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# VERTICAL AND TORSIONAL VIBRATION OF FOUNDATIONS IN INHOMOGENEOUS MEDIA

by

Anestis S. Veletsos and Kirk W. Dotson

Department of Civil Engineering

Rice University

Houston, TX 77251

Technical Report NCEER-87-0010

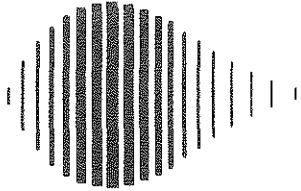
June 1, 1987

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Anestis S. Veletsos<sup>1</sup> and Kirk W. Dotson<sup>2</sup>

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## ABSTRACT

A study of the dynamic impedances for a radially inhomogeneous, infinite viscoelastic layer with a circular hole, and for piles embedded in a medium represented by such layers is made. Systems in vertical and torsional modes of vibration are examined considering both a discontinuous variation in shear modulus and an exponentially increasing, continuous variation. The results are evaluated over wide ranges of the parameters involved and compared with those obtained for a homogeneous layer. In addition, the harmonic response of strip foundations supported at the surface of a vertically inhomogeneous viscoelastic halfspace is examined, and it is shown that within the framework of a previously proposed approximation, the impedances of such foundations for both horizontal and vertical modes of vibration may be deduced from those obtained for the vertically excited, radially inhomogeneous layer.



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## SECTION 1 INTRODUCTION

One of the more efficient and reliable approximate methods for evaluating the dynamic response of piles and other embedded foundations is the one proposed by Novak and his associates (e.g., Ref. 4), in which the restraining action of the surrounding soil is expressed in the spirit of the Winkler approach by a series of independent springs and dashpots arranged in parallel. The dynamic properties of these elements are computed considering the soil to act as a series of thin independent layers of infinite extent in the horizontal plane. This approach effectively assumes that waves in the soil propagate only horizontally. The response quantity of interest in these analyses is the impedance of the soil layer, defined as the complex-valued amplitude of the harmonic force which when applied along an inner circular boundary is necessary to induce a steady-state harmonic displacement of unit amplitude in the direction of the excitation.

In early analyses of this problem (4,5,6,7,9), the impedances were computed from Baranov's solution which is applicable to a layer with uniform properties. The assumption of radial homogeneity may not be realistic, however, because the effective moduli of soils in the immediate vicinity of a foundation are typically lower than for more distant locations. There are two reasons for this: (1) Construction operations disturb and weaken the soil around the foundation; and (2) the strain levels in the immediate vicinity of the foundation are generally larger.

To account for the effect of the reduced resistance of the soil around the foundation, Novak and Sheta (8) have proposed that the shear modulus for the soil within a narrow annular boundary zone be taken lower than for

the rest of the medium. However, in an apparent effort to simplify the analysis of the problem, the boundary zone of the composite layer was presumed to be massless in their solution. A recent rigorous analysis of the problem (11) has revealed that the inertia effects of the thin boundary zone may be quite substantial and should not, in general, be neglected.

The primary objectives of this paper are: (1) To evaluate the correct dynamic impedances of vertically and torsionally excited composite layers over wide ranges of the parameters involved, and to compare the results with those obtained for a homogeneous layer; (2) to evaluate the corresponding layer impedances considering a continuous rather than a discontinuous variation of shear modulus in the radial direction; and (3) to assess the effects of horizontal inhomogeneity in soil properties on the dynamic impedances of representative piles. Both floating and end bearing piles excited vertically and in torsion are examined.

A secondary objective of the paper is to study, by the use of the approximation proposed by Gazetas and Dobry (1), the harmonic response of strip foundations supported at the surface of a vertically inhomogeneous viscoelastic halfspace. It is shown that, within the framework of this approximation, the dynamic impedances of such foundations for both horizontal and vertical modes of vibration may be deduced from those computed for the radially inhomogeneous, vertically excited layer.

The response of horizontally inhomogeneous soil layers was also examined in Lakshmanan and Minai (3) assuming a linear variation of shear modulus within the boundary zone. Although the inertia of the boundary zone was provided for in this study, the relative values of the shear moduli and the width of the boundary zone could not be varied arbitrarily.

## SECTION 2 IMPEDANCES OF COMPOSITE LAYER

**System Considered.**— The system examined is a linear viscoelastic layer of unit depth and infinite extent with a circular hole of radius  $R$ , as shown in Fig. 1. The layer consists of two concentric zones: a thin, annular inner zone of disturbed, generally weakened, material, and an outer zone of undisturbed material. The width of the annular zone and the radius of the interface of the two zones are denoted by  $\Delta R$  and  $R_0$ , respectively. The layer is excited either in axisymmetric vertical harmonic motion by a vertical force which is uniformly distributed along the boundary of the central hole, or in torsional harmonic motion by a torque applied along the same boundary. For each mode of vibration, the layer is presumed to deform in pure shear and only small amplitude oscillations are considered. It is desired to evaluate the dynamic impedance of the layer,  $K$ , defined as the complex-valued amplitude of the vertical force (or torque) which is necessary to induce a vertical displacement (or rotation) of unit amplitude along the boundary of the central hole.

The properties of the soil medium for each region are defined by the complex-valued modulus

$$G^* = G(1 + i \tan \delta) \quad (1)$$

in which  $G$  = the shear modulus of elasticity for the medium;  $i = \sqrt{-1}$ ; and  $\tan \delta$  is a dimensionless measure of the hysteretic material damping capacity of the medium. The subscripts  $i$  and  $o$  will identify properties for the inner and outer zones, respectively. For the solutions presented herein, the mass density for both zones is considered to be the same and is denoted by  $\rho$ .

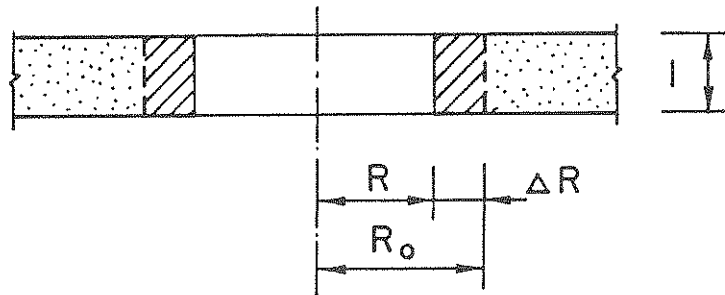
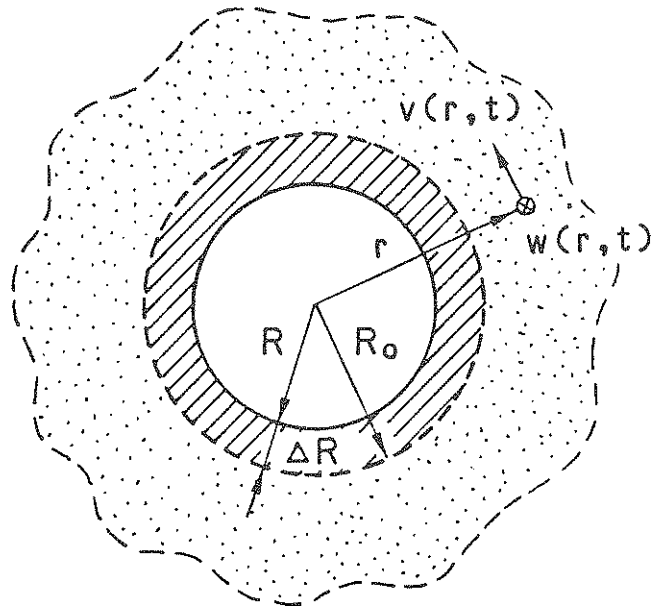


FIGURE 2-1 Composite Layer Considered



**Analysis of System.**— Let  $w(r,t)$  be the vertical displacement at a distance  $r$  from the center of the hole at any time,  $t$ , and let  $v(r,t)$  be the corresponding circumferential displacement. These displacements are of the form

$$w(r,t) = w e^{i\omega t} \quad \text{and} \quad v(r,t) = v e^{i\omega t} \quad (2)$$

in which  $w$  and  $v$  are functions of  $r$ .

The differential equations governing the motions of the composite layer and the expressions for  $w$  and  $v$  have been presented in Ref. 11. By application of these expressions and the procedure outlined therein, it is a simple matter to show that the impedances of the layer for vertical and torsional modes of vibration,  $K_w$  and  $K_\theta$ , are given by

$$K_w = \pi G_i \{ 2\lambda_i (1 + i \tan \delta_i) [A_w K_1(\lambda_i) - B_w I_1(\lambda_i)] \} \quad (3)$$

$$\text{and } K_\theta = 3\pi G_i R \left\{ \frac{2}{3} \lambda_i (1 + i \tan \delta_i) [A_\theta K_2(\lambda_i) - B_\theta I_2(\lambda_i)] \right\} \quad (4)$$

in which  $I_1$  and  $K_1$  are modified Bessel functions of the first and second kind of the first order;  $I_2$  and  $K_2$  are the corresponding functions of the second order;

$$\lambda_i = \frac{ia_i}{\sqrt{1 + i \tan \delta_i}} \quad (5)$$

$$a_i = \frac{\omega R}{v_{si}} \quad (6)$$

$v_{si} = \sqrt{G_i/\rho}$  = the shear wave velocity for the inner or boundary zone;  $A_w$  and  $B_w$  are two of the three non-zero complex-valued constants of integration in the expression for  $w$ ; and  $A_\theta$  and  $B_\theta$  are the corresponding constants in the expression for  $v$ . The constants  $A_w$  and  $B_w$  are determined from the solution of the following system of algebraic equations

