vector $\vec{B}_1$. In a similar manner, in the case of the R-R dyad we can either choose the free-choice variables λ_1 and λ_2 in Eq. (15) or add two auxiliary equations to Eq. (7*c*) to impose additional criteria and make the rank of the resulting system of equations equal to six. The two conditions to be satisfied in choosing the auxiliary equations are: (i) independence with respect to the equations in Eq. (7*c*); and (ii) linearity in the unknown vector variables $\vec{B}_1$ and $\vec{C}_1$. There are many ways of adding the auxiliary equations. For example, Tsai and Roth [14] and Lee and Yoon [12] required that the first link be perpendicular to the fixed and moving revolute axes to obtain the shortest length for $\vec{B}_1$. That is,

$$\begin{aligned}\vec{B}_1\cdot\hat{u}&=0\\ \vec{B}_1\cdot\hat{v}&=0\end{aligned}\tag{19}$$

Sandor [8] and Sandor and Bisshopp [9] used the following conditions that enabled the development of the quaternion rotation operator, but made it necessary to include three auxiliary equations even though two are enough for the reason stated above.

$$\begin{aligned}\vec{B}_1\cdot\hat{u}&=0\\ \vec{C}_1\cdot\hat{w}&=0\quad i=2,3\end{aligned}\tag{20}$$

It should be noted that the nullspace vectors not only furnish the rationale for why and how many auxiliary equations are necessary but also provide guidance in formulating them in an appropriate form. In this paper, the following conditions were used to ensure that the solution vectors of $\vec{B}_1$ and $\vec{C}_1$ have no components along the nullspace vectors and therefore yielding shortest lengths for each link.

$$\begin{aligned}\vec{B}_1\cdot\hat{u}&=0\\ (\vec{B}_1-\vec{C}_1)\cdot\hat{v}_1&=0\end{aligned}\tag{21}$$

4 Consistency Conditions

The fundamental theorem of linear algebra states that for a linear matrix system with non-zero right hand side vector to have a solution, the vector on the right hand side must lie in the column space of the matrix [15]. Therefore, for arbitrarily specified position vectors $\vec{P}_1$, $\vec{P}_2$, and $\vec{P}_3$, the free-choice variables must be chosen such that the condition set forth by the fundamental theorem of linear algebra is satisfied by Eq. (7*c*). One method of verifying this condition is to make sure that the rank of the augmented matrix M_{aug} (shown below) is same as the rank of the matrix M.

$$M_{aug}=\left[[M]_{6\times 6}\ \begin{Bmatrix}\vec{P}_2-\vec{P}_1\\ \vec{P}_3-\vec{P}_1\end{Bmatrix}_{6\times 1}\right]\tag{22}$$

The above condition is less amenable for symbolic algebraic manipulation and for the appropriate selection of free choice variables than an alternative condition discussed next. The indirect form of the fundamental theorem which states that the left nullspace vectors of M should be orthogonal to the right hand side vector provides this condition [15]. Since the matrix M is rank-deficient by two, its left nullspace has a dimension of two. Therefore, two 6×1 vectors span the left nullspace of M. By expressing them in following form with six unknown coefficients, two basis vectors can be obtained as shown in the Appendix. The procedure described in the appendix can easily be extended to the case of multiple links connected with revolute joints.

$$\vec{Y}_{6\times 1}=\begin{Bmatrix}p_1\hat{u}+q_1\hat{v}_1+r_1\hat{w}_2\\ p_2\hat{u}+q_2\hat{v}_1+r_2\hat{w}_3\end{Bmatrix}\tag{23}$$

Thus, if Y_1 and Y_2 span the left nullspace of M, the orthogonality condition gives two scalar equations that must be satisfied in order to ensure the consistency of Eq. (7*c*):

$$Y_i\cdot\begin{Bmatrix}\vec{P}_2-\vec{P}_1\\ \vec{P}_3-\vec{P}_1\end{Bmatrix}=0\quad i=1,2\tag{24}$$

It should be noted that the consistency equations are independent of the loop-closure equations and the auxiliary equations. An attractive feature of the form obtained here is that the consistency equations involve only a subset of the unknown variables ($\hat{u},\hat{v}_1,\alpha_2,\alpha_3,\beta_2$, and β_3). This decoupling of the consistency equations and design Eqs. (7) makes it easy to solve for any two of $\hat{u},\hat{v}_1,\alpha_2,\alpha_3,\beta_2$, and β_3 while assuming six of them freely.

