

NATIONAL CENTER FOR EARTHQUAKE  
ENGINEERING RESEARCH

State University of New York at Buffalo

---

---

# A NONSTATIONARY SOLUTION IN RANDOM VIBRATION THEORY

by

J.R. Red-Horse and P.D. Spanos

Department of Civil Engineering and Mechanical Engineering

Rice University

Houston, Texas 77251

Technical Report NCEER-87-0020

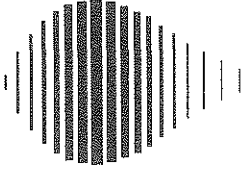
November 3, 1987

This research was conducted at Rice University and was partially supported by the National Science Foundation under Grant No. ECE 86-07591.

## NOTICE

This report was prepared by Rice University as a result of research sponsored by the National Center for Earthquake Engineering Research (NCEER). Neither NCEER, associates of NCEER, its sponsors, Rice University, nor any person acting on their behalf:

- a. makes any warranty, express or implied, with respect to the use of any information, apparatus, method, or process disclosed in this report or that such use may not infringe upon privately owned rights; or
- b. assumes any liabilities of whatsoever kind with respect to the use of, or for damages resulting from the use of, any information, apparatus, method or process disclosed in this report.



---

A NONSTATIONARY SOLUTION IN RANDOM VIBRATION THEORY

by

J.R. Red-Horse<sup>1</sup> and P.D. Spanos<sup>2</sup>

November 3, 1987

Technical Report NCEER-87-0020

NCEER Contract Number 86-3024

NSF Master Contract Number ECE-86-07591

1 Graduate Student, Engineering Mechanics, University of Texas at Austin

2 Professor, Dept. of Civil Engineering and Dept. of Mechanical Engineering, Rice University

NATIONAL CENTER FOR EARTHQUAKE ENGINEERING RESEARCH

State University of New York at Buffalo

Red Jacket Quadrangle, Buffalo, NY 14261

---



## Abstract

An approximate Markov process for a function of the system response of a class of lightly damped, nonlinear random vibration problems is derived using the method of stochastic averaging. The solution of the associated forward Kolmogorov equation for the transition probability density function is presented. The merit of this solution is demonstrated by deriving the spectral density of the system stationary response to step modulated white noise. The reliability of the results is assessed by performing appropriate Monte Carlo simulations.



## **Acknowledgement**

The joint financial support of this work from a PYI-1984 National Science Foundation award and the 86-3024 project from the National Center for Earthquake Research at the University of New York at Buffalo is gratefully acknowledged.





## TABLE OF CONTENTS

SECTION	TITLE	PAGE
1	Introduction.....	1-1
2	Deviation of an Approximate Markov Process.....	2-1
3	Solution of the Forward Kolmogorov Equation .....	3-1
4	Monte Carlo Simulation.....	4-1
5	An Application - Calculation of the Stationary Power Spectral Density.....	5-1
6	Concluding Remarks.....	6-1
	Acknowledgement .....	6-3
7	Notation.....	7-1
8	References.....	8-1



## LIST OF ILLUSTRATIONS

FIGURE	TITLE	PAGE
4-1	Energy Marginal Probability Density; $t = 2\pi\omega_0$ .....	4-2
4-2	Energy Marginal Probability Density; $t = 4\pi\omega_0$ .....	4-3
4-3	Energy Marginal Probability Density; $t = 10\pi\omega_0$ .....	4-4
4-4	Energy Marginal Probability Density; $t = 20\pi\omega_0$ .....	4-5
4-5	Energy Marginal Probability Density; $t = 40\pi\omega_0$ .....	4-6
5-1	Spectral Density of the Zero-Mean Energy Component, $\hat{v}(t)$ .....	5-3



## SECTION 1

### Introduction

Over a recent number of years, the stochastic averaging technique has been applied to a great variety of problems in mathematics and engineering. The method, for which one can trace the development to the Bogoliubov-Mitropolskii [2] theory of averaging for deterministic systems, was introduced based on physical arguments by Stratonovich [14] who applied it to problems in radio engineering. Soon after its initial postulation, stochastic averaging was placed on a sound mathematical foundation by Khasminskii [4] who introduced what is referred to as the stochastic averaging theorem. Further development of the theorem, in the form of more relaxed hypotheses, has followed. Examples of this can be found in the works of Papanicolaou and Kohler [7] and, more recently, Borodin [3]. While the attention of these and other mathematicians has been directed primarily toward the theoretical development of stochastic averaging, the focus of engineers has been on applications of it. See, for example, Roberts and Spanos [10] for a review and a list of references relating to work on the various topics.

The two main advantages gained through the use of stochastic averaging are as follows. First, in the most general case its use allows a  $2n$  first order non-Markovian system to be modeled, under certain conditions, approximately by an  $n$ -order Markovian one. Thus, not only is the order of the system reduced by half, but the Markovian character of the approximation makes it possible to apply well known solution techniques to it. Second, it allows the effects of some types of nonstationary external disturbances to be

incorporated into the approximate system.

In the following discussion, a class of nonlinear, single-degree-of-freedom oscillators subjected to a particular type of nonstationary external excitation is considered. For this problem, the exact solution for the complete statistical description of the two dimensional response vector,  $(x, \dot{x})$ , is unknown. However, by introducing an appropriate transformation, the stochastic averaging theorem can be applied to the system. In this manner, the problem is reduced to that of solving for a one dimensional Markov response vector. The exact solution of this approximate system is attempted via solution techniques for processes of this kind. Specifically, the solution of its associated forward Kolmogorov equation with appropriate initial and boundary conditions is derived. An application of the derived solution is also considered. Proper Monte Carlo simulations are performed to assess the accuracy of the derived analytical results.

## SECTION 2

### Derivation of an approximate Markov process

Consider the following class of single degree of freedom oscillators

$$\ddot{x} + 2\zeta\sqrt{k}\dot{x} + k|x|^\nu \text{sgn}(x) = w(t) \quad (1)$$

where  $\zeta$  is a parameter which represents the system damping. The symbols  $\nu$  and  $k$  characterize the type and strength of the nonlinear term, respectively, and  $\text{sgn}(x)$  is the signum function defined as

$$\text{sgn}(x) = \begin{cases} 1 & x \geq 0 \\ -1 & x < 0. \end{cases} \quad (2)$$

Further, in eqn (1) the symbol  $w(t)$  represents an external, zero mean, Gaussian random excitation which has statistical properties assumed to vary with time. Specifically,  $w(t)$  can be written as

$$w(t) = f(t) \eta(t) \quad (3)$$

where  $f(t)$  is a slowly varying deterministic function of time, and  $\eta(t)$  is a stationary Gaussian random process with the properties

$$E[\eta(t)] = 0 \quad (4)$$

$$E[\eta(t)\eta(t+\tau)] = 2\pi S_0 \delta(\tau) . \quad (5)$$

The symbol  $E[\cdot]$  represents the operator of mathematical expectation and  $\delta(\cdot)$  is the Dirac delta function.