$$\begin{bmatrix} K_0(\lambda_i) & I_0(\lambda_i) & 0 \\ K_0(b\lambda_i) & I_0(b\lambda_i) & -K_0(b\lambda_0) \\ -K_1(b\lambda_i) & I_1(b\lambda_i) & \chi K_1(b\lambda_0) \end{bmatrix} \begin{Bmatrix} A_w \\ B_w \\ C_w \end{Bmatrix} = \begin{Bmatrix} 1 \\ 0 \\ 0 \end{Bmatrix} \quad (7)$$

in which  $C_w$  = the third non-zero constant of integration;  $I_0$  and  $K_0$  are modified Bessel functions of the first and second kind of zero order;  $b = 1 + \Delta R/R$ ;  $\chi = \sqrt{G_0^*/G_i^*}$ ; and  $\lambda_0$  = the factor obtained from Eqs. 5 and 6 using the values of  $\tan \delta$  and  $v_s$  for the outer rather than the inner zone. In an analogous manner, the constants  $A_\theta$  and  $B_\theta$  in Eq. 4 are determined from the solution of the system of equations

$$\begin{bmatrix} K_1(\lambda_i) & I_1(\lambda_i) & 0 \\ K_1(b\lambda_i) & I_1(b\lambda_i) & -K_1(b\lambda_0) \\ -K_2(b\lambda_i) & I_2(b\lambda_i) & \chi K_2(b\lambda_0) \end{bmatrix} \begin{Bmatrix} A_\theta \\ B_\theta \\ C_\theta \end{Bmatrix} = \begin{Bmatrix} R \\ 0 \\ 0 \end{Bmatrix} \quad (8)$$

These equations are obtained by requiring that, for each mode of vibration, displacements and stresses be continuous at the interface of the two zones and that the amplitude of the imposed displacement (or rotation) be unity.

**Format of Presentation.**— Eqs. 3 and 4 will be expressed in the form

$$K_w = \pi G_i (\alpha_w + ia_i \beta_w) \quad (9)$$

$$\text{and } K_\theta = 3\pi G_i R^2 (\alpha_\theta + ia_i \beta_\theta) \quad (10)$$

in which  $\alpha_w$ ,  $\alpha_\theta$ ,  $\beta_w$  and  $\beta_\theta$  are dimensionless factors that depend on  $a_i$ ,  $\Delta R/R$ ,  $G_0/G_i$ ,  $\tan \delta_i$  and  $\tan \delta_0$ . In equivalent spring-dashpot representations of the soil layer, the spring stiffnesses are represented by the real parts of Eqs. 9 and 10, and the damping coefficients are related to  $\beta_w$  and  $\beta_\theta$ . The factors  $\alpha_w$  and  $\alpha_\theta$ , and  $\beta_w$  and  $\beta_\theta$  will be referred to as the stiffness and damping factors, respectively.

Note that unlike the format employed in Ref. 11, in which  $K_w$  and  $K_\theta$  were expressed in terms of the shear modulus for the outer region, Eqs. 9 and 10 are expressed in terms of the shear modulus of the inner region. Note further that the frequency parameter,  $a_i$ , has been added in front of the factors  $\beta_w$  and  $\beta_\theta$ . When expressed in this manner, the high-frequency limits of  $\alpha_w$  and  $\alpha_\theta$  for a homogeneous layer with no material damping are unity, and the corresponding values of  $\beta_w$  and  $\beta_\theta$  are 2 and 2/3, respectively. The variations of these factors with  $a_i = a_0 = \omega R/v_s$  are shown in Fig. 2 for several different values of  $\tan \delta$ .

**Results for Vertically Excited Layer.**— The stiffness and damping factors for a vertically excited layer with no material damping are plotted in Fig. 3 as a function of  $a_i$  for several different combinations of  $\Delta R/R$  and  $G_0/G_i$ . The following trends should be observed:

1. The results for the composite layer may be greater or smaller than for the reference homogeneous layer ( $G_0/G_i = 1$ ) depending on the value of  $a_i$  involved. The undulations in these curves are the result of wave reflections from the discontinuous interface of the two zones, and they generally increase with increasing values of  $G_0/G_i$  or  $\Delta R/R$ .

2. For small values of  $a_i$ , increasing  $G_0/G_i$  increases both  $\alpha_w$  and  $\beta_w$ , the increase in  $\beta_w$  being less pronounced than that in  $\alpha_w$ . These trends may be understood by reference to the limiting behavior of a layer for which  $\Delta R/R$  tends to zero, that is, a homogeneous layer with the properties of the outer region. Since the stiffness of such a layer is proportional to  $G_0$  and its damping is proportional to  $G_0/v_{s0}$  or  $\sqrt{G_0}$  (see Eqs. 9, 10 and 6), the low-frequency values of  $\alpha_w$  for a composite layer should be approximately proportional to  $G_0/G_i$ , and those of  $\beta_w$  should be proportional to