5 Solution Procedure

The first step in the solution procedure is to assign values to six of the eight free-choice variables in Eq. (7*c*) and determine the other two in accordance with the two consistency conditions of Eq. (24). If the variables $\hat{u},\alpha_2,\alpha_3,\beta_2$, and β_3 are chosen to assign arbitrary values, and if $\hat{v}_1$ is represented in terms of ξ_1 and ξ_2 which denote its elevation and azimuthal angles in the global reference frame, the consistency Eq. (24) can be written as

$$\begin{aligned}A_1\sin\xi_1+B_1\sin\xi_2\cos\xi_1+C_1\cos\xi_2\cos\xi_1+D_1=0\\ A_2\sin\xi_1+B_2\sin\xi_2\cos\xi_1+C_2\cos\xi_2\cos\xi_1+D_2=0\end{aligned}\tag{25}$$

where the coefficients are expressed in terms of $\hat{u},\alpha_2,\alpha_3$, $\beta_2,\vec{P}_1,\vec{P}_2$, and $\vec{P}_3$.

The two trigonometric equations in Eq. (25) can be solved for ξ_1 and ξ_2 as

$$\begin{aligned}\xi_1&=\tan^{-1}(\arg_1,\arg_2)\\ \xi_2&=\tan^{-1}(\arg_1,\psi)\end{aligned}\tag{26}$$

where $\arg_1$ and $\arg_2$ are in terms of known quantities and ψ is a solution of the following quartic equation that essentially has a quadratic form:

$$K_a\psi^4+K_b\psi^2+K_c=0\tag{27}$$

with K_a, K_b, and K_c as functions of the coefficients of Eq. (25). The solution of Eq. (27) leads to only two unique real solutions for $\hat{v}_1$.

The second step is to select any four linear equations from Eq. (7*c*) which are now consistent, and solve them in conjunction with two linear auxiliary equations of Eq. (21) for the link vectors $\vec{B}_1$ and $\vec{C}_1$. Since for any consistent choice of $\hat{u},\alpha_2,\alpha_3,\beta_2$, and β_3, two solutions can exist for $\hat{v}_1$, two R-R dyads that pass through the three design points are obtained. It was observed that these two dyad solutions do not form a Bennett linkage unlike the rigid-body guidance case. The solution procedure and the related issues are discussed in detail by Krovi [2].

6 Coupled Spatial R-R Dyad

The synthesis procedure described above is now applied to the design of a coupled spatial R-R dyad mechanism shown in Fig.1(*b*). The kinematic design of R-R dyad and coupled R-R mechanism are identical with a minor difference in the total number of variables. Since the rotations of the two revolute axes are coupled linearly with a coupling ratio g, i.e.,

$$\beta_i=g\,\alpha_i\quad i=2,3\tag{28}$$

for three design positions there will be only three unknowns α_2, α_3, and g instead of four ($\alpha_2, \alpha_3, \beta_2$, and β_3) as in the case of the R-R dyad. Therefore, only five variables can be assigned arbitrary values.