Define  $g(x)$  and  $u(x)$  so that,

$$g(x) = k |x|^v \text{sgn}(x) \quad (6)$$

and

$$u(x) = \int_0^x g(\lambda) d\lambda. \quad (7)$$

Then, consider the following transformation,

$$\dot{x} = -\sqrt{2V} \sin\Phi \quad (8)$$

$$u(x) = V \cos^2\Phi. \quad (9)$$

Eqns. (8) and (9) can be combined to yield,

$$V = \frac{\dot{x}^2}{2} + u(x) \quad (10)$$

$$\Phi = -\tan^{-1} \left[ \frac{\dot{x}}{\sqrt{2u(x)}} \right] \quad (11)$$

where  $V$  represents the energy per unit mass associated with eqn (1). After eqns (10) and (11) are differentiated with respect to time and back substituted into with eqns (8) and (9), the following is derived,

$$\dot{V} = -2\zeta\sqrt{k} [2V \sin^2\Phi] - \sqrt{2V} \sin\Phi w(t). \quad (12)$$

Further, setting  $\Phi = \phi + \theta_0$  where

$$\dot{\phi} = -2\zeta\sqrt{k} \sin\Phi \cos\Phi - \frac{\cos\Phi}{\sqrt{2V}} w(t), \quad (13)$$

yields



$$\dot{\theta}_0 = k \frac{1}{v+1} \frac{v}{(v+1)^{v+1}} \frac{v-1}{\sqrt{2}} V^{\frac{v-1}{2(v+1)}} |\cos\Phi|^{\frac{v-1}{v+1}}. \quad (14)$$

Then, if  $\zeta$  is taken to be small and  $w(t)$  has the property that

$$S_0 f^2(t) = o(\zeta) \text{ as } \zeta \rightarrow 0 \quad \forall t, \quad (15)$$

the right hand sides of eqns (12) and (13) will be small. This implies that both  $V$  and  $\phi$  will be slowly varying, at least in some macroscopic sense. If the above conditions on  $\zeta$  and  $w(t)$  hold, then eqns (12) and (13) satisfy the hypotheses of the stochastic averaging theorem. Application of the theorem consists of evaluating incremental moments which correspond to those of an approximate Itô system. These moments are

$$E[\Delta V(t) \mid V(t)=V, \phi(t)=\phi] = [\pi S_0 f^2(t) - 2\zeta \sqrt{k} \gamma V] \Delta t + o(\Delta t) \quad (16)$$

$$E[\Delta \phi(t) \mid V(t)=V, \phi(t)=\phi] = o(\Delta t) \quad (17)$$

$$E[(\Delta V(t))^2 \mid V(t)=V, \phi(t)=\phi] = [2\gamma \pi S_0 f^2(t) V] \Delta t + o(\Delta t) \quad (18)$$

$$E[(\Delta \phi(t))^2 \mid V(t)=V, \phi(t)=\phi] = \left[ \frac{\pi S_0 f^2(t)}{V} \frac{\gamma}{v+1} \right] \Delta t + o(\Delta t) \quad (19)$$

$$E[\Delta V(t) \Delta \phi(t) \mid V(t)=V, \phi(t)=\phi] = o(\Delta t) \quad (20)$$

where, in the calculation of these expressions,  $V(t)$  and  $\phi(t)$  are treated as deterministic with respect to the expectation operator. Also, use is made of eqns (3) and (5), and time averages are taken with respect to an undamped period of the response at an energy level corresponding to that at the beginning of the interval per eqn (14). The approximate Itô system, therefore, is given by the equation

$$\dot{V} = -[2\zeta\sqrt{k}\gamma V - \pi S_0 f^2(t)] + \sqrt{2\pi S_0 f^2(t)\gamma V} \xi(t) \quad (21)$$

$$\dot{\phi} = \sqrt{\frac{\pi S_0 f^2(t)}{V} \frac{\gamma}{\nu+1}} \chi(t) \quad (22)$$

where  $\xi(t)$  and  $\chi(t)$  represent stationary, independent, Gaussian white noise random processes with the properties

$$E[\xi(t)] = E[\chi(t)] = 0 \quad (23)$$

$$E[\xi(t)\xi(t+\tau)] = E[\chi(t)\chi(t+\tau)] = \delta(\tau) \quad (24)$$

$$E[\xi(t)\chi(t+\tau)] = 0 \quad (25)$$

It is observed that since the moments given by eqns (16) and (18) are independent of  $\phi(t)$  and eqn (20) holds, eqn (21) represents a one-dimensional Markov process in the energy  $V$ . The forward Kolmogorov equation associated with this process is

$$\frac{\partial q_V}{\partial t} = \frac{\partial}{\partial V} \left\{ \left[ 2\zeta\sqrt{k}\gamma V - \pi S_0 f^2(t) \right] q_V \right\} + \gamma\pi S_0 f^2(t) \frac{\partial^2}{\partial V^2} (Vq_V) \quad (26)$$

where  $q_V$  is the transition probability density function with  $q_V(V, t | V_1, t_1) dV =$

Prob[  $V \leq \text{energy} < V + dV$  at time  $t$  given that it was  $V_1$  at time  $t_1$  ], and

$$\gamma = \frac{2(\nu+1)}{(\nu+3)}. \quad (27)$$

For the case of stationary white noise excitation,  $f(t)=1$ , eqn (26) reduces to that derived earlier by Zhu [15] and Roberts [9] using different means. Further, if  $f(t)=1$  and  $\nu=1$ , eqn (26) is in agreement with results given by Spanos [12]. And lastly, if the excitation is taken to be modulated white noise and  $\nu=1$  then eqn (26) agrees, after an appropriate

change of variables, with the solution in Spanos and Solomos [13].