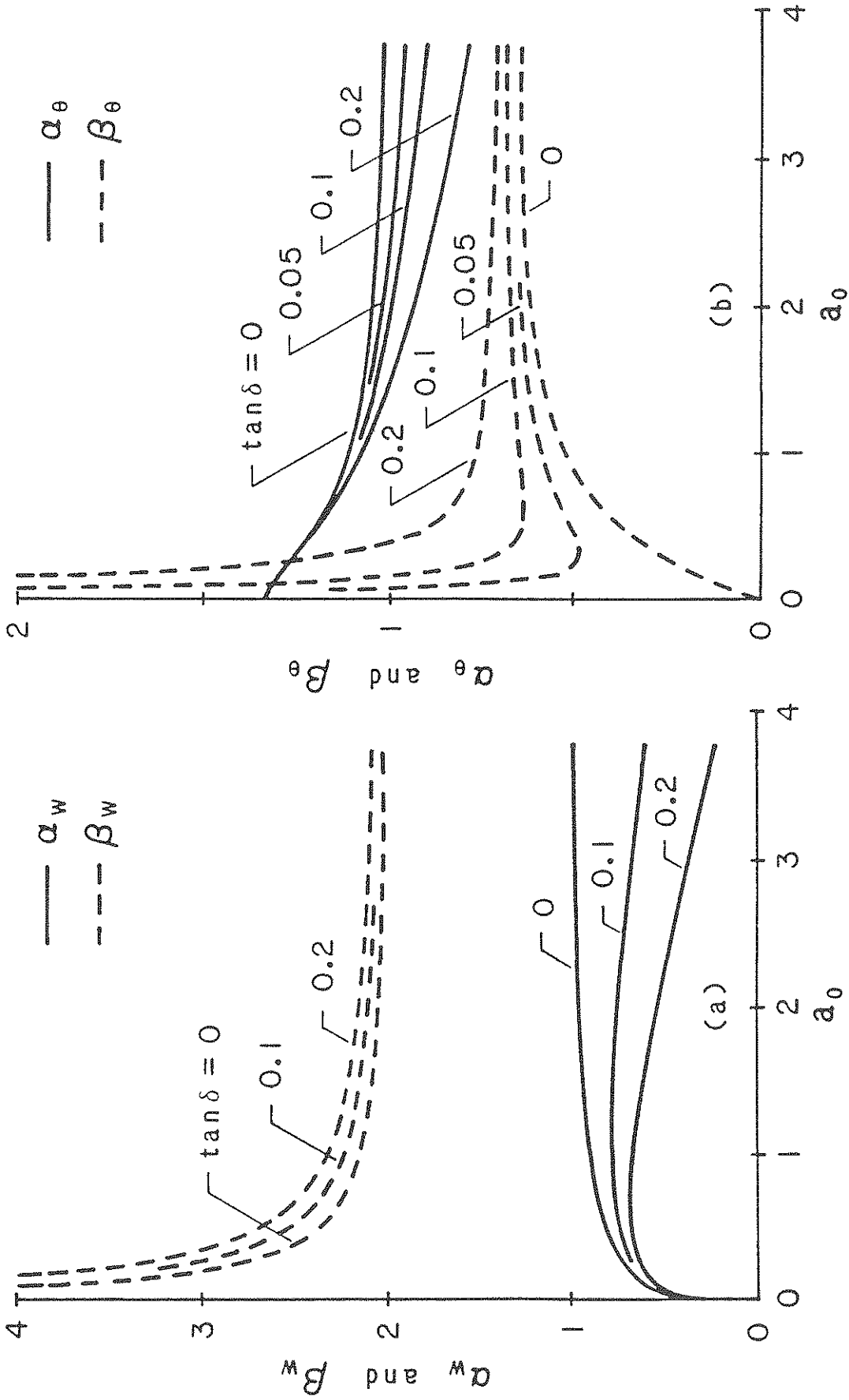


FIGURE 2-2 Impedances for Homogeneous Viscoelastic Soil Layer

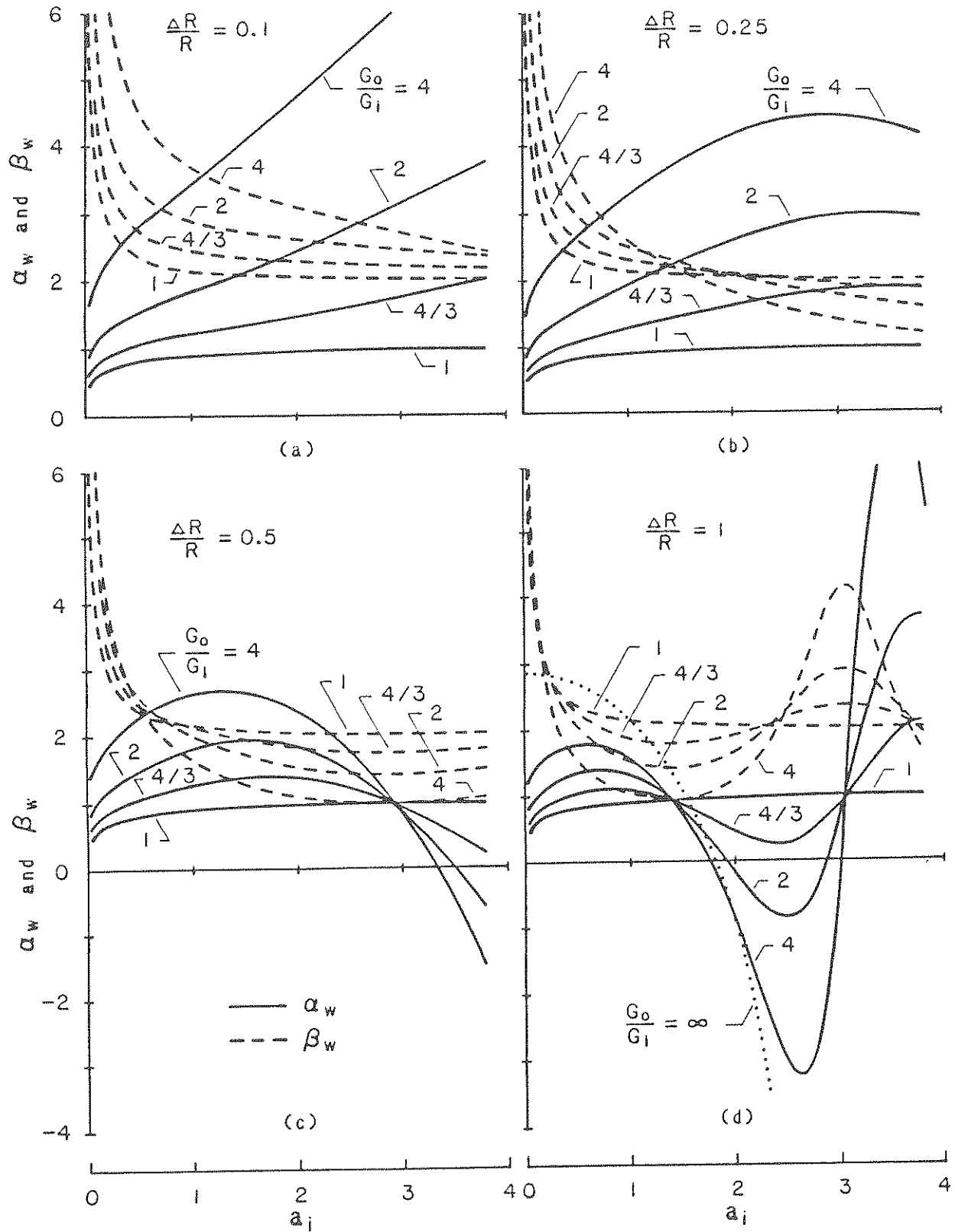


FIGURE 2-3 Impedances For A Vertically Excited Composite Elastic Layer

TABLE 1.— Values of  $c_1$  in Eq. 11 for Vertical and Torsional Modes of Vibration

$\frac{\Delta R}{R}$	Value of $c_1$	
	Vertical	Torsional
(1)	(2)	(3)
Value $\rightarrow 0$	$\pi/2$	$\pi/2$
0.1	1.60	1.66
0.25	1.64	1.79
0.5	1.70	1.98
1.0	1.79	2.27

$\sqrt{G_0/G_i}$ . These expectations are confirmed by the data displayed in Fig. 3(a) for  $\Delta R/R = 0.1$ . For the larger values of  $\Delta R/R$ , the response is clearly influenced by the properties of the inner zone, and the proportionalities referred to do not hold true. However, the general trends of the curves in the low-frequency region of Figs. 3(b) through 3(d) are still similar.

3. The initial ranges of  $a_i$  within which  $\alpha_w$  and  $\beta_w$  for the composite layer are greater than for the homogeneous layer decrease with increasing  $\Delta R/R$ . This is a consequence of the fact that the frequency parameter which controls the behavior of the composite layer is not  $a_i$  but rather the ratio  $\omega/p_1$ , in which  $p_1$  = the fundamental circular natural frequency of the inner zone when its outer boundary is fixed. The latter frequency may conveniently be expressed in the form

$$p_1 = c_1 \frac{v_{si}}{\Delta R} \quad (11)$$

in which  $c_1$  is a dimensionless coefficient that varies with  $\Delta R/R$  as indicated in Table 1. (As  $\Delta R/R$  tends to zero,  $c_1$  tends to  $\pi/2$ , the value obtained by considering the inner zone to act as a series of uniform radial strips deforming in shear.) With  $p_1$  defined by Eq. 11, the frequency ratio,  $\omega/p_1$ , may be expressed in terms of  $a_i$  as follows

$$\frac{\omega}{p_1} = \frac{a_i}{c_1} \frac{\Delta R}{R} \quad (12)$$

Note that the larger the ratio  $\Delta R/R$ , the smaller is the value of  $a_i$  corresponding to a prescribed  $\omega/p_1$ .

4. As  $G_0/G_i$  tends to infinity, the curve for  $\alpha_w$  approaches that for a layer fixed at the far end (see dotted curve in part (d) of Fig. 3). The first zero crossing of this curve occurs at the value of  $a_i$  for which  $\omega = p_1$ .

**Results for Torsionally Excited Layer.**— In Fig. 4 are displayed the stiffness and damping factors for a torsionally excited layer. The general trends of these plots are similar to those identified for the vertically excited layer except for the following:

1. Whereas the vertically excited layer offers no resistance to static loads, with the result that the value of  $\alpha_w \rightarrow 0$  as  $a_i \rightarrow 0$ , the torsionally excited layer is statically stable and the curves for  $\alpha_\theta$  approach non-zero values at  $a_i = 0$ . In fact, the low-frequency limit of  $K_\theta$  can be shown to be given by

$$(K_\theta)_{a_i=0} = 4\pi G_i R^2 \frac{1 + i \tan \delta_i}{1 - (1 - G_i^*/G_0^*)/(1 + \Delta R/R)^2} \quad (13)$$

2. Whereas  $\beta_w$  at small values of  $a_i$  increases with increasing  $G_0/G_i$ ,  $\beta_\theta$  seems to exhibit the opposite trend. The torsionally excited layer is generally less sensitive to the properties of its outer region, and much smaller values of  $\Delta R/R$  than those considered are necessary to reveal the trends identified previously for the vertically excited layer. Indeed, had the results corresponding to  $\Delta R/R = 0.01$  been displayed, the damping curves for  $G_0/G_i > 1$  for the torsionally excited layer would have been higher than those for the homogeneous layer. As a further manifestation

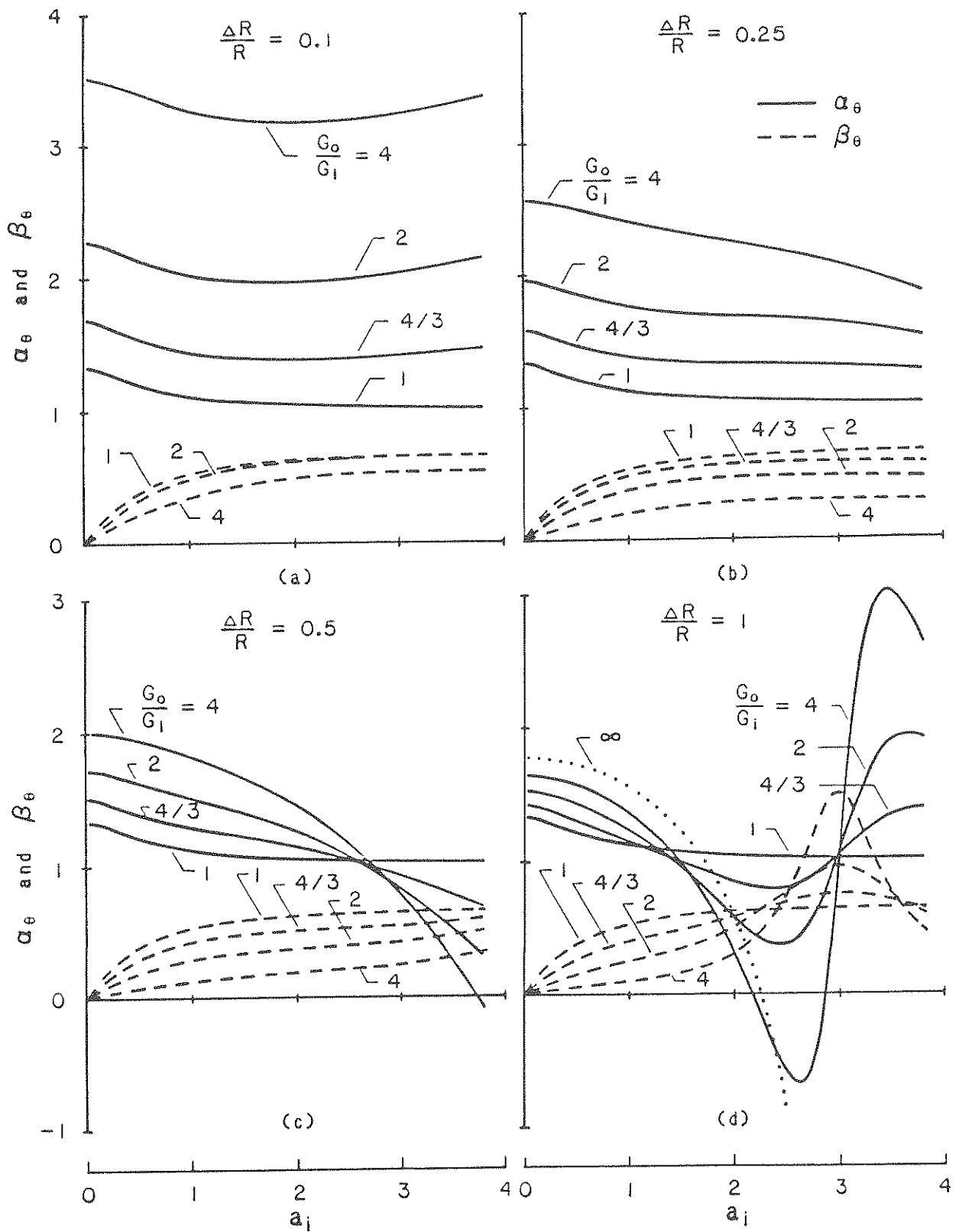


FIGURE 2-4 Impedances For A Torsionally Excited Composite Elastic Layer



of the reduced sensitivity of the response of the torsionally excited layer to the properties of its outer region, the curves in Fig. 4 are not as sensitive to variations in  $G_o/G_i$  as are those in Fig. 3.