Using the five free-choice variables, the three design position synthesis procedure was used in optimizing the coupled R-R dyad to interpolate a specified 3-D path as closely as possible by minimizing the least-square-error. The optimal synthesis procedure consisted of the following steps. Given a path in the 3-D space, three points were chosen as the design positions, and five free-choice variables were used as the optimization variables. The objective function to be minimized is the least squares error between ''correspondence points'' on the specified 3-D path and the end-effector path of the designed mechanisms, which quantifies the deviation between the two paths. The correspondence points are determined on the basis of equal arc-length segments [2]. The sequential quadratic programming algorithm implemented in the Matlab software (The Math Works Inc.) was used to solve the optimization problem. It should be noted that in each function evaluation, the three design position synthesis of the coupled R-R dyad needs to be performed. Two example studies are described in the next section.

7 Numerical Examples

The path to be traced by the coupled R-R dyad is specified in the form of the X, Y, and Z coordinates of 10 points on the path. Cubic spline interpolation is used to obtain the intermediate point data during the iterative optimization procedure. The data for examples 1 and 2 is shown in Tables 1 and 3 respectively. The data for the first example was created using a known coupled R-R dyad to check if the optimization method returns the known mechanism as the solution. The data of the second example was created arbitrarily by choosing a plane curve and raising it into the third dimension smoothly. Figures 6 and 7 show the coupled R-R dyad solution for the first and second examples. In the first example, the known mechanism solution was not recovered although the optimized path is very close to the specified path as shown in Fig. 6. This is attributed to the nonlinearity and non-convexity of the design space and illustrates the difficulties that are typical in

Table 1 The numerical data for the path specified in example 1

Point #	X-coordinate	Y-coordinate	Z-coordinate
1	-1.9932215	7.8333192	18.185092
2	-4.4458151	7.3039887	18.436291
3	-6.5853712	8.0486525	17.778859
4	-7.7020111	9.8229218	16.445882
5	-7.3624875	12.051120	14.742146
6	-5.5421344	14.023706	12.957265
7	-2.6094600	15.136156	11.300676
8	0.8245457	15.074413	9.8800020
9	4.12527190	13.877389	8.7244706
10	6.82969900	11.861732	7.8355153

Table 2 The parameters of the optimized mechanism solution of example 1

$\hat{u}^T = \{0.7753\ 0.0659\ 0.6281\}$

$\hat{v}_1^T = \{-0.4492\ -0.5257\ -0.7224\}$

$\alpha_2 = 29.4244°$

$\alpha_3 = 65.9517°$

$g = 4.5593$

$\bar{A}^T = \{17.4131\ 18.2729\ 23.1507\}$

$\bar{B}_1^T = \{-16.7044\ -0.2804\ -1.3045\}$

$\bar{C}_1^T = \{-2.7018\ -10.1591\ -3.6611\}$

Table 3 The numerical data for the path specified in example 2

Point #	X-coordinate	Y-coordinate	Z-coordinate
1	3.5000000	-4.5000000	0.0000000
2	4.1742945	-4.1577586	0.3135854
3	4.7453369	-3.6168820	0.6237350
4	5.1614497	-2.9101903	0.9270510
5	5.3811611	-2.0831597	1.2202099
6	5.3761753	-1.1908529	1.5000000
7	5.1334465	-0.2941807	1.7633558
8	4.6562145	0.5442498	2.0073918
9	3.9639299	1.2644187	2.2294345
10	3.0910716	1.8130146	2.4270510