### SECTION 3

#### Solution of the forward Kolmogorov equation

Recall eqn (26) governing the transition probability density function,  $q(V, t | V_1, t_1)$ , and, consider the following change of variables

$$V(t) = \frac{1}{2} a^2(t). \quad (28)$$

This leads to

$$\frac{\partial q_a}{\partial t} = \frac{\partial}{\partial a} \left\{ \left[ \zeta \sqrt{k} \gamma a - \frac{(1 - \frac{\gamma}{2}) \pi S_0 f^2(t)}{a} \right] q_a \right\} + \frac{\gamma \pi S_0 f^2(t)}{2} \frac{\partial^2 q_a}{\partial a^2} \quad (29)$$

where  $q_a = q_a(a, t | a_1, t_1)$  is defined via the following identity

$$q_a(a, t | a_1, t_1) da = q_V(V, t | V_1, t_1) dV. \quad (30)$$

Eqn (29) has appended to it the following initial and boundary conditions

$$q_a(a, t_1 | a_1, t_1) = \delta(a - a_1) \quad (31)$$

$$q_a(0, t | a_1, t_1) = \text{finite} \quad (32)$$

$$\lim_{a \rightarrow \infty} q_a(a, t | a_1, t_1) = 0. \quad (33)$$

Next an algebraic change of variables is performed with

$$z = \frac{a^2}{2c(t)} \quad (34)$$

$$s = t. \quad (35)$$

After chain rule differentiating and term rearranging, eqn (29) reduces to

$$\frac{c(s)}{\gamma\pi S_0 f^2(s)} \frac{\partial q_a}{\partial s} = z \frac{\partial^2 q_a}{\partial z^2} + \left[ z + \left(1 - \frac{1}{\gamma}\right) \right] \frac{\partial q_a}{\partial z} + \left[ \frac{\zeta \sqrt{k} \gamma c(s)}{\gamma\pi S_0 f^2(s)} + \frac{(1 - \frac{\gamma}{2})}{2\gamma z} \right] q_a \quad (36)$$

where  $c(s)$  has been selected to satisfy the ordinary differential equation

$$\frac{dc}{ds} + 2\zeta \sqrt{k} \gamma c - \gamma\pi S_0 f^2(s) = 0. \quad (37)$$

Also, eqns (31)-(33) become

$$q_a(z, s_1 | z_1, s_1) = \sqrt{\frac{2z_1}{c_1}} \delta(z - z_1) \quad (38)$$

$$q_a(0, s | z_1, s_1) = \text{finite} \quad (39)$$

$$\lim_{z \rightarrow \infty} q_a(z, s | z_1, s_1) = 0 \quad (40)$$

where  $c_1 = c(s_1)$  and  $z_1 = z(s_1)$ . Attempt a separation of variables approach in eqn (36) by setting  $q_a = Z(z)T(s)$ . This yields

$$\frac{c(s)}{\gamma\pi S_0 f^2(s)} \frac{\dot{T}}{T} - \frac{\zeta \sqrt{k} c(s)}{\pi S_0 f^2(s)} \frac{T}{T} = z \frac{Z''}{Z} + \left[ z + \left(1 - \frac{1}{\gamma}\right) \right] \frac{Z'}{Z} + \frac{(1 - \frac{\gamma}{2})}{2\gamma z} \frac{Z}{Z} = -\lambda \quad (41)$$

with  $\lambda$  denoting the separation constant. Thus, eqn (41) can be separated into two ordinary differential equations,

$$\dot{T} - \left[ \zeta \sqrt{k} \gamma - \frac{\lambda \gamma \pi S_0 f^2(s)}{c(s)} \right] T = 0 \quad (42)$$

and

$$zZ'' + \left[ z + \left(1 - \frac{1}{\gamma}\right) \right] Z' + \frac{(1-\frac{\gamma}{2})}{2\gamma z} Z + \lambda Z = 0. \quad (43)$$

Consider first the solution of eqn (42), for which it is noted that eqn (37) gives

$$c(s) = \gamma \pi e^{-2\zeta\sqrt{k}\gamma s} \int_{s_0}^s S_0 f^2(\xi) e^{2\zeta\sqrt{k}\gamma\xi} d\xi. \quad (44)$$

Here  $s_0$  has been selected so that the constant of integration is zero. Eqn (42) can be recast as

$$\begin{aligned} \frac{dT}{T} &= \zeta\sqrt{k}\gamma ds - \frac{\lambda S_0 f^2(\xi) e^{2\zeta\sqrt{k}\gamma s} ds}{\int_{s_0}^s S_0 f^2(\xi) e^{2\zeta\sqrt{k}\gamma\xi} d\xi} \\ &= \zeta\sqrt{k}\gamma ds - \lambda \frac{d \left[ \int_{s_0}^s S_0 f^2(\xi) e^{2\zeta\sqrt{k}\gamma\xi} d\xi \right]}{\int_{s_0}^s S_0 f^2(\xi) e^{2\zeta\sqrt{k}\gamma\xi} d\xi}. \end{aligned} \quad (45)$$

Therefore,

$$T(s) = \exp \left[ \zeta\sqrt{k}\gamma(s-s_0) \right] \left[ K(s_1) \int_{s_0}^s S_0 f^2(\xi) e^{2\zeta\sqrt{k}\gamma\xi} d\xi \right]^{-\lambda} \quad (46)$$

where  $K(s_1)$  is the constant of integration. Select it, for simplicity, such that

$$\left[ K(s_1) \int_{s_0}^s S_0 f^2(\xi) e^{2\xi\sqrt{k}\gamma} d\xi \right] \rightarrow 1 \text{ as } s \rightarrow s_1 . \quad (47)$$

That is,

$$K(s_1) = \frac{1}{\int_{s_0}^{s_1} S_0 f^2(\xi) e^{2\xi\sqrt{k}\gamma} d\xi} . \quad (48)$$

Then,

$$\begin{aligned} T(s) &= \exp \left[ \zeta\sqrt{k}\gamma(s-s_0) \right] \left[ \frac{\int_{s_0}^{s_1} S_0 f^2(\xi) e^{2\xi\sqrt{k}\gamma} d\xi}{\int_{s_0}^s S_0 f^2(\xi) e^{2\xi\sqrt{k}\gamma} d\xi} \right]^\lambda \\ &= \exp \left[ \zeta\sqrt{k}\gamma(s-s_0) \right] r^{2\lambda} \end{aligned} \quad (49)$$

where, taking into consideration eqn (44),

$$r^2 = \frac{\int_{s_0}^{s_1} S_0 f^2(\xi) e^{2\xi\sqrt{k}\gamma} d\xi}{\int_{s_0}^s S_0 f^2(\xi) e^{2\xi\sqrt{k}\gamma} d\xi} = \frac{c_1}{c} e^{-2\xi\sqrt{k}\gamma(s-s_1)} . \quad (50)$$