**Effect of Material Damping.**— Two sets of the impedance functions for layers without material damping presented in Figs. 3 and 4 are compared in Fig. 5 with those obtained for a viscoelastic layer having two different values of  $\tan \delta_i$ . The material damping factor for the outer layer in these solutions is considered to be one-half as large as that for the inner region, and the values of  $\Delta R/R$  and  $G_o/G_i$  are taken as 1 and 2, respectively. It can be seen that material damping increases the damping capacity and decreases the stiffness of the layer at low values of the frequency parameter. The increase in damping is particularly large for the torsional mode of vibration for which the effects of radiation damping are small. These trends are similar to those reported in Ref. 9 for a homogeneous layer. Material damping also suppresses the undulations of both the stiffness and damping curves; the reductions in amplitude are largest for the higher values of the frequency parameter,  $a_i$ , and for composite layers with large values of  $\Delta R/R$ .

The undulations in the results for layers with wide boundary zones can more clearly be seen in Fig. 6, in which the impedance functions for a layer with  $G_o/G_i = 2$  and values of  $\tan \delta_i = 0.10$  and  $\tan \delta_o = 0.05$  are plotted for three different values of  $\Delta R/R$  in the range between 1 and 4. Included for comparison in these plots is also the solution for a homogeneous layer having the properties of the inner region. As would be expected, the curves for the composite layer oscillate about those for the homogeneous layer, the undulations of the curves for the torsionally excited layer being substantially less pronounced than those for the vertically excited layer.

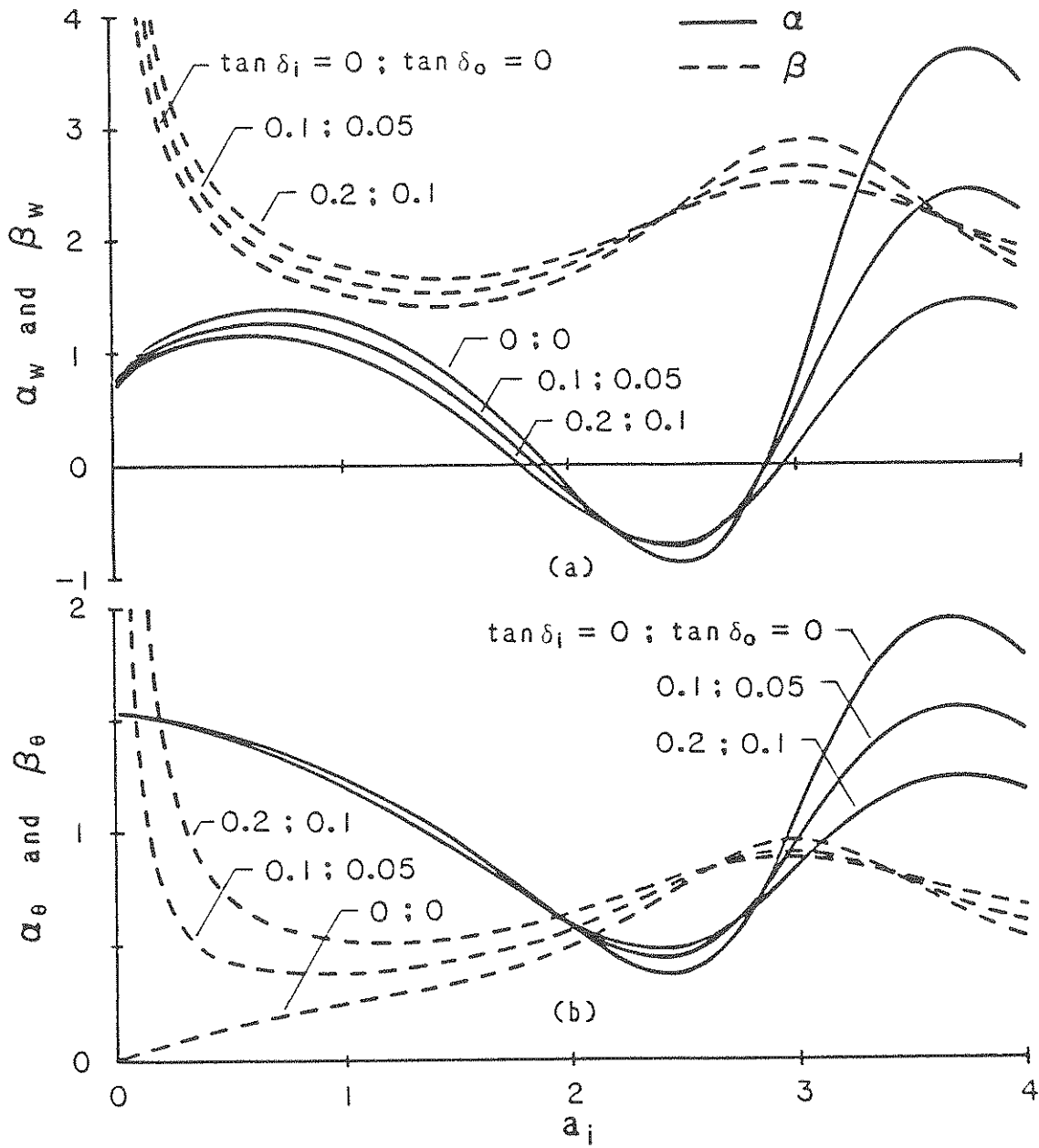


FIGURE 2-5 Effect Of Soil Material Damping On Composite Layer Impedances

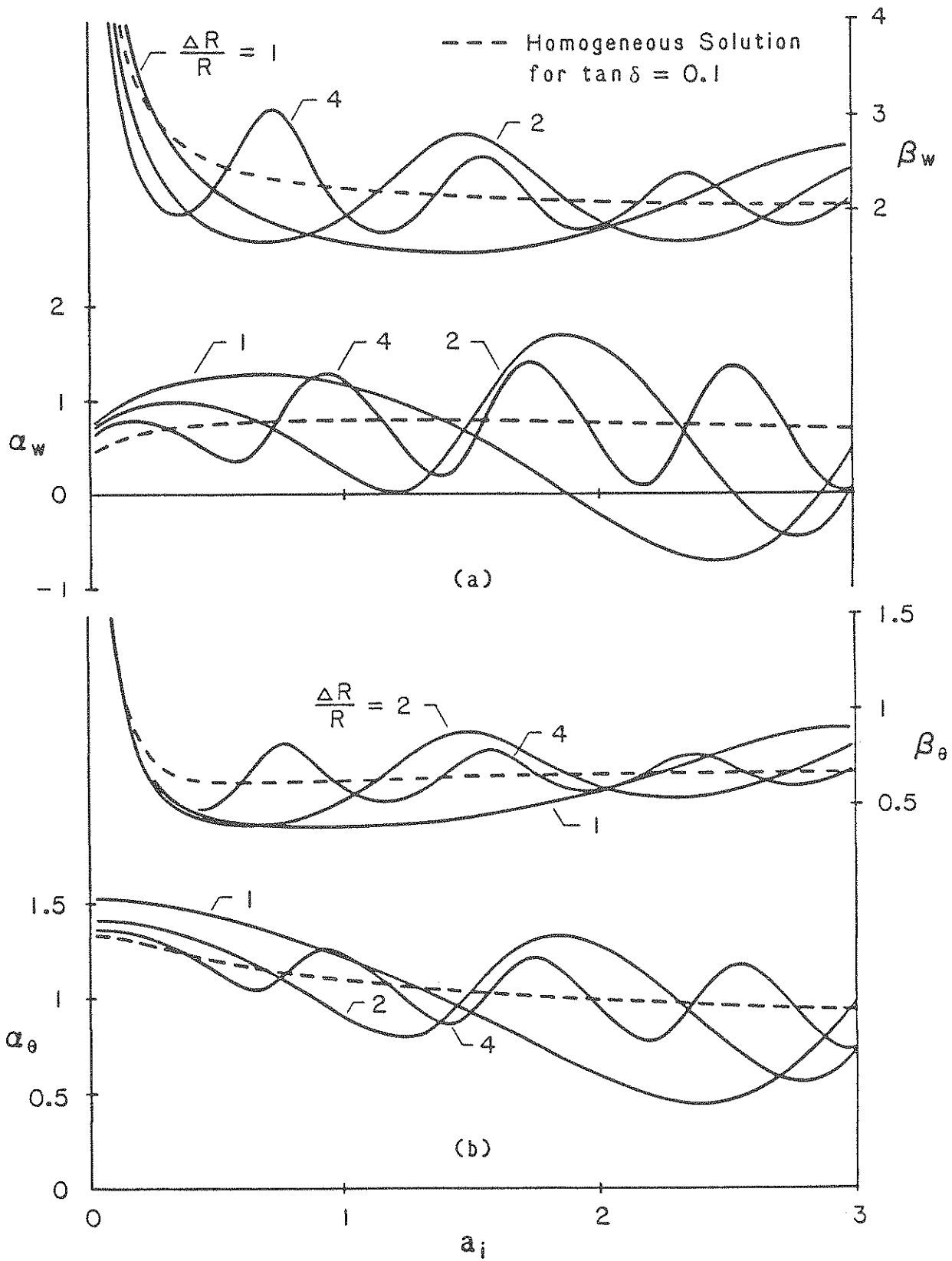


FIGURE 2-6 Limiting Behavior For Composite Layer Impedances



### SECTION 3

#### IMPEDANCES OF LAYERS WITH CONTINUOUS VARIATION IN SHEAR MODULUS

Inasmuch as they are due to wave reflections from the discontinuous interface of the two zones, the undulations in the impedance functions for the composite layers presented in the preceding sections would not be expected to occur for layers with continuous variations in properties. In this section, the complex-valued shear modulus of the viscoelastic layer at a distance  $r$  from the origin is taken in the form

$$G(r) = G_i(1 + i \tan \delta) f(r) \quad (14)$$

in which  $G_i$  = a real-valued quantity representing the shear modulus along the inner circular boundary;  $\tan \delta$  = the soil material damping factor, considered to be independent of  $r$ ; and  $f(r) = f$  is a dimensionless function of  $r$  with a unit value at  $r = R$ . As before, the mass density of the layer,  $\rho$ , is considered to be constant.