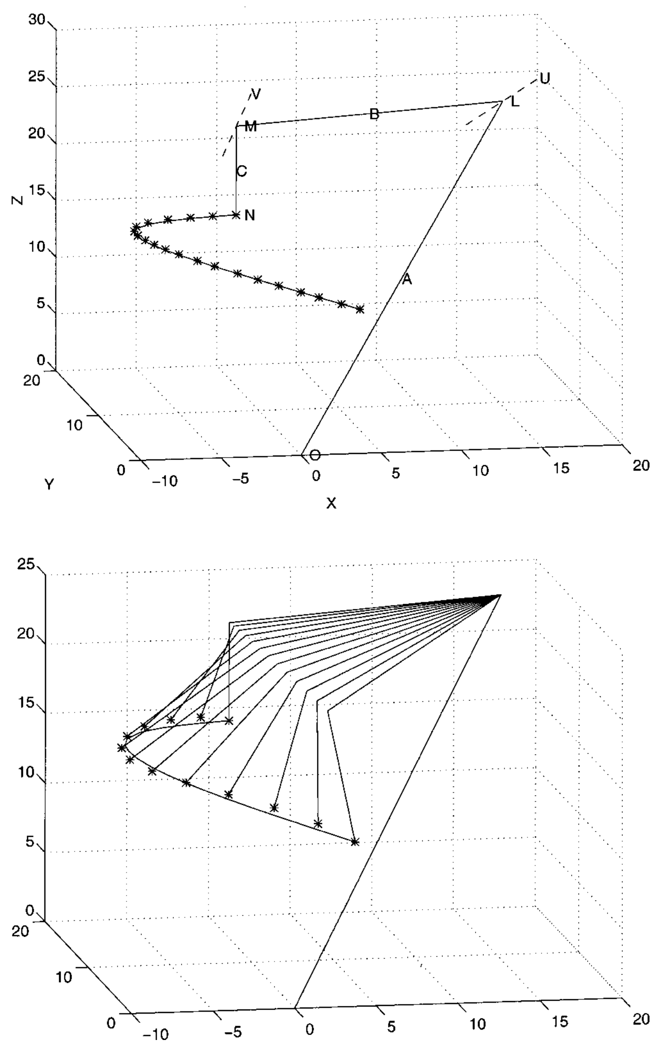

Fig. 6 (*a*) Optimized coupled R-R dyad OLMN of example 1 shown in its first position; (*b*) snapshots of the optimized dyad of example 1 during its entire motion

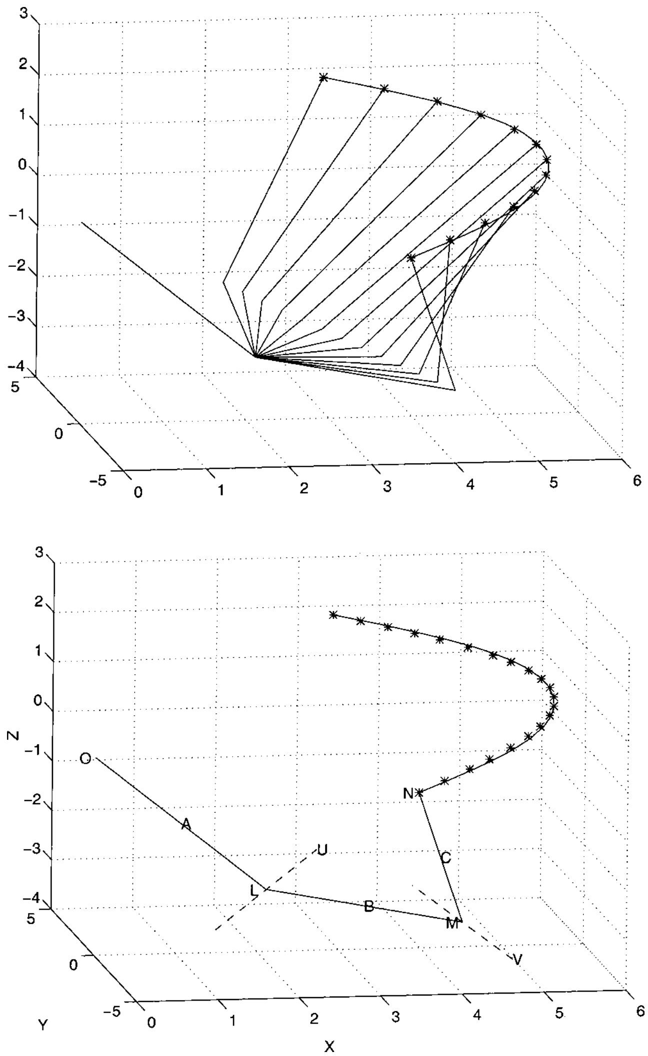

Fig. 7 (*a*) Optimized coupled R-R dyad OLMN of example 2 shown in its first position; (*b*) snapshots of the optimized dyad of example 2 during its entire motion