Consider next the solution of eqn (43), which after dividing through by  $z$  becomes



$$Z'' + \left[ 1 + \frac{(1 - \frac{1}{\gamma})}{z} \right] Z' + \left[ \frac{\lambda}{z} + \frac{1 - \frac{\gamma}{2}}{2\gamma z^2} \right] Z = 0. \quad (51)$$

If the change of variables,

$$Z = z^\beta e^{-z} X, \quad \beta = \frac{1}{\gamma} - \frac{1}{2} \quad (52)$$

is performed, then eqn (51) becomes

$$zX'' + \left[ \left( \frac{1}{\gamma} - 1 \right) + 1 - z \right] X' + \left( \lambda - \frac{1}{2} \right) X = 0, \quad (53)$$

which can be written as

$$zX'' + \left( \frac{1}{\gamma} - z \right) X' + \left( \lambda - \frac{1}{2} \right) X = 0. \quad (54)$$

Eqn (54) is recognized as the confluent hypergeometric equation and is known to have one solution of the form

$$X(z) = M \left[ -\left( \lambda - \frac{1}{2} \right), \frac{1}{\gamma}; z \right] \quad (55)$$

where  $M[a, b; z]$  is the confluent hypergeometric function [1]. The boundary condition as  $z$  becomes unbounded, which is expressed by eqn (40), requires that

$$\sqrt{h(z)} M \left[ -\left( \lambda - \frac{1}{2} \right), \frac{1}{\gamma}; z \right] \rightarrow 0 \quad (56)$$

where

$$h(z) = e^{-z} z^{\frac{1}{\gamma} - 1}. \quad (57)$$

The asymptotic behavior of  $M \left[ -\left( \lambda - \frac{1}{2} \right), \frac{1}{\gamma}; z \right]$  as  $z \rightarrow \infty$  is

$$\frac{\Gamma(\frac{1}{\gamma})}{\Gamma[-(\lambda-\frac{1}{2})]} e^{zz} z^{-(\lambda-\frac{1}{2})-\frac{1}{\gamma}} \quad (58)$$

where  $\Gamma(\cdot)$  is the usual gamma function. This implies that  $\Gamma[-(\lambda-\frac{1}{2})] \rightarrow \infty$ . Consequently,

$$-(\lambda-\frac{1}{2}) = -n, \quad n=0,1,\dots \quad (59)$$

Therefore,

$$X(z) = L_n^{(\frac{1}{\gamma}-1)}(z) \quad (60)$$

where the

$$L_n^{(\alpha)}(z) = \frac{1}{n!} e^{zz} z^{-\alpha} \frac{d^n}{dz^n} (e^{-z} z^{n+\alpha}) \quad (61)$$

are the generalized Laguerre polynomials of order  $\alpha$ . Note here that the  $L_n^{(\alpha)}(z)$  form a complete basis in  $[0,\infty)$  and can be shown via the self-adjoint form of eqn (54) to be orthogonal with respect to  $e^{-z} z^\alpha$  such that

$$\int_0^\infty e^{-x} x^\alpha L_n^{(\alpha)}(x) L_m^{(\alpha)}(x) dx = \begin{cases} 0 & , m \neq n \\ \frac{\Gamma(\alpha+n+1)}{n!} & , m = n. \end{cases} \quad (62)$$

It should also be noted here that since the considered boundary value problem is singular, i.e. it has an infinite domain, the classical approach does not always prove to be a fruitful one. As a digression consider an outline for an alternative method of solution. First a transformation can be applied to eqn (54) in order to eliminate the first order term.

The behavior of the zero-th order term at infinity would determine whether the eigenvalue spectrum was discrete or continuous, which in this case is the former. Then, since it's well known that the solutions to this system consist of at least the generalized Laguerre polynomials which correspond to non-negative integral eigenvalues, it remains to show that no eigenfunctions other than these can exist. To achieve this, a power series solution to eqn (54) would be constructed and shown to converge at infinity to a function which grows at a rate which is greater than that of  $e^{z/z^l}$  where  $l$  is some integer. This would imply that these functions are not square integrable relative to the weight function given in eqn (62) and therefore cannot be eigenfunctions for the system. Thus, it is shown that the eigenvalues can coincide only with the non-negative integers and the eigenfunctions are the generalized Laguerre polynomials. For a more detailed discussion of problems of this kind see Levitan and Sargsjan [5].

Now, in order to proceed with the solution procedure, note that based on eqns (49), (52), and (60),

$$q_a(z,s|z_1,s_1) = \sum_{n=0}^{\infty} H_n \exp \left[ \zeta \sqrt{k} \gamma (s-s_0) \right] r^{2n+1} z^{\frac{1}{\gamma} - \frac{1}{2}} e^{-z} L_n^{(\frac{1}{\gamma}-1)}(z). \quad (63)$$

From eqn (50), it is seen that  $r \rightarrow 1$  as  $s \rightarrow s_1$ . Therefore, after imposing the initial condition, eqn (38), the following can be written

$$\sqrt{\frac{2z_1}{c_1}} \delta(z-z_1) = \sum_{n=0}^{\infty} H_n \exp \left[ \zeta \sqrt{k} \gamma (s_1-s_0) \right] z^{\frac{1}{\gamma} - \frac{1}{2}} e^{-z} L_n^{(\frac{1}{\gamma}-1)}(z). \quad (64)$$

If both sides of eqn (64) are multiplied by

$$z^{-\frac{1}{2}} L_n^{(\frac{1}{\gamma}-1)}(z) \quad (65)$$

and integrated from zero to infinity, the orthogonality condition stated in eqn (62) can be exploited to express  $H_n$  as

$$H_n = \sqrt{\frac{2}{c_1}} L_n^{(\frac{1}{\gamma}-1)}(z_1) \exp \left[ -\zeta \sqrt{k} \gamma (s_1 - s_0) \right] \frac{n!}{\Gamma(\frac{1}{\gamma} + n)}. \quad (66)$$