The differential equations for the displacement amplitudes of the vertically and torsionally excited layers in this case are (see Ref. 3)

$$r^2 f \frac{d^2 w}{dr^2} + [r^2 \frac{df}{dr} + rf] \frac{dw}{dr} + \frac{\omega^2 r^2}{v_{si}^2 (1 + i \tan \delta)} w = 0 \quad (15)$$

$$\text{and } r^2 f \frac{d^2 v}{dr^2} + [r^2 \frac{df}{dr} + rf] \frac{dv}{dr} + \left[ \frac{\omega^2 r^2}{v_{si}^2 (1 + i \tan \delta)} - r \frac{df}{dr} - f \right] v = 0 \quad (16)$$

in which  $v_{si} = \sqrt{G_i/\rho}$ .

For the solutions presented herein,  $f(r)$  is taken in the form

$$f(r) = \left(\frac{r}{R}\right)^m \quad (17)$$

in which  $m$  is a constant in the range between zero and unity. A value of  $m=0$  defines a homogeneous viscoelastic layer, whereas  $m=1$  defines a layer for which the shear modulus increases linearly with  $r$ . The unbounded

increase in modulus represented by Eq. 17 is clearly not realistic for an actual soil. However, this variation is still of value because the associated layer impedances are generally not sensitive to the characteristics of the layer at large distances from the inner boundary, and are controlled instead by the soil properties in the immediate vicinity of the central hole. Another reason for considering this particular function is that it leads to relatively simple closed-form expressions from which the effects of the various parameters may be assessed readily. The same variation was also considered in an approximate treatment of the problem in Ref. 1.

**Solutions of Equations.**— For the variation of shear modulus represented by Eq. 17, the solutions of Eqs. 15 and 16 may be written as

$$w(\xi) = r^{-m/2} [E_w K_{\kappa-1}(\kappa\lambda_i \xi^{1/\kappa}) + F_w I_{\kappa-1}(\kappa\lambda_i \xi^{1/\kappa})] \quad (18)$$

$$\text{and } v(\xi) = r^{-m/2} [E_\theta K_{2\kappa-1}(\kappa\lambda_i \xi^{1/\kappa}) + F_\theta I_{2\kappa-1}(\kappa\lambda_i \xi^{1/\kappa})] \quad (19)$$

in which  $\xi = r/R$ ;  $\kappa = 2/(2-m)$ ;  $I_\kappa$  and  $K_\kappa$  are modified Bessel functions of the first and second kind of order  $\kappa$ ; and  $E_w$ ,  $F_w$ ,  $E_\theta$  and  $F_\theta$  are constants of integration which must be determined from the appropriate boundary conditions. The condition of vanishing displacements at  $r \rightarrow \infty$  requires that  $F_w = F_\theta = 0$ . The constant  $E_w$  is determined by requiring that  $w$  be unity at  $\xi = 1$ , and  $E_\theta$  is determined by requiring that the rotation amplitude,  $v(1)/R = 1$ . The resulting displacements are

$$w(\xi) = \xi^{-m/2} \frac{K_{\kappa-1}(\kappa\lambda_i \xi^{1/\kappa})}{K_{\kappa-1}(\kappa\lambda_i)} \quad (20)$$

$$\text{and } v(\xi) = R \xi^{-m/2} \frac{K_{2\kappa-1}(\kappa\lambda_i \xi^{1/\kappa})}{K_{2\kappa-1}(\kappa\lambda_i)} \quad (21)$$

Determined by application of Eqs. 8 and 9 in Ref. 11, the impedances for the vertically and torsionally excited layers, are given by

$$K_w = \pi G_i \left[ 2 \lambda_i (1 + i \tan \delta) \frac{K_{\kappa}(\kappa \lambda_i)}{K_{\kappa-1}(\kappa \lambda_i)} \right] \quad (22)$$

$$\text{and } K_{\theta} = 3\pi G_i R^2 \left[ \frac{2}{3} \lambda_i (1 + i \tan \delta) \frac{K_{2\kappa}(\kappa \lambda_i)}{K_{2\kappa-1}(\kappa \lambda_i)} \right] \quad (23)$$

On separating their real and imaginary parts, Eqs. 22 and 23 may be recast in the form of Eqs. 9 and 10, respectively.

**Presentation and Analysis of Data.**— The impedance factors for an inhomogeneous layer with no material damping are displayed in Fig. 7 for several different values of  $m$  in the range between zero and unity. As anticipated, these curves, unlike the corresponding curves for the composite layer examined previously, are non-oscillatory and approach monotonically their high frequency limits. Note further that whereas the stiffness factor  $\alpha_w$  for the vertically excited homogeneous layer tends to zero as  $a_i \rightarrow 0$ , for the inhomogeneous layers it tends to finite values.

The curves for the damping factor of the vertically excited layer,  $\beta_w$ , are essentially the same as those presented in Ref. 1 for a related problem which is examined in a later section. Inasmuch as the trends of these curves and the explanations for them have already been identified in (1), they will not be repeated here. It is worth emphasizing, however, that the high-frequency limit of these curves, like that of the corresponding curves for the torsionally excited layer, is independent of the inhomogeneity parameter  $m$ . As noted in (1), "the source" in this case "sees the transmitting medium as a homogeneous medium" with properties equal to those of the medium in the immediate vicinity of the source.

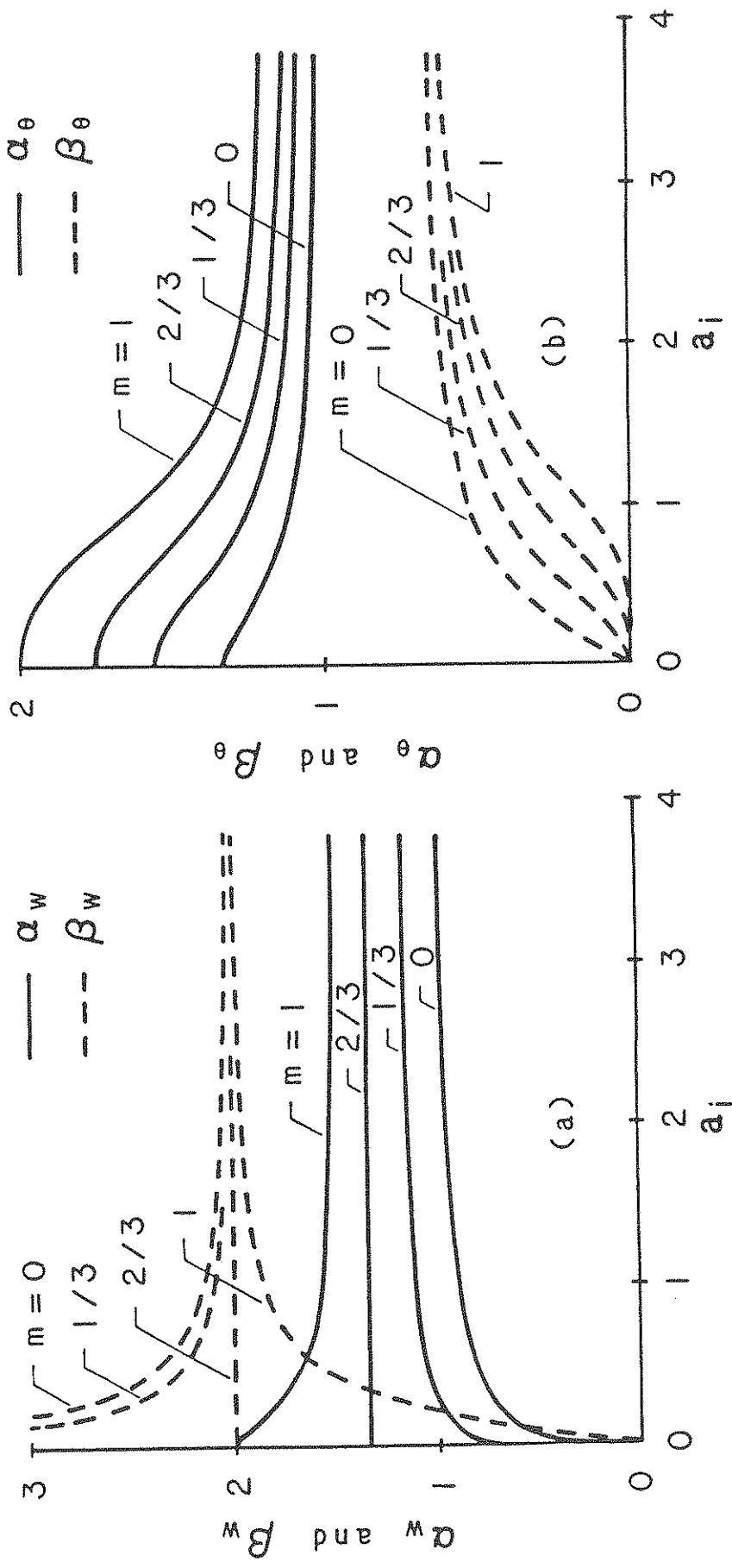


FIGURE 3-1 Impedances For Soil Layer With Exponentially Increasing Variation In Shear Modulus