Table 4 The parameters of the optimized mechanism solution of example 2

$\hat{u}^T$	$= \{0.6273\ 0.1061\ 0.7715\}$
$\hat{v}_1^T$	$= \{-0.5216\ -0.6564\ -0.5450\}$
α_2	$= 21.1360°$
α_3	$= 47.5419°$
g	$= 2.1177$
$\bar{A}^T$	$= \{-8.4619\ 7.5123\ 10.2564\}$
$\bar{B}_1{}^T$	$= \{8.5909\ -6.6822\ -4.7148\}$
$\bar{C}_1{}^T$	$= \{3.3710\ -5.3301\ -5.5416\}$

mechanism synthesis problems. The optimum solution was found to be dependent on the initial guess assigned to the design variables. The parameters of the optimized solutions of examples a and 2 are given in Tables 2 and 4 respectively.

8 Conclusions

In this paper, the necessary theory and optimal design procedure for the kinematic synthesis of spatial coupled R-R dyads for path following applications was presented by building upon previous work on spatial R-R dyads. The rotation matrix and vector notation was used to write the loop-closure equations so that the design equations are linear in the link vector components. This linear system is rank-deficient by two and therefore two extra auxiliary equations need to be added. In contrast to previous work, a systematic approach to selecting the auxiliary equations was described along with a geometric interpretation based on the nullspace of the matrix in the design equations. A nullspace dyad that can be generalized to spatial revolute chains of any number of links, with or without coupling, was presented. The second key concept presented in this paper is the technique of ensuring the consistency of the linear system of design equations in choosing values for the free-choice variables. This was accomplished using the indirect form of the fundamental theorem of linear algebra. A systematic synthesis procedure was then outlined. Optimization of a coupled R-R dyad with the free-choice variables as optimization design variables was performed to generate a specified path in the 3-D space. Two numerical examples were presented to demonstrate the effectiveness of the method.

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Appendix

Derivation of the Left Nullspace of $[M]$. Let the basis vectors that span the left nullspace of $[M]$ (i.e., the nullspace of $[M]^T$) be of the following form with six unknown coefficients:

$$\vec{Y}_{6\times 1} = \begin{Bmatrix} p_1\hat{u} + q_1\hat{v}_1 + r_1\hat{w}_2 \\ p_2\hat{u} + q_2\hat{v}_1 + r_2\hat{w}_2 \end{Bmatrix} \tag{A1}$$

Noting that $(R[\hat{u},\alpha])^T = (R[\hat{u},-\alpha])$, and using $[M]$ defined in Eq. ($7c$) and $\vec{Y}$ defined above, for a homogeneous solution we have

$$[M]^T\{Y\} = \{0\}$$

$$\begin{bmatrix} (R[\hat{u},-\alpha_2]-I) & (R[\hat{u},-\alpha_3]-I) \\ (R[\hat{v}_1,-\beta_2]R[\hat{u},-\alpha_2]-I) & (R[\hat{v}_1,-\beta_3]R[\hat{u},-\alpha_3]-I) \end{bmatrix} \times \begin{Bmatrix} p_1\hat{u} + q_1\hat{v}_1 + r_1\hat{w}_2 \\ p_2\hat{u} + q_2\hat{v}_1 + r_2\hat{w}_2 \end{Bmatrix} = \begin{Bmatrix} \vec{0} \\ \vec{0} \end{Bmatrix} \tag{A2}$$