Then, eqn (63) can be written as

$$q_a(z, s | z_1, s_1) = \sum_{n=0}^{\infty} \frac{n!}{\Gamma(\frac{1}{\gamma} + n)} z^{\frac{1}{\gamma}-\frac{1}{2}} e^{-z r} r^{2n} L_n^{(\frac{1}{\gamma}-1)}(z) L_n^{(\frac{1}{\gamma}-1)}(z_1) \sqrt{\frac{2}{c}}. \quad (67)$$

Further, if eqn (67) is expressed in terms of the original  $a, t$  variables and transformed using the Hille Hardy generating function formula, it can be rewritten as

$$q_a(a, t | a_1, t_1) = \left[ \frac{a^2}{2c} \right]^{\alpha+\frac{1}{2}} \sqrt{\frac{2}{c}} e^{-\frac{a^2}{2c}} \left\{ \frac{\left[ \frac{a^2}{2c} \frac{a_1^2}{2c_1} r^2 \right]}{1-r^2} \exp \left[ \frac{-r^2(a^2 c_1 + a_1^2)}{2c c_1 (1-r^2)} \right] \right\}$$

$$I_\alpha \left[ \frac{a a_1 r}{\sqrt{c c_1 (1-r^2)}} \right] \quad \forall t > t_1 \quad (68)$$

where  $\alpha = \frac{1}{\gamma} - 1$  and  $I_\alpha(\cdot)$  is the Bessel function of the first kind of order  $\alpha$ .

Note that in the linear case,  $\gamma=1, \alpha=0$  eqn (68) agrees with the corresponding result of Spanos and Solomos [13] for the modulated white noise case. Also for  $f(t)=1$ , eqn (68) agrees with the solution of Roberts [8].

Lastly, if  $q_a$  is transformed into  $q_V$  then the following expression is obtained,

$$q_V(V, t | V_1, t_1) = \frac{1}{c^*(t_1, t)} \left[ \frac{V}{V_1 e^{-2\zeta\sqrt{k}\gamma(t-t_1)}} \right]^{\frac{\alpha}{2}} \exp \left[ \frac{-1}{c^*(t_1, t)} (V + V_1 e^{-2\zeta\sqrt{k}\gamma(t-t_1)}) \right] I_{\alpha} \left[ \frac{2\sqrt{VV_1} e^{-2\zeta\sqrt{k}\gamma(t-t_1)}}{c^*(t_1, t)} \right], \quad (69)$$

where

$$c^*(t_1, t) = \gamma \pi e^{-2\zeta\sqrt{k}\gamma t} \int_{t_1}^t S_0 f^2(\xi) e^{2\zeta\sqrt{k}\gamma\xi} d\xi. \quad (70)$$



## SECTION 4

### Monte Carlo Simulation

In order to assess the reliability of the derived solution, a Monte Carlo simulation, Shinozuka [11], was performed. An oscillator, characterized by the parameters  $\nu=3$ ,  $k=\omega_0^2$ ,  $\omega_0=2\pi$  and  $\zeta=0.02$  was subjected to an excitation with

$$f(t) = \frac{1}{4}(e^{-\frac{t}{4}} - e^{-\frac{t}{2}}). \quad (71)$$

For instances in which the initial conditions are deterministic and non-zero, the functional form of the marginal probability density,  $p(V,t)$ , can be recognized as being identical to that of the transition, eqn (69). With this in mind each realization was started with the initial conditions

$$x_0 = 1 \quad (72)$$

$$\dot{x}_0 = -\sqrt{\frac{k}{3}} \quad (73)$$

at time  $t_0=t_1=0$ . Note that for this case  $V_1=V_0=\frac{5k}{12}$ . In figures 4-1 through 4-5 the evolution of the exact and estimated marginal probabilities is shown at 1, 2, 5, 10 and 20 cycles of the oscillator response. These figures show the evolution of the marginal probability density from a delta function at  $V=V_0$  toward another delta function at  $V=0$ , with excellent agreement between the theoretical solution and the simulated data.