By contrast, the high-frequency limits of the stiffness factors,  $\alpha_w$  and  $\alpha_\theta$ , do depend on  $m$ . On making use of the asymptotic expansions of modified Bessel functions for large arguments, the high-frequency limits of Eqs. 22 and 23 for a layer with no material damping can be shown to be given by

$$(K_w)_{a_i \rightarrow \infty} = \pi G_i \left[ 1 + \frac{m}{2} + i2a_i \right] \quad (24)$$

$$\text{and } (K_\theta)_{a_i \rightarrow \infty} = 3\pi G_i R^2 \left[ 1 + \frac{m}{6} + i\frac{2}{3}a_i \right] \quad (25)$$

Additional studies of this issue (to be reported elsewhere) suggest that the high-frequency limits of  $\alpha_w$  and  $\alpha_\theta$  are controlled by the quantity  $f'(1)$  = the derivative of  $f(\xi)$  with respect to  $\xi$  evaluated at  $\xi = 1$ . For the particular variation of shear modulus defined by Eq. 17,  $f'(1) = m$ , and the values of  $\alpha_w$  and  $\alpha_\theta$  in Eqs. 24 and 25 may be expressed as  $1 + (1/2)f'(1)$  and  $1 + (1/6)f'(1)$ , respectively.

It is also of interest to examine the low-frequency limiting behavior of the impedance curves. Because of the long-period oscillations involved, the results in this case would be expected to depend on the shear modulus of the layer over a rather wide boundary zone and thus be functions of the inhomogeneity parameter,  $m$ . It can indeed be shown that for  $a_i = 0$ , Eqs. 22 and 23 reduce to:

$$(K_w)_{a_i=0} = \pi G_i [2m (1 + i \tan \delta)] \quad (26)$$

$$\text{and } (K_\theta)_{a_i=0} = 4\pi G_i R^2 \left[ \left( 1 + \frac{m}{2} \right) (1 + i \tan \delta) \right] \quad (27)$$



**SECTION 4**  
**PILE HEAD IMPEDANCES**

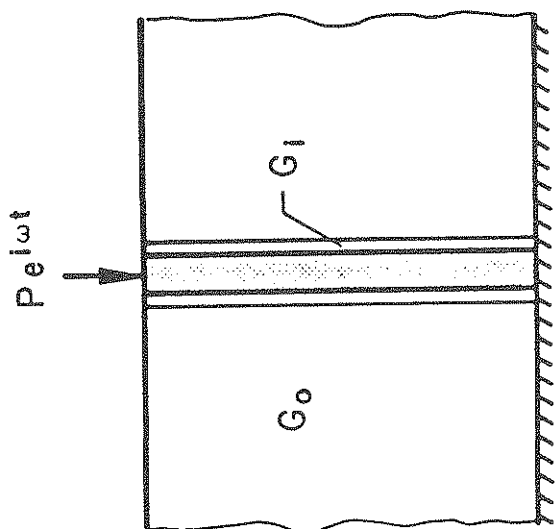
This section demonstrates the influence that radial inhomogeneity in soil properties may have on the dynamic impedances of piles excited vertically or in torsion. Both floating piles and piles supported on a rigid base are considered as shown in Fig. 8. The soil properties above the pile tip are presumed to be independent of depth, and for the floating piles, those beneath the tip are considered to be the same as for the undisturbed outer soil region. The piles are presumed to have a solid circular cross section; their length-to-diameter ratio is taken as  $L/d = 20$ ; the ratio of the mass densities for the pile and soil is taken as  $\rho_p/\rho = 1.6$ , a value that is representative of concrete piles; and the relative stiffnesses of the soil around the pile and of the pile itself are expressed by the ratio  $v_{si}/v_c = 0.025$ , in which  $v_c = \sqrt{E_p/\rho_p}$  = the compressional wave velocity in the pile, and  $E_p$  = Young's modulus of elasticity for the pile. Poisson's ratios for the soil and pile is taken as 1/3 and 1/6, respectively. No material damping is assumed for either the pile or the soil.

The impedances for the vertically and torsionally excited piles,  $\tilde{K}_w$  and  $\tilde{K}_\theta$ , are expressed in the form

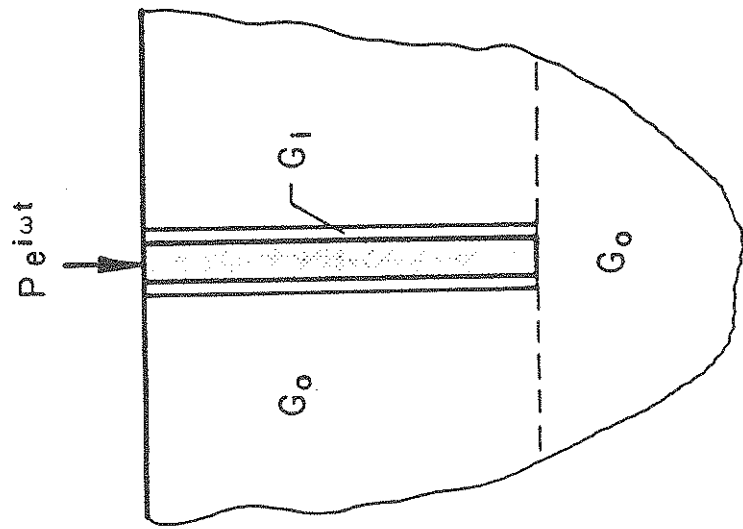
$$\tilde{K}_w = (\tilde{K}_{st})_w [\tilde{\alpha}_w + i a_i \tilde{\beta}_w] \quad (28)$$

$$\text{and } \tilde{K}_\theta = (\tilde{K}_{st})_\theta [\tilde{\alpha}_\theta + i a_i \tilde{\beta}_\theta] \quad (29)$$

in which  $(\tilde{K}_{st})_w$  = the vertical static stiffness of a **floating pile in a homogeneous halfspace** the properties of which are equal to those of the soil in contact with the pile;  $(\tilde{K}_{st})_\theta$  = the corresponding torsional stiffness; and  $\tilde{\alpha}_w$ ,  $\tilde{\beta}_w$ ,  $\tilde{\alpha}_\theta$  and  $\tilde{\beta}_\theta$  are dimensionless factors that depend on the relative properties of the soil and pile.



Rigidly Supported Pile



Floating Pile

FIGURE 4-1 Pile-Soil Systems Considered

For the particular conditions considered, the static stiffnesses were determined from Ref. 10 to be  $(\tilde{K}_{st})_w = 30.1 G_i d$  and  $(\tilde{K}_{st})_\theta = 18.5 G_i d^3$ . The factors  $\tilde{\alpha}$  and  $\tilde{\beta}$  were evaluated from the solutions of the differential equation of motion for the pile given in Refs. 4, 5 and 7, making use of the appropriate soil layer impedances presented in the preceding sections.

**Presentation of Results.**— In Fig. 9 are presented the results obtained for vertically excited piles in a medium with a disturbed zone of the type shown in Fig. 1 and several different values of  $G_0/G_i$ . Both floating and rigidly supported piles are considered, with the width of the boundary zone taken as  $\Delta R/R = 0.5$ . Note that changes in the ratio  $G_0/G_i$  in each case affect the pile head impedance coefficients in much the same way as they do those for the soil layer (see Fig. 3c).

As would be expected, the stiffness of the rigidly supported pile is generally greater than that of the floating pile, whereas the damping is lower. The added resistance of the rigid base decreases the extent to which the soil participates in the overall motion and hence its ability to dissipate energy by radiation of waves. As a further consequence of the reduced importance of the properties of the surrounding soil to the response of rigidly supported piles, the impedances of such piles are less sensitive to variations in  $G_0/G_i$  than are those of floating piles. Included as a frame of reference in these plots is also the normalized stiffness curve (shown dotted) for a rigidly supported, free standing pile excited vertically at its head.

The dynamic impedances of the rigidly supported pile were also evaluated on the assumption that the shear modulus of the soil increases in the radial direction in accordance with Eq. 17. The results for both the vertically and torsionally excited piles are presented in Fig. 10 for sev-

eral different values of the inhomogeneity parameter  $m$ . It can again be seen that changes in the value of  $m$  affect the pile impedance factors in essentially the same manner as they do the corresponding factors for the soil layer (see Fig. 7).