Noting that for $i=2$, and 3, $(R[\hat{u},-\alpha_i]-I)\hat{u}=0$ and $(R[\hat{v}_1, -\beta_i]R[\hat{u},-\alpha_i]-I)\hat{w}_i=0$, the following relationships can be obtained.

$$\begin{aligned} \hat{u}^T R[\hat{v}_1,\beta_i]\hat{w}_i &= \hat{u}^T\hat{w}_i \quad i=2,3 \\ \hat{v}_1^T R[u,-\beta_i]\hat{w}_i &= \hat{v}_1^T\hat{w}_i \quad i=2,3 \\ \hat{w}_i^T R[\hat{v}_1,-\beta_i]\hat{u} &= \hat{w}_i^T\hat{u} \quad i=2,3 \\ \hat{w}_i^T R[\hat{u},\beta_i]\hat{v}_1^T &= \hat{w}_i^T\hat{v}_1^T \quad i=2,3 \end{aligned} \tag{A3}$$

The block multiplication of the top and bottom rows in Eq. ($A2$) gives

$$\begin{aligned} &(R[\hat{u},-\alpha_2]-I)(q_1\hat{v}_1 + r_1\hat{w}_2) \\ &\quad + (R[\hat{u},-\alpha_3]-I)(q_2\hat{v}_1 + r_2\hat{w}_2) = \vec{0} \\ &(R[\hat{v}_1,-\beta_2]R[\hat{u},-\alpha_2]-I)(p_1\hat{u} + q_1\hat{v}_1) \\ &\quad + (R[\hat{v}_1,-\beta_3]R[\hat{u},-\alpha_3]-I)(p_2\hat{u} + q_2\hat{v}_1) = \vec{0} \end{aligned} \tag{A4}$$

Pre-multiplying the first equation of ($A4$) by $\hat{v}_1^T$ and simplifying using Eq. ($A3$) yields

$$q_1\hat{v}_1^T(R[\hat{u},-\alpha_2]-I)\hat{v}_1+q_2\hat{v}_1^T(R[\hat{u},-\alpha_3]-I)\hat{v}_1=\vec{0} \quad (A5)$$

Pre-multiplying the second equation of ($A4$) by $\hat{w}_2^T$ and simplifying using Eqs. ($A3$) yields

$$p_2\hat{w}_2^T(R[\hat{v}_1,-\beta_3]-I)\hat{u} + q_2\hat{w}_2^T(R[\hat{v}_1,-\beta_3]R[\hat{u},-\alpha_3]-I)\hat{v}_1=\vec{0} \quad (A6)$$

Similarly, pre-multiplying the second equation of ($A4$) by $\hat{w}_3^T$ and simplifying using Eq. ($A3$) yields

$$p_1\hat{w}_3^T(R[\hat{v}_1,-\beta_2]-I)\hat{u} + q_1\hat{w}_3^T(R[\hat{v}_1,-\beta_2]R[\hat{u},-\alpha_2]-I)\hat{v}_1=\vec{0} \quad (A7)$$

Since the rank of $[M]^T$ is only four, two variables among p_1, q_1, r_1, p_2, q_2, and r_2 can be chosen arbitrarily to find the basis vectors of the left nullspace of $[M]$, and the remaining can be solved using Eqs. ($A4$) to ($A7$). For example, $r_1=1$, and $q_1=0$ gives a basis vector in a very simple form:

$$\vec{Y}_1=\begin{Bmatrix}\hat{w}_2\\ k\hat{w}_2\end{Bmatrix} \quad (A8)$$

where $k=-\{\hat{a}^T(R[\hat{u},-\alpha_2]-I)\hat{w}_2\}/\{\hat{a}^T(R[\hat{u},-\alpha_3]-I)\hat{w}_3\}$ and $\hat{a}$ is any vector that has a non-zero component along $\hat{u}\times\hat{v}_1$. The second basis vector can be chosen with $r_1=0$, and $q_1=1$ which does not reduce to as simple a form as in Eq. ($A8$) but is available in the symbolic form.

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