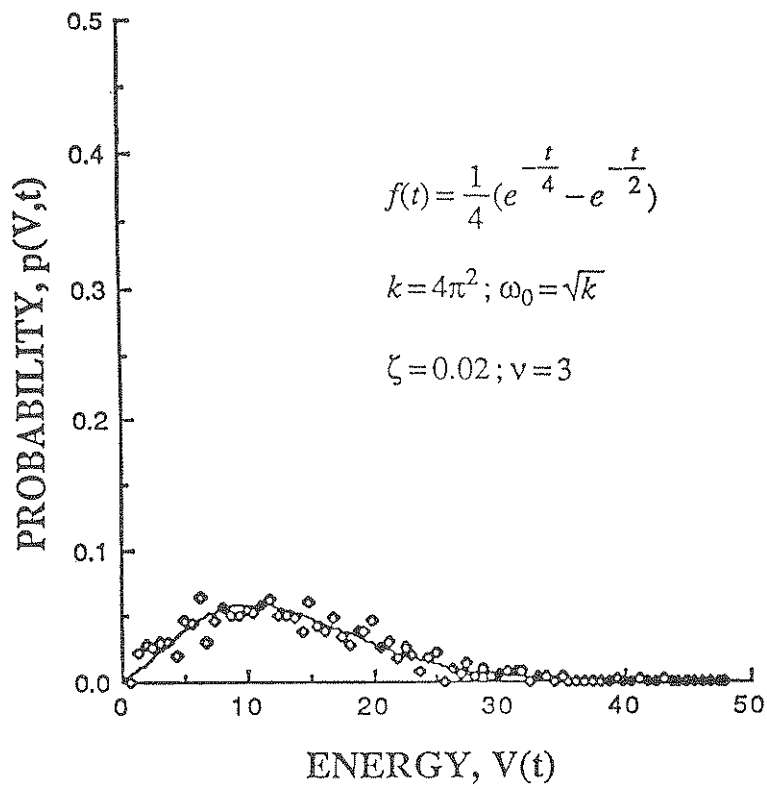


Figure 4-1 Energy Marginal Probability Density;  $t=2\pi\omega_0$



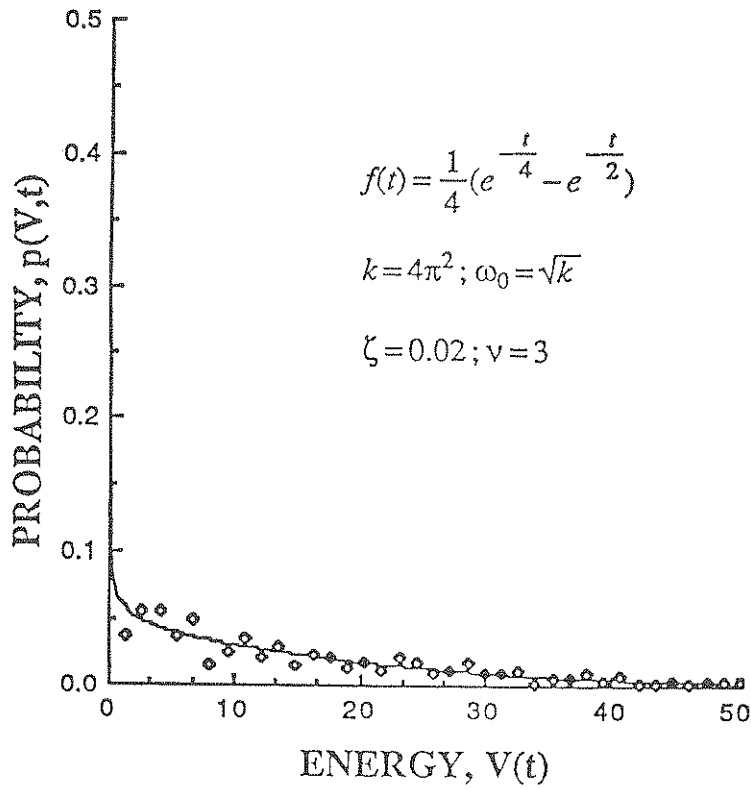


Figure 4-2 Energy Marginal Probability Density;  $t=4\pi\omega_0$

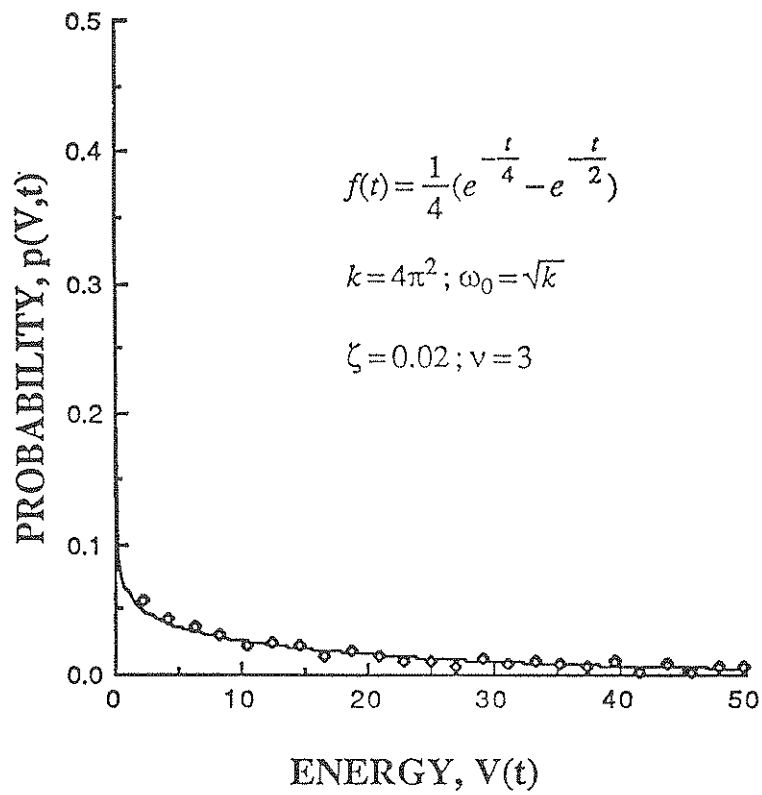


Figure 4-3 Energy Marginal Probability Density;  $t=10\pi\omega_0$

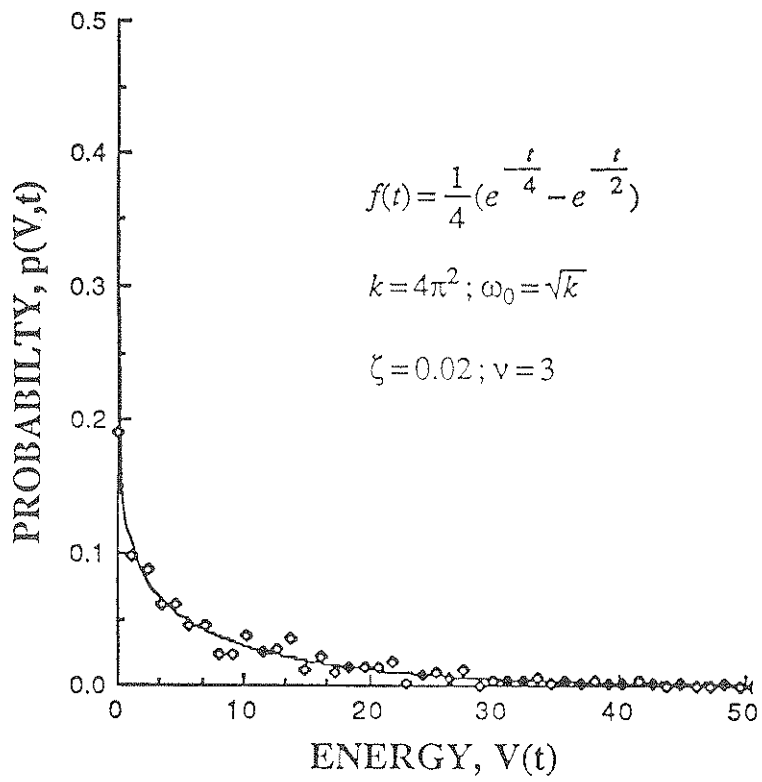


Figure 4-4 Energy Marginal Probability Density;  $t=20\pi\omega_0$

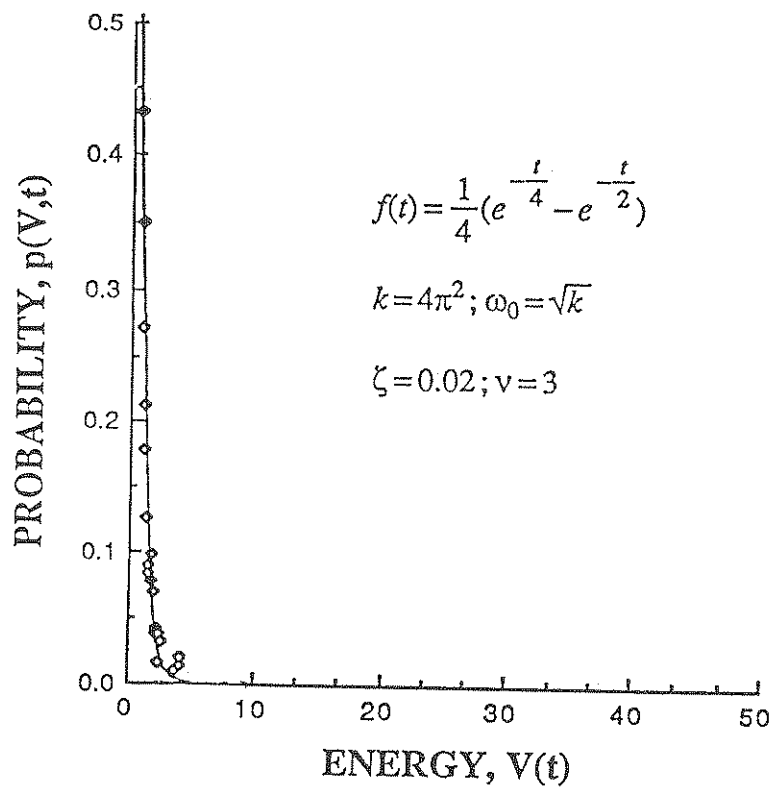


Figure 4-5 Energy Marginal Probability Density;  $t=40\pi\omega_0$

## SECTION 5

### An Application - Calculation of the Stationary Power Spectral Density

As an application of the derived result, eqn (69), consider the calculation of the power spectral density for the zero mean component of the stationary response energy to step modulated,  $f(t) = 1$ , excitation. That is, set

$$V(t) = \bar{V} + \hat{V}(t) \quad (74)$$

where  $\bar{V}$  is the stationary mean energy. Then, the autocorrelation of  $\hat{V}(t)$  is

$$R_{\hat{V}}(\tau) = E[V(t)V(t+\tau)] - E^2[V] \quad (75)$$

and  $R_{\hat{V}}(\tau) \rightarrow 0$  as  $\tau \rightarrow \infty$ . To determine  $R_{\hat{V}}(\tau)$ , the joint probability density function,  $p(V, V_1)$ , for the energy will be needed. To address this, recall eqn (70). For  $f(t) = 1$  it is found that

$$c^*(t_1, t) = \frac{\pi S_0}{2\zeta\sqrt{k}} \left[ 1 - e^{-2\zeta\sqrt{k}\gamma(t-t_1)} \right]. \quad (76)$$

Further, if  $t$  is allowed to become unbounded while  $t_1$  is held fixed, an expression is found for the stationary marginal probability function,  $p(V)$ . Specifically, in this case it is possible to express  $p(V, V_1)$  as