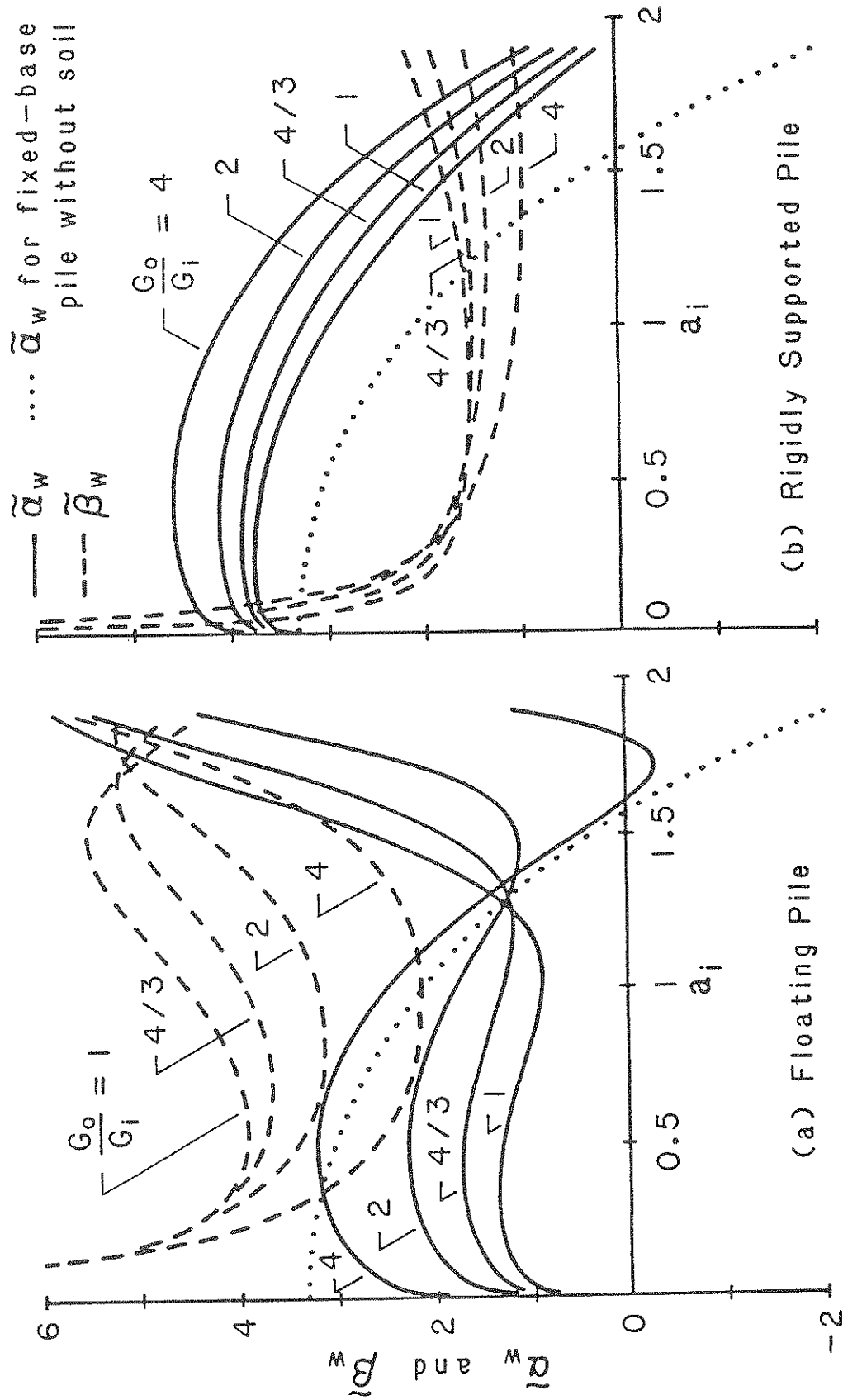


FIGURE 4-2 Impedances For Vertically Excited Piles In A Composite Medium

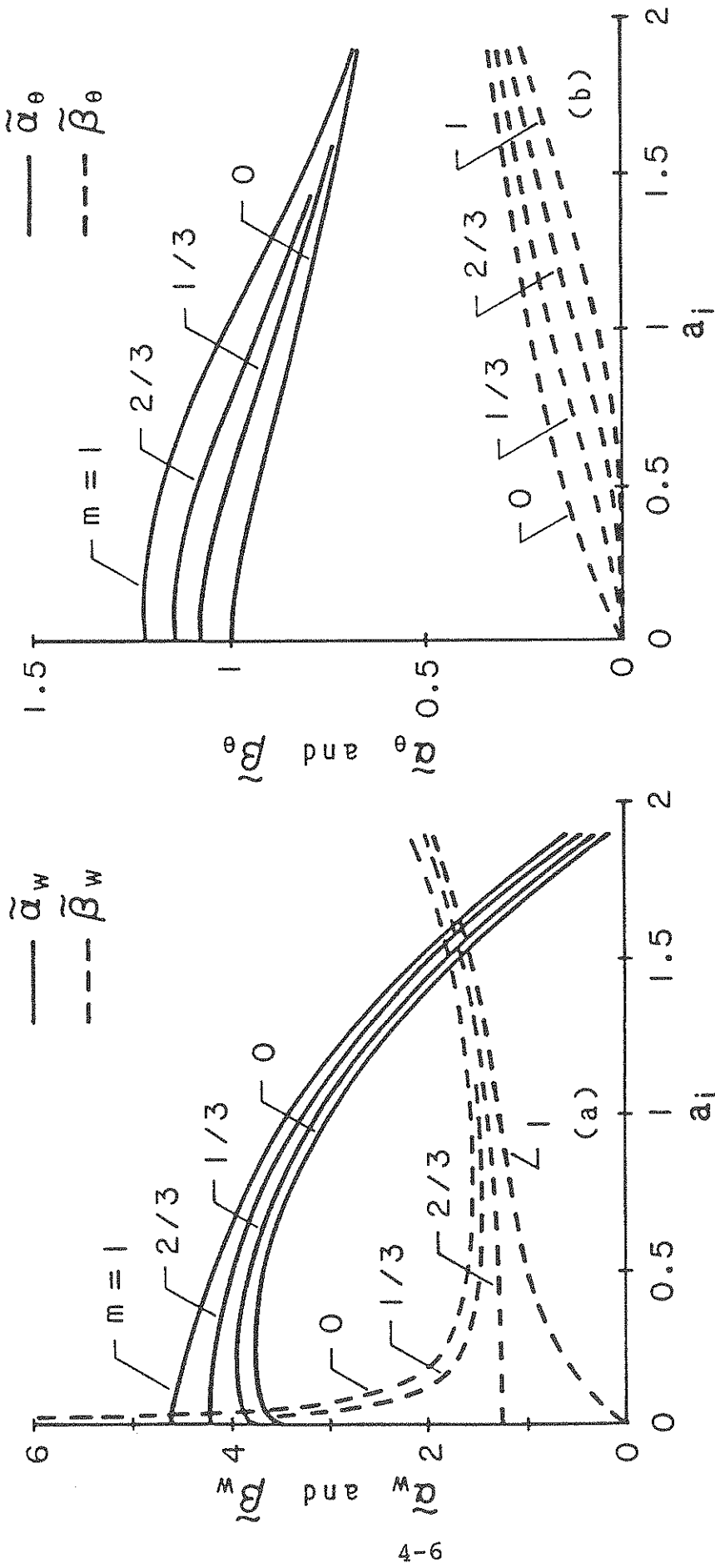


FIGURE 4-3 Impedances For Rigidly Supported Piles In A Stratum With Exponentially Increasing Horizontal Variation In Shear Modulus



**SECTION 5**  
**VIBRATION OF STRIP FOUNDATIONS**  
**ON VERTICALLY INHOMOGENOUS HALFSPACE**

Consider an infinitely long, rigid foundation of width  $2B$  which is supported at the surface of a viscoelastic halfspace for which the complex-valued shear modulus is constant in the horizontal direction but varies in the vertical or  $z$ -direction according to

$$G(z) = G_1(1 + i \tan \delta) \left(\frac{z}{B}\right)^m \quad (30)$$

The foundation is presumed to be massless. In Eq. 30,  $G_1$  = the real-valued shear modulus of the medium at the surface;  $\tan \delta$  = the material damping factor;  $m$  = a non-negative exponent; and the origin of  $z$  is taken at a distance  $B$  above the centerline of the foundation. Like  $\tan \delta$ , the mass density for the medium,  $\rho$ , and Poisson's ratio,  $\nu$ , are considered to be independent of  $z$ . The foundation is presumed to be excited harmonically either by a horizontal force or by a vertical force. The vertical variation of shear modulus defined by Eq. 30 is a generalization of that employed for this problem in Ref. 1 and analogous to the radial variation of  $G(r)$  used in the analysis of the radially inhomogeneous soil layer examined in previous sections (Eqs. 14 and 17).

If one adopts the simplified analysis for the system proposed by Gazetas and Dobry (1), according to which only the quarter-plane that is darkened in Fig. 11 is effective in transmitting the energy imparted to the foundation, it is a simple matter to show that the governing equation of motion for both the horizontally and vertically excited systems may be written as

$$\eta^2 \frac{d^2 u}{d\eta^2} + (m+1) \eta \frac{du}{d\eta} - \lambda^2 \eta^{2-m} u = 0 \quad (31)$$

in which  $\eta = z/B$ ,  $u = u(\eta)$  = the displacement at  $\eta$  in the direction of the excitation

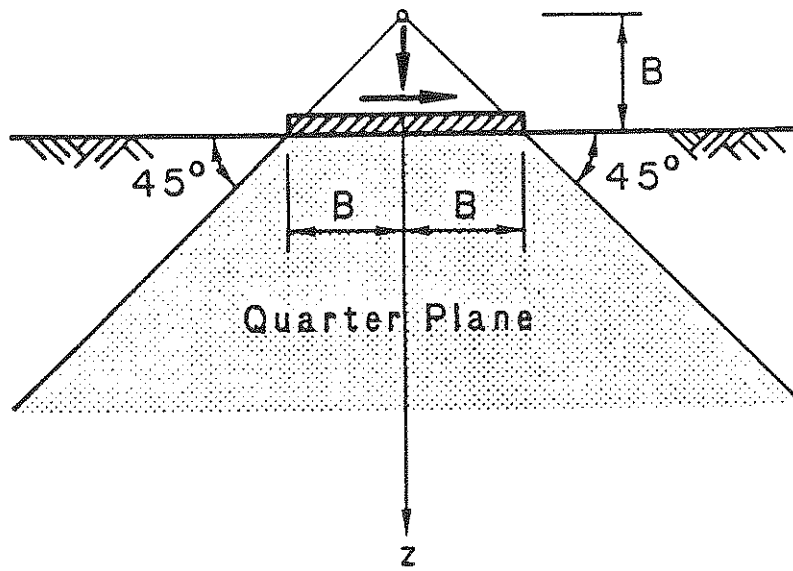


FIGURE 5-1 Approximation For Strip Foundation

$$\lambda = \frac{i \omega B}{v \sqrt{1 + i \tan \delta}} \quad (32)$$

and  $v$  = the appropriate wave propagation velocity for the medium.

