Introduction to (Great Lakes) Water Quality Modeling

Ecosystem Restoration Workshop
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Basic idea

- ❖ Based on fundamental "truths", usually stated in terms of conservation of a particular property
  - ➢ e.g. mass, momentum, energy
    - Mass is neither created nor destroyed, but may change form
    - Momentum is governed by certain physical laws (such as Newton's laws), reacts to forces
    - Energy also is conserved, but may change form, such as potential ⇔ kinetic

- ❖ Most physical models based on <u>rate expressions</u> –
  - ➢ For example, consider the process of filling a tank with water; how would you answer the question, "What is the volume of water in the tank at any given time?"
    - The <u>rate</u> at which water is being added to (or leaking from) the tank should, intuitively, be part of the answer
    - In general, some initial starting point (initial condition) also needs to be known in order to calculate a history of water volume

❖ Formalization of problem in mathematical terms –

- We can't start with *V* (volume) = ______ , since we don't know enough to start with
- What do we know?
  - [Rate of change of *V*] = [filling rate] – [leaking rate]

$$\frac{dV}{dt} = Q_{in} - Q_{out}$$

(Q = flowrate)

$$\frac{dV}{dt} = Q_{in} - Q_{out}$$

❖ This is a simple first-order initial value problem, which can be solved if we know the flow values and initial condition

- Let the initial volume be V_0
- Assume the inflow rate is constant
- Assume the outflow rate is proportional to *V*
- Then,

$$Q_{out} = kV \qquad \frac{dV}{dt} + kV = Q_{in}$$

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❖ With the above assumptions, and if *k* is approximately constant (in reality it is a function of *V*), this equation happens to have an analytical solution

❖ More complicated problems that may not have an analytical solution are usually solved using numerical integration

Concentration changes

❖ Now consider the problem of changing concentration of a particular substance in a reactor (jar, tank, stream, lake, etc.);

➢ [rate of change of mass] =

[rate of transport in]

- [rate of transport out]

+ (-) [net rate of transformation or production]

Note concept of <u>control volume</u> used here; also, "transport" includes both advective and diffusive processes.

Mathematical statement

$$\frac{dM}{dt} = \frac{d(CV)}{dt} = (QC)_{in} - (QC)_{out} - w_s CA - KCV$$

(+ dispersion)

M = mass
C = concentration
w_s = settling rate (if applicable)
A = bottom area
K = first-order decay (growth) rate (if applicable)

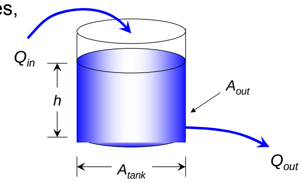
Stella application

❖ Tank filling example:

Recall $$\frac{dV}{dt} + kV = Q_{in}$$

From basic application of energy and mass conservation principles,

$$k \cong \sqrt{\frac{2g}{h}}\frac{A_{out}}{A_{tank}}$$



- If *h* is assumed to change only very slowly, then *k* is approximately constant (*k* is really a function of *V*) and an analytical solution may be obtained,

$$V \cong V_0 e^{-kt} + \frac{Q_{in}}{k}$$

- A steady state solution is found when d/dt → 0 or *t* → ∞,

$$V_{steady} = \frac{Q_{in}}{k}$$

Numerical solution

- (perhaps better in this case, since fewer approximations)
- write

$$\frac{dV}{dt} + kV = Q_{in} \quad \Rightarrow dV = dt\underbrace{\left(Q_{in} - kV\right)}_{\text{rate}}$$

$$V_{i+1} = V_i + dt * \left(Q_{in} - k_i V_i\right)$$

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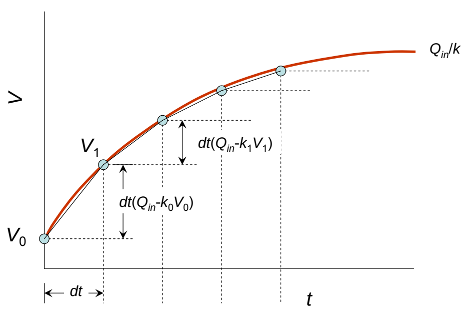


Concentration model

- Now consider the issue of changing concentration, still in a mixed reactor
  - From before, assume $Q_{in} = Q_{out} = Q$ (so, V is constant)

$$\frac{dM}{dt} = \frac{d(CV)}{dt} = (QC)_{in} - (QC)_{out} - w_s CA - KCV$$

Often called loading, *W*

$$\frac{dC}{dt} = \frac{QC_{in}}{V} - C\frac{(Q + w_s A + KV)}{V}$$

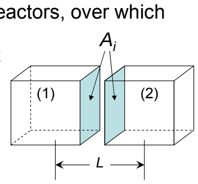
$$\Rightarrow C = C_0 e^{-\lambda t} + \frac{QC_{in}}{\lambda V}, \quad \lambda = \frac{Q + w_s}{V} + K$$

(for constant coefficients)

Multiple reactors

- Add dispersion to account for exchange between reactors
  - Dispersive transport:
    L = distance over which change in C occurs (usually taken as distance between centers of reactors)
    A_i = interfacial area between reactors, over which dispersion occurs
    E' = bulk dispersion coefficient

$$\frac{E A_i}{L}(C_2 - C_1) = E'(C_2 - C_1)$$



Example applications

- Two reactors consist of Lake Huron and Saginaw Bay
  - Assuming steady state, estimate the exchange coefficient value E' using data from Table 1 (handout) for chloride

$$V_2 \frac{dC_2}{dt} = W_2 - Q_2 C_2 + E'(C_1 - C_2)$$

  - Formulate the full unsteady model (in Stella) and ensure that it reproduces the steady state concentrations

Ref: Chapra, Steven C., Surface Water Quality Modeling, McGraw-Hill, 1997

- Consider a model for Lake Erie that is based on a division of the lake into its three main basins (see Figure 1 on handout). Develop a model for phosphorus in the three basins using data from Table1 (handout).
  - You will need to choose settling velocities to try to match the data shown in Table 2 (handout).