$$p(V, V_1) = q_V(V|V_1)p(V_1). \quad (77)$$

If eqn (77) is used in eqn (75), the following equation for  $R_{\hat{V}}(\tau)$  is derived

$$R_{\hat{V}}(\tau) = \mu^2(\alpha + 1)e^{-2\zeta\sqrt{k}\gamma\tau} \quad (78)$$

where  $\mu = \frac{\pi S_0}{2\zeta\sqrt{k}}$  and the remaining parameters are as previously defined. This result yields the following power spectral density function

$$S_{\dot{v}}(\omega) = \frac{1}{\pi} \mu^2 (\alpha + 1) \frac{(2\zeta\sqrt{k}\gamma)}{\omega^2 + (2\zeta\sqrt{k}\gamma)^2}. \quad (79)$$

Again, in order to assess the validity of the result, eqn (78), a Monte Carlo simulation was performed. Here the previous oscillator parameters were chosen with the exception that  $f(t) = 1$ . Also,  $x_0$  and  $\dot{x}_0$  were sampled from their respective stationary distributions, see for example, Lin [6]. An FFT algorithm was employed to estimate the spectral characteristics of the response energy. The results of this simulation are shown in figure 5-1.

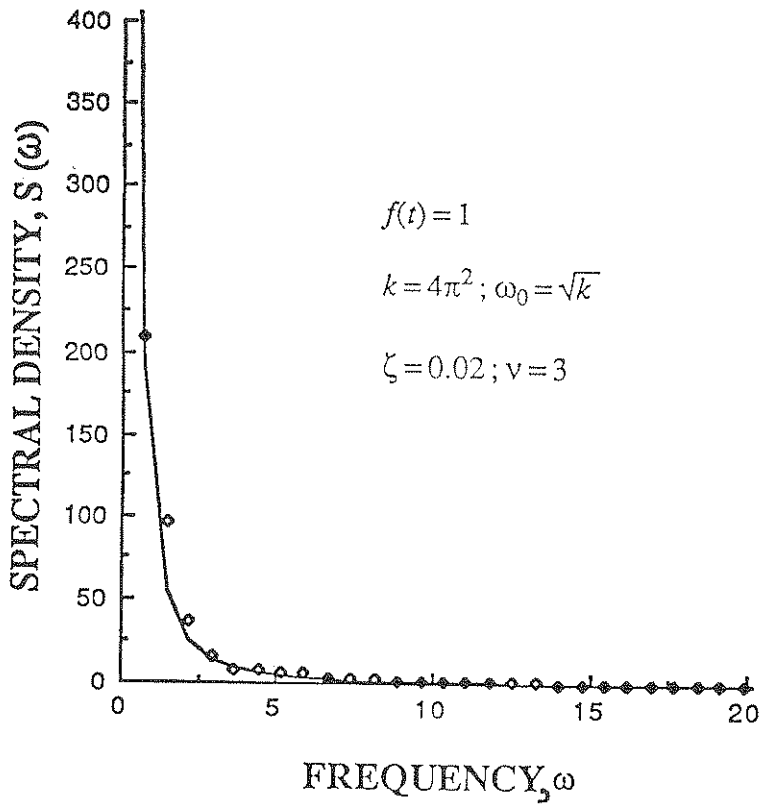


Figure 5-1 Spectral Density of the Zero-Mean Energy Component,  $\hat{V}(t)$





## SECTION 6

### Concluding Remarks

A class of nonlinear single-degree-of-freedom oscillators subjected to zero mean modulated Gaussian white noise excitation was considered. After the energy per unit mass and an associated phase were defined, stochastic averaging was employed to derive an "averaged" energy and phase. Subsequently, this energy was shown to be a one dimensional Markov process. The statistics of the derived one dimensional process converge asymptotically to those of the originally defined energy as the damping coefficient approaches zero.