The assumptions underlying Eq. 31 are that a horizontal excitation induces only horizontal displacements and only shearing stresses on horizontal planes, whereas a vertical excitation induces only vertical displacements and normal stresses. Furthermore, these displacements and stresses are assumed to be uniformly distributed in the horizontal direction.

For the horizontally excited system,  $u$  in Eq. 31 should be interpreted as the amplitude of the horizontal displacement, and  $v$  should be taken as  $v_{si} = \sqrt{G_i/\rho}$  = the shear wave velocity corresponding to the initial or surface properties of the halfspace. For the vertically excited system,  $u$  should be interpreted as the amplitude of the vertical displacement, and  $v$  should be taken as the 'Lysmer analog' wave velocity, given by

$$v = \frac{3.4}{\pi(1-\nu)} v_{si} \quad (33)$$

**Analogy with Vertically Excited Layer.**— Eq. 31 for the displacement amplitudes of the strip foundation is of the same form as the corresponding equation obtained from Eq. 15 for the radially inhomogeneous, vertically excited layer defined by Eqs. 14 and 17. Furthermore, the boundary conditions for the two problems, and hence the solutions of the two governing equations, are similar. It follows then that the complex-valued impedances for both the horizontally and vertically excited strip foundations,  $K_s$ , may be expressed in a form analogous to Eq. 9 as

$$K_s = G_s (\alpha_s + i a_s \beta_s) \quad (34)$$

in which  $G_s = \rho v^2$ ;

$$a_s = \frac{\omega B}{v} \quad (35)$$

and  $\alpha_s$  and  $\beta_s$  are dimensionless parameters that are functions of  $a_s$ ,  $\tan \delta$  and  $m$ .

With  $G_s$  interpreted in this manner, the factors  $\alpha_s$  and  $\beta_s$  in Eq. 34 are identical to the factors  $\alpha_w$  and  $\beta_w$  presented in Fig. 7a, provided, of course, that the frequency parameter,  $a_i$ , in the latter figure is taken as  $a_s$ . In fact, the values of  $\beta_s$  for the strip foundation on an elastic halfspace reported by Gazetas and Dobry (1) are identical to those presented in Fig. 7a.

If the quarter-plane idealization of the soil medium is presumed to be valid for all depthwise variations of the shear modulus, the data for the impedances of the composite layer presented in Figs. 3, 5a and 6a could also be used to evaluate the impedances of a strip foundation on an elastic stratum that is underlain by a halfspace of higher shear modulus.

For a homogeneous halfspace and for a halfspace for which the increase in shear modulus with depth is represented by a value of  $m = 1/2$  in Eq. 30, Gazetas and Dobry (1) have shown that the **damping values** for strip foundations obtained by the quarter plane idealization compare favorably with those obtained by more nearly exact methods. Notwithstanding this demonstration, additional studies are needed to assess the range of applicability and accuracy of the quarter-plane idealization for vertically inhomogeneous media, particularly those involving abrupt variations in shear modulus. It would also be desirable to assess the accuracy with which the **dynamic stiffnesses** of the foundation may be determined by this approach. Detailed studies of these issues are beyond the scope of the work reported herein, but a comparison of one set of results is presented in Fig. 12.

The data in Fig. 12 are for a vertically excited strip foundation supported at the surface of a viscoelastic stratum which is, in turn sup-

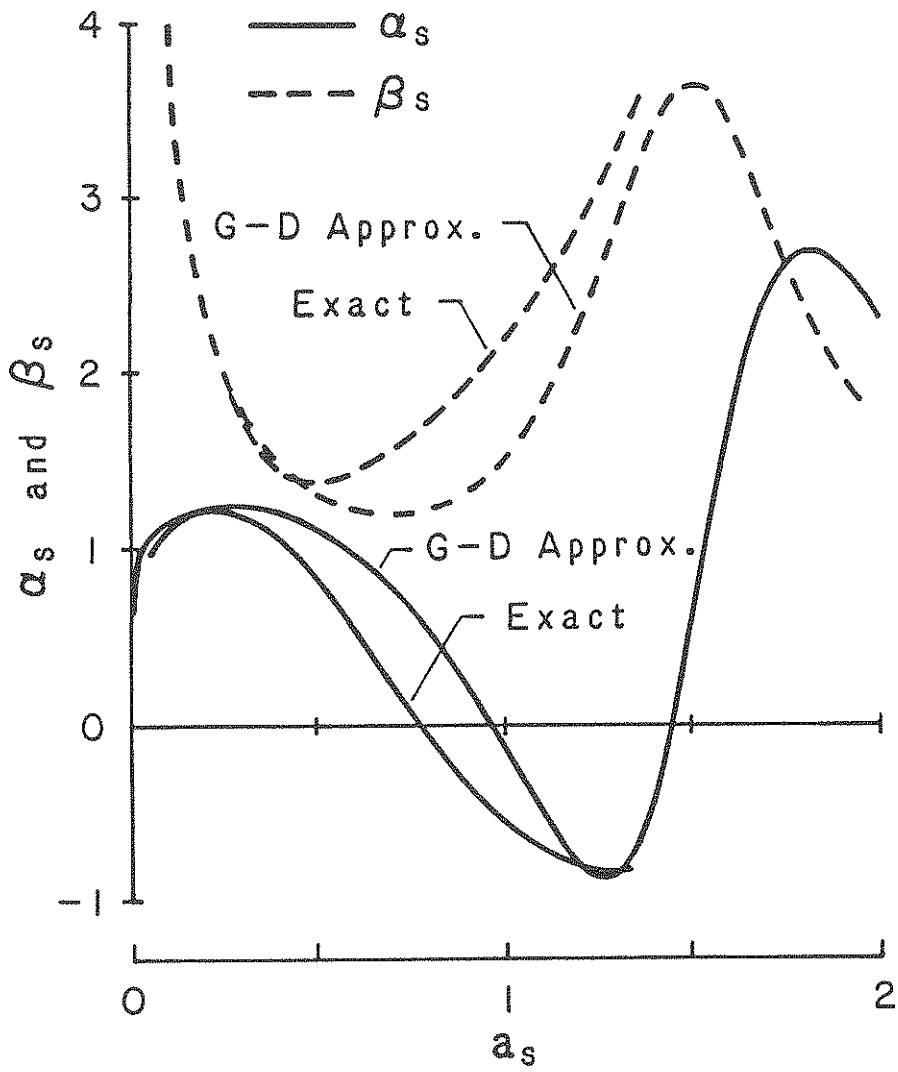


FIGURE 5-2 Comparison Of Exact Strip Foundation Impedances With Those For Quarter-Plane Approximation

ported by a halfspace for which the shear modulus is four times greater than that for the stratum. The thickness of the stratum is considered to be equal to the width of the foundation; Poisson's ratio for the soil is taken as  $\nu = 0.4$ ; and the value of  $\tan\delta$  for the stratum and the halfspace is taken as 0.10. The exact solution is reproduced from Ref. 2. The agreement between the two sets of results in this case is reasonable.

The analogy between the approximate solution for a strip foundation supported on a vertically inhomogeneous medium and the exact solution for a radially inhomogeneous layer does not appear to have been recognized in the past. In fact, in investigating the radiation damping effects in vertically excited soil layers, the damping factor for the layer in Ref. 1 was evaluated by use of the quarter-plane idealization for the medium in the radial direction rather than from the corresponding solution for the strip foundation presented therein. Incidentally, the previously reported expression for the damping coefficient (Eq. 39 in Ref. 1) is in error. The correct expression, in the notation of the original reference, is

$$\frac{C_v}{2\pi B \Delta z \rho V_{s0}} = \left(\frac{\pi}{4}\right)^{1/6} \operatorname{Re} \left\{ -i \frac{H_{6/5}^{(2)} \left( \left(\frac{\pi}{4}\right)^{5/6} \frac{6}{5} \frac{\omega B}{v_{s0}} \right)}{H_{1/5}^{(2)} \left( \left(\frac{\pi}{4}\right)^{5/6} \frac{6}{5} \frac{\omega B}{v_{s0}} \right)} \right\} \quad (36)$$

## SECTION 6 CONCLUSION

With the information and insight into the response that have been provided in this paper, the dynamic impedances of a radially inhomogeneous viscoelastic soil layer with a central hole, and of piles in a medium represented by a series of such layers may be evaluated readily. Both vertically and torsionally excited systems have been examined. Subject to the approximation noted in the text, the impedances for the vertically excited layer may also be used to determine the impedances of horizontally or vertically excited strip foundations supported at the surface of a vertically inhomogeneous halfspace.





**SECTION 7**  
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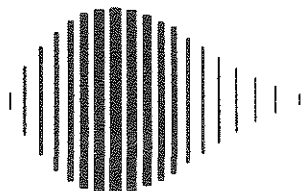
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