The forward Kolmogorov equation associated with the energy Markov process was solved explicitly, with the appropriate initial and boundary conditions appended, for the transition probability density. This solution yields a complete statistical description of the approximate process. The reliability of this result was assessed by performing a Monte Carlo simulation for a representative case of the oscillators.

As a special application of the general solution, the stationary response to a step modulated noise was considered. For this case pertinent autocorrelation and power spectral density functions were determined. Again, as a test of these results, a Monte Carlo simulation was performed from which spectral estimates were extracted and compared with those of the power spectral density function mentioned above.

It appears that this result is the sole available closed form solution for the probability density of nonlinear oscillators exposed to excitations possessing non-stationary

power spectra. Clearly, it is a quite useful result since the considered class of oscillators encompasses a variety of physical problems. Further, the result has purely mathematical merit since it can be used to gauge the performance of other, numerical or semi analytical techniques of dealing either with the equation of motion or the associated forward Kolmogorov equation for a particular nonlinear oscillator under random excitation.

## SECTION 7

### Notation

*The following symbols are used in this paper:*

- $a$  =  $\sqrt{2V}$ ;
- $c(t)$  = change of variable parameter;
- $E[\cdot]$  = operator of mathematical expectation;
- $f(t)$  = modulating envelope;
- $g(x)$  = nonlinear stiffness term;
- $h(z)$  = weighting function;
- $k$  = strength of nonlinearity;
- $L_n^{(\alpha)}(z)$  = generalized Laguerre polynomials of order  $\alpha$ ;
- $M[a,b;z]$  = confluent hypergeometric function;
- $o(\cdot)$  = order of magnitude symbol;
- $p(V,t)$  = nonstationary marginal probability density for energy;
- $p(V)$  = stationary marginal probability density for energy;
- $p(V,V_1)$  = joint probability density;
- $q_a$  = transition probability density for  $a$ ;
- $q_V$  = transition probability density for  $V$ ;
- $r$  = parameter in separated time equation;
- $R_{\hat{V}}(\tau)$  = autocorrelation for the zero mean component of energy;

$s$	=	dummy time variable;
$\text{sgn}(x)$	=	signum function;
$S_0$	=	spectral density of $\eta(t)$ ;
$S_{\dot{\gamma}}(\omega)$	=	spectral density for the zero mean component of energy;
$t$	=	time;
$T$	=	temporal separation function;
$u(x)$	=	integral of the stiffness term;
$V$	=	energy per unit mass;
$w(t)$	=	modulated Gaussian white noise excitation;
$x(t)$	=	displacement;
$\dot{x}(t)$	=	velocity;
$Z$	=	spatial separation function;
$\beta$	=	$\frac{1}{\gamma} - \frac{1}{2}$ ;
$\gamma$	=	nonlinearity parameter;
$\Gamma$	=	gamma function;
$\delta(t)$	=	Dirac delta function;
$\zeta$	=	ratio of critical damping;
$\eta(t)$	=	stationary Gaussian white noise;
$\dot{\theta}_0$	=	frequency of the undamped, free vibration;
$\lambda$	=	separation constant;
$\nu$	=	type of nonlinearity;
$\xi$	=	stationary unit Gaussian white noise;

- $\phi$  = random phase of  $x(t)$ ;
- $\Phi$  =  $\theta_0 + \phi$ ;
- $\chi$  = stationary unit Gaussian white noise.



## SECTION 8

### References

1. Abramowitz, M. and Stegun, I. A. eds., *Handbook of Mathematical Functions*, 9th Printing, Dover Publications, Inc., New York (1970).
2. Bogoliubov, N. and Mitropolskii, A., *Asymptotic Methods in the Theory of Non-linear Oscillations*, Gordon and Breach, New York (1961).
3. Borodin, A. N., "Limit Theorem for Solutions of Differential Equations with Random Right-Hand Sides," *Theory Probab. Appl.* **22**(3) pp. 482-497 (1977).
4. Khasminskii, R. Z., "A Limit Theorem for the Solutions of Differential Equations with Random Right-Hand Sides," *Theory of Probability and its Applications* **11**(3) pp. 390-405 (1966).
5. Levitan, B. M. and Sargsjan, I. S., *Introduction to Spectral Theory*, AMS Translations Vol. 39, Providence (1975).
6. Lin, Y. K., *Probabilistic Theory of Structural Dynamics*, reprint, R. E. Krieger, Huntington, N.Y. (1976).
7. Papanicolaou, G. C. and Kohler, W., "Asymptotic Theory of Mixing Stochastic Ordinary Differential Equations," *Commun. pure appl. Math.* **27** pp. 642-668 (1974).
8. Roberts, J. B., "Probability of First Passage Failure for Stationary Random Vibration," *AIAA* **12**(12) pp. 1636-1643 (1974).

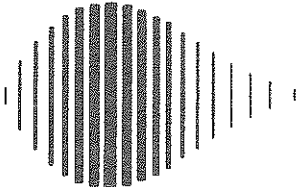
9. Roberts, J. B., "First Passage Probability for Nonlinear Oscillators," *J. Eng. Mech. Div., ASCE* **102**(EM5) pp. 851-866 (1976).
10. Roberts, J. B. and Spanos, P. D., "Stochastic Averaging: An Approximate Method of Solving Random Vibration Problems," *Int. J. Non-linear Mech.* **21**(2) pp. 111-134 (1986).
11. Shinozuka, M., "Monte Carlo Solution of Structural Dynamics," *Computers and Structures* **2** pp. 855-874 (1972).
12. Spanos, P. D., "Energy Analysis of Structural Vibrations under Modulated Random Excitation," *J. Struct. Eng.* **6**(3) pp. 289-302 (1978).
13. Spanos, P. D. and Solomos, G. P., "Markov Approximation to Transient Vibration," *J. Eng. Mech. Div., ASCE* **109**(EM4) pp. 1134-1150 (1983).
14. Stratonovich, R. L., *Topics in the Theory of Random Noise*, Vols 1 and 2, Gordon and Breach, New York (1963).
15. Zhu, W. Q., "Stochastic Averaging of the Energy Envelope of Nearly Lyapunov Systems," *Proceedings of the IUTAM Symposium on Random Vibrations and Reliability*, pp. 347-357 (Edited by K. Hennig) Akademie, (1983).











National Center for Earthquake Engineering Research  
State University of New York at Buffalo